# 2023 NYCU-Math TA training

(1) **The definition of limits** Let $L \in \mathbb{R}$. We write

$$\lim_{x \to a} f(x) = L$$

if, for any $\epsilon > 0$, there is $\delta > 0$ such that

$$|f(x) - L| < \epsilon, \quad \forall 0 < |x - a| < \delta.$$

(2) **The intermediate value theorem** If $f : [a, b] \to \mathbb{R}$ is continuous, then, for any $L$ between $f(a)$ and $f(b)$, there is $c \in (a, b)$ such that $f(c) = L$.

(3) **The extremum value theorem** If $f : [a, b] \to \mathbb{R}$ is continuous, then $f$ attains its maximum and minimum values. That is, there are $\alpha, \beta \in [a, b]$ such that

$$f(\alpha) \leq f(x) \leq f(\beta), \quad \forall x \in [a, b].$$

(4) **The definition of differentiation** Let $a \in \mathbb{R}$. A function $f$ is differentiable at $a$ if

$$f'(a) := \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \quad \text{exists,}$$

or equivalently if there is (unique) $M \in \mathbb{R}$ such that

$$\lim_{x \to a} \frac{|f(x) - (f(a) + M(x - a))|}{|x - a|} = 0.$$

In particular, $M = f'(a)$.

(5) **The mean value theorem for derivatives** Let $f$ be a function defined on $[a, b]$. If $f$ is continuous on $[a, b]$ and differentiable on $(a, b)$, then there is $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

(6) **The chain rule** If $f$ is differentiable at $x$ and $g$ is differentiable at $f(x)$, then $g \circ f$ is differentiable at $x$ and $(g \circ f)'(x) = g'(f(x))f'(x)$.

(7) **The inverse function theorem** Suppose $f : (a, b) \to \mathbb{R}$ is one-to-one and differentiable on $(a, b)$. If $f'(x) \neq 0$, then $f^{-1}$ is differentiable at $f(x)$ and

$$(f^{-1})'(f(x)) = \frac{1}{f'(x)}.$$

(8) **The l'Hospital's rule** Let $f, g$ be differentiable functions defined on $(a, b)$. Assume that

$$\lim_{x\to a^+} f(x) = \lim_{x\to a^+} g(x) \in \{0, \pm\infty\}, \quad \lim_{x\to a^+} \frac{f'(x)}{g'(x)} \in \mathbb{R}\cup\{\pm\infty\}.$$

Then,

$$\lim_{x\to a^+} \frac{f(x)}{g(x)} = \lim_{x\to a^+} \frac{f'(x)}{g'(x)}.$$

(9) **The fundamental theorem of calculus** Let $f : [a, b] \to \mathbb{R}$ be continuous.

(1) If $F(x) = \int_a^x f(t)dt$, then $F$ is an antiderivative of $f$ on $(a, b)$ and continuous on $[a.b]$.

(2) If $G : [a, b] \to \mathbb{R}$ is an antiderivative of $f$ on $(a, b)$ and continuous on $[a, b]$, then $\int_a^b f(t)dt = G(b) - G(a)$.

(10) **The mean value theorem for integrals** If $f : [a, b] \to \mathbb{R}$ is continuous, then there is $c \in (a, b)$ such that

$$\int_a^b f(x)dx = f(c)(b-a).$$

(11) **Lagrange multipliers** Let $f, g_1, ..., g_k : \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable functions with $k \le n$ and $E = \{x \in \mathbb{R}^n | g_i(x) = 0, \forall 1 \le i \le k\}$. Suppose $\nabla g_1(x), ..., \nabla g_k(x)$ are linearly independent for all $x \in E$. If $f$, restricted to $E$, attains its maximum or minimum at $P$, then there are constants $c_1, ..., c_k$ such that

$$\nabla f(P) = c_1\nabla g_1(P) + \cdots + c_k\nabla g_k(P).$$

(12) **Taylor series and analyticity** Let $f$ be a function which is infinitely differentiable at $a$. The Taylor series of $f$ centered at $a$ refers to the following series

(1) The Taylor series of $f$ centered at $a$ refers to the following series

$$T(x) := \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

(2) $f$ is analytic at $a$ if there is $\epsilon > 0$ such that

$$f(x) = T(x), \quad \forall x \in (a-\epsilon, a+\epsilon).$$

(13) **Fubini's Theorem** Let $D$ is a rectangle region be contained in $\mathbb{R}^2$

$$D = \{(x, y) \mid a \le x \le b,\ c \le y \le d\}$$

and let $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ is integrable. If $f_y(x) = f(x, y)$ is integrable on $[a, b]$ for each $y \in [c, d]$, and if $g(y) = \int_a^b f(x, y)dx$ is integrable on $[c, d]$. Then the Riemann integral of $f$ over $D$ equals to the iterated integral

$$\int\int_D f(x,y)dA = \int_c^d \left( \int_a^b f(x,y)dx \right) dy$$

(14) **Change of Variables for multiple integrals** Given $U$ is a open set in $\mathbb{R}^n$, let $g : U \to \mathbb{R}^n$ is one to one and continuously differentiable on $U$. If the Jacobian of $g$, $Jg(\mathbf{x}) \neq 0$ on $U$, if $A$ is a Jordan measurable and $\overline{A} \subseteq U$, if $f$ is bounded and integrable on $g(A)$, then $f \circ g$ is integrable on $A$ and

$$\int_{g(A)} f(\mathbf{y})d\mathbf{y} = \int_A f(g(\mathbf{x}))\ |Jg(\mathbf{x})|\ d\mathbf{x}$$