國 立 交 通 大 學
應用數學系
碩 士 論 文

The Weakly Cyclic Pairs of Linear Transformations

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指導老師：翁 志 文　教 授

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一對廣義圈型線性變換的研究

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摘要

令 **X** 是個方陣，當主對角線正下方及第一列最後一行的元素都非零，其他非對角線之元素皆為零，我們稱 **X** 為廣義圈型。在有限維的向量空間 **V** 中，如果兩線性變換 **A**：**V** → **V**、**B**：**V** → **V** 滿足下列條件**(1),(2)**則我們稱**(A,B)**為一對廣義圈型線性變換，

(1) **V** 中存在一個基底可以使的 **A** 的矩陣表示為對角矩陣，**B** 的矩陣表示為廣義圈型。

(2) **V** 中存在一個基底可以使的 **B** 的矩陣表示為對角矩陣，**A** 的矩陣表示為廣義圈型。

我們將會給一對廣義圈型線性變換存在的兩個必要條件。

中 華 民 國 九 十 三 年 六 月

The Weakly Cyclic Pairs of Linear Transformations

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Abstract

Let **X** be a square matrix. We say **X** is weak cyclic when each of the entries in the lower diagonal and in the last column of the lower diagonal are nonzero and all the other nondiagonal entries of **X** are zero. Let **V** denote a vector space over C with finite positive dimension. By a weakly cyclic pair on **V** we mean an ordered pair of linear transformations **A**：**V** → **V** and **B**：**V** → **V** that satisfies conditions **(i), (ii)** below.

(i). There exists a basis for **V** with respect to which the matrix representing **A** is diagonal and the matrix representing **B** is weakly cyclic.

(ii). There exists a basis for **V** with respect to which the matrix representing **B** is diagonal and the matrix representing **A** is weakly cyclic.

We give two necessary conditions among the eigenvalues and the coefficients in some representing matrix of a weak cyclic pair.

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1 Introduction

The study of a pair of linear transformations with specified properties occurred in [1]−[18]. In [3], a pair of linear transformations called *cyclic pair* is given. We generalize the idea of cyclic pairs to *weakly cyclic pairs*. See Section 2 for formal definition.

We choose a nice basis such that the matrix forms of these two linear transformations are simplified. Theorem 2.5 is the result. In Theorem 2.6, we find two constraints on the entries of these two matrices. Together with previous result from [3], we can complete determine all the cyclic pairs. We also characterized the cyclic pair by their multiplication rules.

2 Weakly Cyclic Pair

Let $\mathbb{C}$ denote the field of complex numbers and let $\mathrm{Mat}_{d+1}(\mathbb{C})$ denote the set of $(d+1) \times (d+1)$ matrices over $\mathbb{C}$ with index set $\{0, 1, \cdots, d\}$.

Definition 2.1. For $A \in \mathrm{Mat}_{d+1}(\mathbb{C})$, We say A is *weakly cyclic* when each of the entries $A_{10}, A_{21}, \cdots, A_{d,d-1}, A_{0d}$ is nonzero and all other nondiagonal entries of A are zero.

Lemma 2.2. *Let A be a weakly cyclic matrix. The minimal polynomial of A is the characteristic polynomial of A.*

Proof. Using the nonzero coefficients $A_{10}, A_{21}, ..., A_{d,d-1}$, one can find for each i $(1 \leq i \leq d)$, $A^i_{i0} \neq 0$ and $A^j_{i0} = 0$ $(1 \leq j < i)$. Hence A^i is not in the span of I,A,A^2,...,A^{i-1} $(1 \leq i \leq d)$. That implies I,A,A^2,...,A^d are linear independent. Since A is a $(d+1) \times (d+1)$ matrix, the minimal polynomial of A has degree $d+1$. □

Definition 2.3. Let V denote a vector space over $\mathbb{C}$ with finite positive dimension. By a weakly cyclic pair on V we mean an ordered pair of linear transformations $A : V \rightarrow V$ and $B : V \rightarrow V$ that satisfies conditions (i), (ii) below.

(i) There exists a basis for V with respect to which the matrix representing A is diagonal and the matrix representing B is weakly cyclic.

(ii) There exists a basis for V with respect to which the matrix representing B is diagonal and the matrix representing A is weakly cyclic.

