國 立 交 通 大 學
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碩 士 論 文

一對圈型的線性變換

A Cyclic Pair of Linear Transformations

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摘 要

若一個方陣 X，其所有在對角線下方和最後一行第一列的項是非零，我們稱其為 cyclic。令 **C** 代表一個體，V 代表一個有限維佈於 **C** 的向量空間。我們稱一個在 V 上的 cyclic pair，意思是一個有序對的線性變換 A:V→V 和 B:V→V 滿足下面(i), (ii)的條件。

(i)存在一組 V 的基底使 A 在此基底的矩陣表示法為對角矩陣和 B 在此基底的矩陣表示法為 cyclic 矩陣。

(ii)存在一組 V 的基底使 B 在此基底的矩陣表示法為對角矩陣和 A 在此基底的矩陣表示法為 cyclic 矩陣。

我們藉由他們矩陣係數和乘法運算規則來描繪 cyclic pair。其中一個規則是和二項式定理相關。

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Abstract

A square matrix X is cyclic if all the entries in the lower diagonal and in the last column of the first row are nonzero. Let **C** denote a field and let V denote a vector space over **C** with finite positive dimension. By a cyclic pair on V we mean an ordered pair of linear transformations A:V→V and B:V→V that satisfies conditions (i), (ii) below.

(i) There exists a basis for V with respect to which the matrix representing A is diagonal and the matrix representing B is cyclic.

(ii) There exists a basis for V with respect to which the matrix representing B is diagonal and the matrix representing A is cyclic.

We characterized cyclic pairs by their matrix coefficients, and by their multiplication rules. One of the rules is related to the binomial theorem.

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Contents

1 Introduction

The study of a pair of linear transformations with specified combinatorial properties occurred in [1], [2], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [22]. The study of this pair of linear transformations is related to the study of module structure of some free algebras with 2 generators. Usually the entries in the matrix forms of these two linear transformations can be described by recurrence relations and is related to a class of special functions. One of such a pair of linear transformations occurs in P- and Q-polynomial scheme.

For example, referring to [17], [18], [19], let Y denote a symmetric association scheme, with vertex set X. Assume Y is P-polynomial with respect to an associate matrix A, and Q-polynomial with respect to a primitive idempotent E. Let $Mat_X(\mathbb{C})$ denote the $\mathbb{C}$-algebra consisting of all matrices with rows and columns indexed by X, and entries in $\mathbb{C}$. Fix a vertex $x \in X$, and let $A^* = A^*(x)$ denote the diagonal matrix in $Mat_X(\mathbb{C})$ with diagonal entries

$$A^*_{yy} = |X|E_{xy} (\forall y \in X).$$

Let T denote the subalgebra of $Mat_X(\mathbb{C})$ generated by A and A^*, and let V denote an irreducible T-module. Then the restriction $A|_V$, $A^*|_V$ form a pair of linear transformations on V. This is called a Leonard pair [7]. The study of Leonard pair is connected to a theorem of Leonard [4], [5, p.260] involving the q-Racah and related polynomials of the Askey scheme [20].

The pair of linear transformations we study in this thesis are related to a cycle. See section 2 for formal definition. We characterized them by their matrix coefficients, by their multiplication rules. One of the rules is related to the generalized binomial theorem. See Corollary 2.9 for details.

2 Cyclic pairs

Theorem 2.1. *Let d denote a nonnegative integer. Let V denote a vector space over $\mathbb{C}$ with dimension $d+1$. Let $A : V \to V$ and $B : V \to V$ denote linear transformations. If there exists a basis $u_0, u_1, \cdots, u_d$ for V with respect to which the matrices representing A and B have the following forms,*

$$A : \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & \alpha \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \quad B : \begin{bmatrix} \beta & 0 & 0 & \cdots & 0 \\ 0 & \beta q & 0 & \cdots & 0 \\ 0 & 0 & \beta q^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \beta q^d \end{bmatrix},$$

where α, $\beta \in \mathbb{C}$ are nonzero scalars and $q \in \mathbb{C}$ is a primitive root of unity of order $d+1$. Then there exists two nonzero complex numbers γ, η such that $A^{d+1} = \gamma I$, $B^{d+1} = \eta I$, $BA = qAB$, where q is a primitive root of unity of order $d+1$.

Proof.

