一百學年度國立交通大學應用數學系博士班入學考試試題

科目：分析

Please explain *all* your answers and indicate which theorems you are using!

1. Let $f(x,y)$ be continuous on $[0,1]\times[0,1]$ such that for some finite $C>0$

$$|f(x,y_1)-f(x,y_2)|\le C|y_1-y_2|,\qquad \text{for all } x, y_1, \text{ and } y_2\in[0,1].$$

   (a) (7%) Prove that $\dfrac{\partial f}{\partial y}(x,y)$ exists for almost all $y\in[0,1]$ with each fixed $x\in[0,1]$.

   (b) (7%) Prove that $\dfrac{d}{dy}\displaystyle\int_0^1 f(x,y)\,dx=\int_0^1\frac{\partial f}{\partial y}(x,y)\,dx$.

   (c) (6%) Express $\dfrac{d}{dy}\displaystyle\int_0^{y^2} f(x,y)\,dx$ in terms of the integral of f and $\dfrac{\partial f}{\partial y}$ for $y\in[0,1]$.

2. (a) (10%) Suppose that f is Lebesgue integrable over $[0,1]$. Prove or disprove that there is a decreasing sequence of real numbers such that $\lim_{n\to\infty}a_n=0$ and that $\lim_{n\to\infty}a_n|f(a_n)|=0$.

   (b) (10%) Let $\{f_n\}_n$ be a sequence of Lebesgue integrable functions defined on $[0,1]$ such that $\sup_n\displaystyle\int_0^1|f_n(x)|\,dx<\infty$. Is it true that there is a decreasing sequence of real numbers such that $\lim_{n\to\infty}a_n=0$ and that $\lim_{n\to\infty}a_n|f_n(a_n)|=0$? Justify your answer.

3. Let $\{f_n\}_n$ be a sequence of Lebesgue integrable functions defined on $[0,1]$ with $\lim_{n\to\infty}\displaystyle\int_0^1 f_n(x)\,dx=0$.

   (a) (7%) Is it true that $\lim_{n\to\infty}f_n(x)=0$ for at least one value of $x\in[0,1]$?

   (b) (7%) Is (a) true if the f_n's are further assumed to be continuous on $[0,1]$?

   Justify your answers.

4. (10%) Let f be a Lebesgue integrable function defined on $[0,\infty)$. Compute the limit $\lim_{s\searrow 0}\displaystyle\int_0^\infty f(x)e^{-sx^2}\,dx$.

5. Let f be a function in $L^2[0,1]$ and let $F(x)=\displaystyle\int_0^x f(t)\,dt$ for $x\in[0,1]$.

   (a) (6%) Prove that $\|F\|_2\le\|f\|_2$.

   (b) (4%) Give a necessary and sufficient condition on f for which $\|F\|_2=\|f\|_2$. Prove your assertion.

6. Let $1\le p\le q<\infty$ and $[a,b]\subset\mathbb{R}$.

   (a) (6%) Prove that $L^q[a,b]\subset L^p[a,b]$.

   (b) (6%) Let $X: L^q[a,b]\longrightarrow L^p[a,b]$ be such that $Xf=f$ for $f\in L^q[a,b]$. Find the norm of X.

7. Let $\Pi=\{z\in\mathbb{C}\mid |z|=1\}$ and let $\mathcal{C}(\Pi,\mathbb{C})=\{f\mid f \text{ continuous function from } \Pi \text{ to } \mathbb{C}\}$.

   (a) (8%) Give a complete statement of the *Stone-Weierstrass theorem* for $\mathcal{C}(\Pi,\mathbb{C})$.

   (b) (6%) Use this theorem to prove that, for any f in $\mathcal{C}(\Pi,\mathbb{C})$ and $z_0\in\Pi$, the limit

$$\lim_{n\to\infty}\frac{1}{n}\left(\sum_{k=0}^{n-1}f\left(z_0^k\right)\right)$$

   exists. Also find the limit value.