Lemma 2.4. *Let (A, B) be a weakly cyclic pair on V. Then the eigenvalues of A (resp. B) are distinct.*

Proof. By the above lemma the minimal polynomial of A is the characteristic polynomial of A and by definition of weakly cyclic pair, A is diagonalizable. So A has distinct eigenvalues. □

Theorem 2.5. *Let V denote a vector space over $\mathbb{C}$ with dimension $d+1$. Let $A : V \to V$ and $B : V \to V$ denote linear transformations. Then the following are equivalent.*

(i) (A, B) *is a weakly cyclic pair on V.*

(ii) There exists a basis $v_0, v_1, ..., v_d$ for V with respect to which the matrices representing A and B have the following forms,

$$A : \begin{bmatrix} a_0 & 0 & 0 & \dots & 0 & s \\ 1 & a_1 & 0 & \dots & 0 & 0 \\ 0 & 1 & a_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & a_d \end{bmatrix}, \quad B : \begin{bmatrix} \eta_0 & 0 & 0 & \dots & 0 \\ 0 & \eta_1 & 0 & \dots & 0 \\ 0 & 0 & \eta_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \eta_d \end{bmatrix},$$

and there exists a basis $w_0, w_1, ..., w_d$ for V with respect to which the matrices representing A and B have the following forms,

$$A : \begin{bmatrix} \theta_0 & 0 & 0 & \dots & 0 \\ 0 & \theta_1 & 0 & \dots & 0 \\ 0 & 0 & \theta_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \theta_d \end{bmatrix}, \quad B : \begin{bmatrix} b_0 & 0 & 0 & \dots & 0 & t \\ 1 & b_1 & 0 & \dots & 0 & 0 \\ 0 & 1 & b_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & b_d \end{bmatrix},$$

where $s, t \in \mathbb{C}$ are nonzero scalars, and θ_i are eigenvalues of A and η_i are eigenvalues of B for $0 \le i \le d$.

Proof. (ii)→(i) This is clear. (i)→(ii) Suppose that (A, B) is a weakly cyclic pair. Find a basis $u_0, u_1, ..., u_d$ such that the matrices representing A and B are as follows.

$$A : \begin{bmatrix} a_0 & 0 & 0 & \dots & 0 & c_0 \\ c_1 & a_1 & 0 & \dots & 0 & 0 \\ 0 & c_2 & a_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & c_d & a_d \end{bmatrix}, \quad B : \begin{bmatrix} \eta_0 & 0 & 0 & \dots & 0 \\ 0 & \eta_1 & 0 & \dots & 0 \\ 0 & 0 & \eta_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \eta_d \end{bmatrix}, \tag{2.1}$$

where c_i are not zero $(0 \leq i \leq d)$. So we know that

$$Au_i = a_i u_i + c_{i+1} u_{i+1} \quad (0 \leq i \leq d-1) \tag{2.2}$$

and

$$Au_d = c_0 u_0 + a_d u_d. \tag{2.3}$$

Set

$$v_0 = u_0 \tag{2.4}$$

and

$$v_i = c_1 \cdots c_i u_i \quad (1 \leq i \leq d). \tag{2.5}$$

So we have

$$Av_i = a_i v_i + v_{i+1} \quad (0 \leq i \leq d-1)$$

and

$$Av_d = c_0 c_1 \cdots c_d u_0 + a_d v_d.$$

On the other hand,

$$Bv_0 = Bu_0 = \eta_0 u_0 = \eta_0 v_0$$

and

$$Bv_i = c_1 \cdots c_i Bu_i = \eta_i c_1 \cdots c_i u_i = \eta_i v_i \quad (1 \leq i \leq d).$$

Hence in the basis $v_0, \cdots, v_d$, the matrices representing A, B as follows.

$$A = \begin{bmatrix} a_o & 0 & 0 & \dots & 0 & s \\ 1 & a_1 & 0 & \dots & 0 & 0 \\ 0 & 1 & a_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & a_d \end{bmatrix}, \quad B = \begin{bmatrix} \eta_0 & 0 & 0 & \dots & 0 \\ 0 & \eta_1 & 0 & \dots & 0 \\ 0 & 0 & \eta_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \eta_d \end{bmatrix},$$

where $s = c_0 \cdots c_d \neq 0$. Similarly there exists a basis $w_0, w_1, \cdots, w_d$ of V such that the matrix representing A,B as follows

$$A = \begin{bmatrix} \theta_0 & 0 & 0 & \dots & 0 \\ 0 & \theta_1 & 0 & \dots & 0 \\ 0 & 0 & \theta_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \theta_d \end{bmatrix}, \quad B = \begin{bmatrix} b_0 & 0 & 0 & \dots & 0 & t \\ 1 & b_1 & 0 & \dots & 0 & 0 \\ 0 & 1 & b_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & b_d \end{bmatrix}$$

for some nonzero $t \in \mathbb{C}$. □

Theorem 2.6. *As the notation in Theorem* 2.5*, suppose Theorem* 2.5 *(i) – (ii) hold. Then*

$$(\theta_i - a_{j+1})(\eta_{j+1} - b_{i-1}) = (\eta_j - b_{i-1})(\theta_{i-1} - a_{j+1}) \quad (1 \le i \le d, 0 \le j \le d-1). \tag{2.6}$$

$$(\theta_j - a_i)(\eta_{i+1} - b_j) = (\theta_{j-1} - a_i)(\eta_i - b_j) \quad (0 \le i \le d-1, 1 \le j \le d). \tag{2.7}$$