Observe

$$det(xI - A) = \begin{bmatrix} x & 0 & 0 & \cdots & 0 & -\alpha \\ -1 & x & 0 & \cdots & 0 & 0 \\ 0 & -1 & x & \cdots & 0 & 0 \\ 0 & 0 & -1 & \ddots & \vdots & 0 \\ \vdots & \vdots & \vdots & \ddots & x & \vdots \\ 0 & 0 & 0 & \cdots & -1 & x \end{bmatrix} = x^{d+1} + (-1)^{d+1}\alpha.$$

Then $x^{d+1} + (-1)^{d+1}\alpha = 0$. Set $\gamma = (-1)^d\alpha$. We have $A^{d+1} = \gamma I$ by Cayley-Hamilton theorem. Set $\eta = \beta^{d+1}$. It is easy to check $B^{d+1} = \eta I$. By computing the matrix products AB and BA, we get $BA = qAB$. ∎

Definition 2.2. Let $\mathbb{C}$ denote the field of complex numbers, let d denote a nonnegative integer, and let A denote a matrix in $\mathrm{Mat}_{d+1}(\mathbb{C})$. We say A is *(left)-cyclic* when each of the entries A_{10}, A_{21} ,$\cdots$, $A_{d,d-1}$, A_{0d} is nonzero and all other entries of A are zero.

Definition 2.3. Let V denote a vector space over $\mathbb{C}$ with finite positive dimension. By a cyclic pair on V we mean an ordered pair of linear transformations $A : V \to V$ and $B : V \to V$ that satisfy conditions (i), (ii) below.

(i) There exists a basis for V with respect to which the matrix representing A is diagonal and the matrix representing B is cyclic.

(ii) There exists a basis for V with respect to which the matrix representing B is diagonal and the matrix representing A is cyclic.

Theorem 2.4. *Let d denote a nonnegative integer. Let V denote a vector space over $\mathbb{C}$ with dimension $d+1$. If there exists two nonzero complex scalars γ, η such that*

$$A^{d+1} = \gamma I, B^{d+1} = \eta I, BA = qAB,$$

where q is a primitive root of unity of order $d+1$. Then the pair A, B is a cyclic pair on V.

Proof.

Observe all eigenvalues of A, B are nonzero, since their characteristic polynomials have the form $x^{d+1} - c = 0$ for some $c \neq 0$.

Fix an eigenvalue λ of B and let v be the corresponding eigenvector. Set $v_i = A^i v$, where $0 \leq i \leq d$. Observe $v_i \neq 0$ since A is invertible. We first claim $v_0, v_1, \cdots, v_d$ are linear independent. Suppoe

$$c_0 v_0 + c_1 v_1 + c_2 v_2 + \cdots + c_d v_d = 0,$$

for some $c_0, \cdots, c_d \in \mathbb{C}$. Then

$$c_0 v + c_1 A v + c_2 A^2 v + \cdots + c_d A^d v = 0.$$

Applying B to both sides,

$$c_0 B v + c_1 B A v + c_2 B A^2 v + \cdots + c_d B A^d v = 0.$$

Because of $BA = qAB$,

$$c_0 B v + c_1 q A B v + c_2 q^2 A^2 B v + \cdots + c_d q^d A^d B v = 0.$$

Since v is an eigenvector of B corresponding to eigenvalue λ, we have

$$\lambda(c_0 + c_1 q A + c_2 q^2 A^2 + \cdots + c_d q^d A^d) v = 0.$$

Eliminate λ

$$(c_0 + c_1 q A + c_2 q^2 A^2 + \cdots + c_d q^d A^d) v = 0.$$

Repeating of above steps, we obtain

$$\begin{aligned}
(c_0 + c_1 q^2 A + c_2 q^4 A^2 + \cdots + c_d q^{2d} A^d) v &= 0, \\
(c_0 + c_1 q^3 A + c_2 q^6 A^2 + \cdots + c_d q^{3d} A^d) v &= 0, \\
&\vdots \\
(c_0 + c_1 q^d A + c_2 q^{2d} A^2 + \cdots + c_d q^{d^2} A^d) v &= 0.
\end{aligned}$$

