(i) Syllabus on Algebra and Number Theory

Algebra:

- Group theory: Sylow theorems, p-groups, solvable groups, free groups.
- Rings and modules: tensor products, determinants, Jordan canonical form, PID's, UFD's, polynomials rings.
- Field theory: splitting fields, separable and inseparable extensions.
- Galois theory: Fundamental theorems of Galois theory, finite fields, cyclotomic fields.
- Homological algebra: exact sequences, splittings, snake and five lemmas, projective, injective, and flat modules, complexes, (co)homology.
- Commutative ring: localizations, Hilbert’s basis theorem, integral extensions, radicals of ideals, Zariski topology and Hilbert’s Nullstellensatz, Dedekind rings, DVRs.
- Representations of Finite Groups: character theory, induced representations, structure of the group ring.
- Basics of Lie groups and Lie algebras: exponential map, nilpotent and semi-simple Lie algebras and Lie groups.

References: Dummit and Foote: Abstract Algebra, 2nd edition; Serre: Representations of Finite Groups; Fulton-Harris: Representation Theory:

A First Course (Graduate Texts in Mathematics/Readings in Mathematics); Serge Lang: Algebra.

Number Theory:

Factorization and the primes; congruences; quadratic residues and reciprocity; continued fractions and approximations; eta functions; zeta functions;

Number fields; unique factorization of ideals; finiteness of class group; structure of unit group; Frobenius elements; local fields; ramification; weak approximation.

References: Z.I. Borevich, Igor Shafarevich, Number Theory; Serge Lang, Algebraic Number Theory

(ii) Syllabus on Analysis and Differential Equations

Real Analysis:

- Convergence theorems for integrals, Borel measure, Riesz representation theorem
- L^p space, Duality of L^p space, Jensen inequality
- Lebesgue differentiation theorem, Fubini theorem, Hilbert space
- Complex measures of bounded variation, Radon-Nikodym theorem.

- Hahn-Banah Theorem, open mapping theorem, uniform boundedness theorem, closed graph theorem.
- Basic properties of compact operators, Riesz-Fredholm Theory, spectrum of compact operators.
- Fourier series, Fourier transform, convolution.

References: Rudin: Real and Complex Analysis; Stein and Shakarchi: Real analysis; Stein and Shakarchi: Fourier Analysis.

Complex Analysis:

- Holomorphic and meromorphic functions
- Conformal maps, linear fractional transformations, Schwarz's lemma
- Complex integrals: Cauchy's theorem, Cauchy integral formula, residues
- Harmonic functions: the mean value property; the reflection principle; Dirichlet's problem
- Series and product developments: Laurent series, partial fractions expansions, and canonical products
- Special functions: the Gamma function, the zeta functions and elliptic functions
- Basics of Riemann surfaces
- Riemann mapping theorem, Picard theorems.

References: Ahlfors: Complex Analysis (3rd edition)

Differential Equations:

- Existence and uniqueness theorems for solutions of ODE; explicit solutions of simple equations; self-adjoint boundary value problems on finite intervals; critical points, phase space, stability analysis.
- First order partial differential equations, linear and quasi-linear PDE.
- Phase plane analysis, Burgers equation, Hamilton-Jacobi equation.
- Potential equations: Green functions and existence of solutions of Dirichlet problem, harmonic functions, maximal principle and applications, existence of solutions of Neumann's problem.
- Heat equation, Dirichlet problem, fundamental solutions
- Wave equations: initial condition and boundary condition, well-posedness, Sturm-Liouville eigenvalue problem, energy functional method, uniqueness and stability of solutions
- Distributions, Sobolev embedding theorem.

References: V. I. Arnold: Mathematical Methods of Classical Mechanics; Craig Evans: Partial Differential Equations.

(iii) Syllabus on Geometry and Topology

Differential Geometry:

- Basics of smooth manifolds: Inverse function theorem, implicit function theorem, submanifolds, Sard’s Theorem, embedding theorem, transversality, degree theory, integration on manifolds.
- Basics of matrix Lie groups over R and C: The definitions of Gl(n), SU(n), SO(n), U(n), their manifold structures, Lie algebras, right and left invariant vector fields and differential forms, the exponential map.
- Definition of real and complex vector bundles, tangent and cotangent bundles, basic operations on bundles such as dual bundle, tensor products, exterior products, direct sums, pull-back bundles.
- Definition of differential forms, exterior product, exterior derivative, de Rham cohomology, behavior under pull-back.
- Metrics on vector bundles.
- Riemannian metrics, definition of a geodesic, existence and uniqueness of geodesics.
- Definition of a principal Lie group bundle for matrix groups.
- Associated vector bundles: Relation between principal bundles and vector bundles

- Definition of covariant derivative for a vector bundle and connection on a principal bundle. Relations between the two.
- Definition of curvature, flat connections, parallel transport.
- Definition of Levi-Civita connection and properties of the Riemann curvature tensor, manifolds of constant curvature.
- Jacobi fields, second variation of geodesics
- Manifolds of nonpositive curvature, manifolds of positive curvature

References: V. Guillemin, A. Pollack, Differential topology; J. Milnor, Topology from the differentiable viewpoint; Cliff Taubes: Differential geometry: Bundles, Connections, Metrics and Curvature; John Lee: Introduction to Riemannian manifolds, second edition; S. Kobayashi and K. Nomizu: Foundations of Differential Geometry.

