

# 交通大學應用數學系博士班組合組入學考試試題

2011 年 5 月 3 日

(5 大題, 共計 100 分)

你的答案要有完整的解釋才能得到全部分數, 但部分解答仍能得部分分數。

Let  $[n] = \{1, 2, \dots, n\}$ . The notation  $d|n$  means  $d$  divides  $n$ .  $\text{g.c.d}(n, t)$  is the greatest common divisor of  $n$  and  $t$ .

1. 20分 How many ways of selecting  $r$  integers in  $[n]$  such that no two of selected integers are consecutive?
2. 20分 Determine the number  $\sum_{A \subseteq [n]} \sum_{B \subseteq [n]} |A \cap B|$  for each positive integer  $n$ , and then prove your answer.
3. 20分 Determine the number  $|\{(A_1, A_2, \dots, A_k) \mid \bigcup_{i=1}^k A_i = [n]\}|$ .
4. 20分 Let  $A_1, A_2, \dots, A_n$  be subsets of  $S$  and  $2t \leq n$ , and note that the intersection of no subsets of  $S$  is interpreted as  $S$ . Show that

$$\sum_{k=0}^{2t-1} (-1)^k \sum_{\substack{T \subseteq [n] \\ |T|=k}} \left| \bigcap_{i \in T} A_i \right| \leq \left| \overline{\bigcup_{i=1}^n A_i} \right| \leq \sum_{k=0}^{2t} (-1)^k \sum_{\substack{T \subseteq [n] \\ |T|=k}} \left| \bigcap_{i \in T} A_i \right|.$$

5. 20分 The Euler function  $\phi : \mathbb{N} \rightarrow \mathbb{N}$  is

$$\phi(n) := |\{1 \leq t \leq n \mid \text{g.c.d}(n, t) = 1\}|$$

for  $n \in \mathbb{N}$ . Show that  $\sum_{d|n} \phi(d) = n$ .