Writting these equations in matrix form,

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & q & q^2 & \cdots & q^d \\ 1 & q^2 & q^4 & \cdots & q^{2d} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & q^d & q^{2d} & \cdots & q^{d^2} \end{bmatrix} \begin{bmatrix} c_0 & 0 & 0 & \cdots & 0 \\ 0 & c_1 & 0 & \cdots & 0 \\ 0 & 0 & c_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & c_d \end{bmatrix} \begin{bmatrix} v_0 \\ v_1 \\ v_2 \\ \vdots \\ v_d \end{bmatrix} = 0.$$

Since $1, q, q^2, \cdots, q^d$ are distinct, the above first matrix is invertible. This forces $c_0 v_0 = c_1 v_1 = \cdots = c_d v_d = 0$ and then $c_0 = c_1 = c_2 = \cdots = c_d = 0$.

Now we know $Av_i = v_{i+1}$, $i < d$, $Av_d = AA^d v = A^{d+1} v = \gamma I v = \gamma v_0$, then A is similar to

$$\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & \gamma \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

$Bv_i = BA^i v = q^i A^i B v = \lambda q^i A^i v = \lambda q^i v_i$ and B is similar to

$$\begin{bmatrix} \lambda & 0 & 0 & \cdots & 0 \\ 0 & \lambda q & 0 & \cdots & 0 \\ 0 & 0 & \lambda q^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda q^d \end{bmatrix}.$$

∎

Corollary 2.5. *Let V denote a vector space over $\mathbb{C}$ with dimension $d+1$. Let $A : V \to V$ and $B : V \to V$ denote linear transformations. Then the following (i)-(iii) are equivalent.*

(i) (A, B) is a cyclic pair on V.

(ii) There exists a basis $u_0, u_1, \cdots, u_d$ for V with respect to which the matrices representing A and B have the following form,

$$A: \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & \alpha \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \quad B: \begin{bmatrix} \beta & 0 & 0 & \cdots & 0 \\ 0 & \beta q & 0 & \cdots & 0 \\ 0 & 0 & \beta q^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \beta q^d \end{bmatrix},$$

where α, $\beta \in \mathbb{C}$ are nonzero scalars and $q \in \mathbb{C}$ is a primitive root of unity of order $d+1$.

(iii) There exists two nonzero complex numbers γ, η such that $A^{d+1} = \gamma I$, $B^{d+1} = \eta I$, $BA = qAB$, where q is a primitive root of unity of order $d+1$.

Proof.

(ii)⇒(iii) This is Theorem 2.1. (iii)⇒(i) This is Theorem 2.4. (i)⇒(ii) This is from [3]. ∎

Definition 2.6. [6, p.292]

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q^n - 1)(q^{n-1} - 1) \cdots (q^{n-k+1} - 1)}{(q^k - 1)(q^{k-1} - 1) \cdots (q - 1)} \quad (0 \le k \le n).$$

Lemma 2.7. [6, p.295]

$$\begin{bmatrix} k \\ r \end{bmatrix}_q - \begin{bmatrix} k-1 \\ r \end{bmatrix}_q = q^{k-r} \begin{bmatrix} k-1 \\ r-1 \end{bmatrix}_q (0 \le r \le k).$$

Proof.

$$
\begin{aligned}
&\left[\begin{array}{c} k \\ r \end{array}\right]_q - \left[\begin{array}{c} k-1 \\ r \end{array}\right]_q \\
= \;& \frac{(q^k-1)(q^{k-1}-1)\cdots(q^{k-r+1}-1)}{(q^r-1)(q^{r-1}-1)\cdots(q-1)} - \frac{(q^{k-1}-1)(q^{k-2}-1)\cdots(q^{k-r}-1)}{(q^r-1)(q^{r-1}-1)\cdots(q-1)} \\
= \;& \frac{(q^k-1)-(q^{k-r}-1)}{(q^r-1)} \cdot \frac{(q^{k-1}-1)\cdots(q^{k-r+1}-1)}{(q^{r-1}-1)\cdots(q-1)} \\
= \;& q^{k-r}\left[\begin{array}{c} k-1 \\ r-1 \end{array}\right]_q .
\end{aligned}
$$

∎

Theorem 2.8. *The following are equivalent.*

(i) $BA{=}qAB$.

(ii) $(A+B)^n = \sum_{i=0}^{n} \left[\begin{array}{c} n \\ i \end{array}\right]_q A^{n-i}B^i$ *for all* $n \in \mathbb{N}$.