Algebraic Topology:

- Fundamental groups
- Covering spaces
- Higher homotopy groups
- Fibrations and the long exact sequence of a fibration
- Singular homology and cohomology
- Relative homology

- CW complexes and the homology of CW complexes
- Mayer-Vietoris sequence
- Universal coefficient theorem
- Kunneth formula
- Poincare duality
- Lefschetz fixed point formula
- Hopf index theorem
- Cech cohomology and de Rham cohomology.
- Equivalence between singular, Cech and de Rham cohomology

References: Alan Hatcher: Algebraic Topology; William Fulton: Algebraic Topology; Edwin Spanier: Algebraic Topology; M. Greenberg and J. Harper: Algebraic Topology: A First Course.

(iv) Syllabus on Computational and Applied Mathematics

Interpolation and approximation:

Trigonometric interpolation and approximation, fast Fourier transform; approximations by rational functions; polynomial and spline interpolations and approximation; least-squares approximation.

Nonlinear equation solvers:

Convergence of iterative methods (bisection, Newton's method, quasi-Newton's methods and fixed-point methods) for both scalar equations and systems, finding roots of polynomials.

Linear systems and eigenvalue problems:

Classical and modern iterative method for linear systems and eigenvalue problems, condition number and singular value decomposition, iterative methods for large sparse system of linear equations

Numerical solutions of ordinary differential equations:

Single step methods and multi-step methods, stability, accuracy and convergence; absolute stability, long time behavior; numerical methods for stiff ODE's.

Numerical solutions of partial differential equations:

Finite difference method, finite element method and spectral method: stability, accuracy and convergence, Lax equivalence theorem.

Mathematical modeling, simulation, and applied analysis:

Scaling behavior and asymptotics analysis, stationary phase analysis, boundary layer analysis, qualitative and quantitative analysis of mathematical models, Monte-Carlo method.

Linear and nonlinear programming:

Simplex method, interior method, penalty method, Newton's method, homotopy method and fixed point method, dynamic programming.

References:

[1] C. M. Bender and S. A. Orszag, *Advanced Mathematical Methods for*

Scientists and Engineers, 1999.

[2] C. de Boor and S.D. Conte, *Elementary Numerical Analysis, an algorithmic approach,* McGraw-Hill, 2000.

[3] G.H. Golub and C.F. van Loan, *Matrix Computations, third edition*, Johns Hopkins University Press, 1996.

[4] E. Hairer, P. Syvert and G. Wanner, *Solving Ordinary Differential Equations,* Springer, 1993.

[5] B. Gustafsson, H.-O. Kreiss and J. Oliger, *Time Dependent Problems and Difference Methods*, John Wiley Sons, 1995.

[6] J. Keener, "*Principles of Applied Mathematics*", Addison-Wesley, 1988.

[7] Lloyd N. Trefethen and David Bau, *Numerical linear algebra*, SIAM, 1997.

[8] Susanne Brenner and Ridgway Scott, *The Mathematical Theory of Finite Element Methods*, Springer, 2010.

[9] F.Y.M. Wan, *Introduction to Calculus of Variations and Its Applications*, Chapman & Hall, 1995

(v) Syllabus on Probability and Statistics

Probability:

Random variable, Expectation, Independence

Variance and covariance, correlation, moment

Various distribution functions

Multivariate distribution

Characteristic function, Generating function

Various modes of convergence of random variables

Bayes formula, Conditional probability

Conditional expectation given a sigma-field

Laws of large numbers

Central limit theorems

Martingales

Markov chains

Basic properties of Poisson processes

Basic properties of Brownian motion

References:

Rick Durrett: Probability: Theory and Examples, Cambridge University Press, 2010

Kai-Lai Chung: A Course in Probability Theory, New York, 1968.

Statistics:

Distribution Theory and Basic Statistics

Families of continuous distributions: normal, chi-sq, t, F, gamma, beta;

Families of discrete distributions: multinomial, Poisson, negative binomial; Basic statistics: sample mean, variance, median and quantiles.

Testing

Neyman-Pearson paradigm, null and alternative hypotheses, simple and composite hypotheses, type I and type II errors, power, most powerful test, likelihood ratio test, Neyman-Pearson Theorem, generalized likelihood ratio test.

Estimation

Parameter estimation, method of moments, maximum likelihood estimation, criteria for evaluation of estimators, Fisher information and its use, confidence interval.

Bayesian Statistics

Prior, posterior, conjugate priors, Bayesian estimator.

Large sample properties

Consistency, asymptotic normality, chi-sq approximation to likelihood ratio statistic.

References:

Casella, G. and Berger, R.L. (2002). Statistical Inference (2nd Ed.) Duxbury Press.

茆诗松，程依明，濮晓龙，概率论与数理统计教程（第二版），高等教育出版社，2008.

陈家鼎，孙山泽，李东风，刘力平，数理统计学讲义，高等教育出版

社，2006.

郑明，陈子毅，汪嘉冈，数理统计讲义，复旦大学出版社，2006.

陈希孺，倪国熙，数理统计学教程，中国科学技术大学出版社，2009.