(iii) $(A+B)^2 = A^2 + (q+1)AB + B^2$.

Proof.

(i)⇒(ii)We will prove this by induction on n. Then $n = 1$ is clearly true. Now let k be a natural number such that $n = k$ is true; that is,

$$(A+B)^k = \sum_{i=0}^{k} \left[\begin{array}{c} k \\ i \end{array}\right]_q A^{k-i}B^i.$$

We need to show that $n = k+1$ is true. However,

$$
\begin{aligned}
(A+B)^{k+1} &= (A+B)(A+B)^k \\
&= (A+B)(\sum_{i=0}^{k} \begin{bmatrix} k \\ i \end{bmatrix}_q A^{k-i}B^i) \\
&= (\sum_{i=0}^{k} \begin{bmatrix} k \\ i \end{bmatrix}_q AA^{k-i}B^i) + (\sum_{i=0}^{k} \begin{bmatrix} k \\ i \end{bmatrix}_q BA^{k-i}B^i) \\
&= (\sum_{i=0}^{k} \begin{bmatrix} k \\ i \end{bmatrix}_q A^{(k+1)-i}B^i) + (\sum_{i=0}^{k} \begin{bmatrix} k \\ i \end{bmatrix}_q q^{k-i}A^{k-i}B^{i+1}) \\
&= (A^{k+1} + \sum_{i=1}^{k} \begin{bmatrix} k \\ i \end{bmatrix}_q A^{(k+1)-i}B^i) \\
&\quad +(B^{k+1} + \sum_{i=1}^{k} \begin{bmatrix} k \\ i-1 \end{bmatrix}_q q^{(k+1)-i}A^{(k+1)-i}B^i) \\
&= A^{k+1} + \sum_{i=1}^{k} (\begin{bmatrix} k \\ i \end{bmatrix}_q + \begin{bmatrix} k \\ i-1 \end{bmatrix}_q q^{(k+1)-i})A^{(k+1)-i}B^i + B^{k+1} \\
&= A^{k+1} + \sum_{i=1}^{k} \begin{bmatrix} k+1 \\ i \end{bmatrix}_q A^{(k+1)-i}B^i + B^{k+1} \\
&= \sum_{i=0}^{k+1} \begin{bmatrix} k+1 \\ i \end{bmatrix}_q A^{(k+1)-i}B^i.
\end{aligned}
$$

We see that $n = k+1$ is true.

(ii)⇒(iii)This is clear.

(iii)⇒(i)

$$
\begin{aligned}
(A+B)^2 &= A^2 + BA + AB + B^2 \\
&= A^2 + (q+1)AB + B^2.
\end{aligned}
$$

Hence

$$BA = qAB.$$

∎

Corollary 2.9. *Let V denote a vector space over $\mathbb{C}$ with dimension $d+1$. Let $A : V \to V$ and $B : V \to V$ denote linear transformations. Then the following (i)-(iv) are equivalent.*

(i) (A, B) *is a cyclic pair on* V.

(ii) There exists a basis $u_0, u_1, \cdots, u_d$ for V with respect to which the matrices representing A and B have the following form,

$$A : \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & \alpha \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \quad B : \begin{bmatrix} \beta & 0 & 0 & \cdots & 0 \\ 0 & \beta q & 0 & \cdots & 0 \\ 0 & 0 & \beta q^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \beta q^d \end{bmatrix},$$

where α, $\beta \in \mathbb{C}$ are nonzero scalars and $q \in \mathbb{C}$ is a primitive root of unity of order $d+1$.

(iii) There exists two nonzero complex numbers γ, η such that $A^{d+1} = \gamma I$, $B^{d+1} = \eta I$, $BA = qAB$, where q is a primitive root of unity of order $d+1$.

(iv) There exists two nonzero complex numbers γ, η such that $A^{d+1} = \gamma I$, $B^{d+1} = \eta I$, $(A+B)^n = \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_q A^{n-i}B^i$ for any $n \in \mathbb{N}$, where q is a primitive root of unity of order $d+1$.

Proof.

The equivalence of (i), (ii), (iii) follows from corollary 2.5. (iii)⇒(iv) This is Theorem 2.8. (iv)⇒(iii) This is Theorem 2.8. ∎

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