Proof. Let $v_0, v_1, \cdots, v_d$ and $w_0, w_1, \cdots, w_d$ be the two bases described in Theorem 2.5(*ii*). Suppose

$$w_i = \sum_{j=0}^{d} c_{ij} v_j \tag{2.8}$$

for some $c_{ij} \in \mathbb{C}$. So we have

$$Aw_i = \theta_i w_i = \sum_{j=0}^{d} c_{ij} \theta_i v_j \quad (0 \le i \le d) \tag{2.9}$$

and

$$\begin{aligned} Aw_i &= \sum_{j=0}^{d} c_{ij} A v_j && (2.10) \\ &= \sum_{j=0}^{d-1} c_{ij}(a_j v_j + v_{j+1}) + c_{id}(s v_0 + a_d v_d) && (2.11) \\ &= (c_{i0} a_0 + c_{id} s) v_0 + \sum_{j=1}^{d} (c_{ij} a_j + c_{i\ j-1}) v_j \quad (0 \le i \le d). && (2.12) \end{aligned}$$

Comparing (2.9) – (2.12),

$$c_{ij}\theta_i = c_{ij} a_j + c_{i\ j-1} \quad (1 \le j \le d, 0 \le i \le d),$$

$$c_{i0}\theta_i = c_{i0} a_0 + c_{id} s \quad (0 \le i \le d).$$

Hence

$$c_{ij}(\theta_i - a_j) = c_{i\ j-1} \quad (1 \le j \le d, 0 \le i \le d), \tag{2.13}$$

$$c_{i0}(\theta_i - a_0) = c_{id}s \quad (0 \le i \le d). \tag{2.14}$$

Similarly

$$Bw_i = \sum_{j=0}^{d} c_{ij}Bv_j = \sum_{j=0}^{d} c_{ij}\eta_j v_j \quad (0 \le i \le d) \tag{2.15}$$

and

$$\begin{aligned} Bw_i &= b_i w_i + w_{i+1} & (2.16)\\ &= b_i \sum_{j=0}^{d} c_{ij}v_j + \sum_{j=0}^{d} c_{i+1\ j}v_j & (2.17)\\ &= \sum_{j=0}^{d}(b_i c_{ij} + c_{i+1\ j})v_j. \quad (0 \le i \le d-1), & (2.18)\end{aligned}$$

$$Bw_d = b_d w_d + tw_d = \sum_{j=0}^{d}(b_d c_{dj} + tc_{0j})v_j. \tag{2.19}$$

Comparing (2.15) – (2.19),

$$c_{ij}\eta_j = b_i c_{ij} + c_{i+1\ j} \quad (0 \le i \le d-1, 0 \le j \le d),$$

$$c_{dj}\eta_j = b_d c_{dj} + c_{0j}t \quad (0 \le j \le d).$$

Thus

$$c_{ij}(\eta_j - b_i) = c_{i+1\ j} \quad (0 \le i \le d-1, 0 \le j \le d). \tag{2.20}$$

By (2.13)(2.20)

$$\begin{aligned} c_{ij} &= c_{i\ j+1}(\theta_i - a_{j+1}) & (2.21)\\ &= c_{i-1\ j+1}(\theta_i - a_{j+1})(\eta_{j+1} - b_{i-1})(0 \le j \le d-1, 1 \le i \le d), & \\ c_{ij} &= c_{i-1\ j}(\eta_j - b_{i-1}) & (2.22)\\ &= c_{i-1\ j+1}(\eta_j - b_{i-1})(\theta_{i-1} - a_{j+1})(1 \le i \le d, 0 \le j \le d-1). & \end{aligned}$$

Fix i ($0 \leq i \leq d$). Observe $c_{id} \neq 0$, otherwise $c_{ij} = 0$ by (2.13) and then $w_i = 0$ by (2.8). Observe $c_{i0} \neq 0$, otherwise $c_{id} = 0$ by (2.14) and since $s \neq 0$. Hence $c_{ij} \neq 0$ by (2.13).

By above comments and by (2.21)-(2.22), we have for $1 \leq i \leq d, 0 \leq j \leq d-1$,

$$(\theta_i - a_{j+1})(\eta_{j+1} - b_{i-1}) = (\eta_j - b_{i-1})(\theta_{i-1} - a_{j+1}). \tag{2.23}$$

By the same step with supposing

$$v_i = \sum_{j=0}^{d} d_{ij} w_j, \tag{2.24}$$

we have for $0 \leq i \leq d-1, 1 \leq j \leq d$,

$$(\theta_j - a_i)(\eta_{i+1} - b_j) = (\theta_{j-1} - a_i)(\eta_i - b_j) \quad (0 \leq i \leq d-1, 1 \leq j \leq d). \tag{2.25}$$

□

3 Cyclic Pair

We consider a special case of weakly cyclic pair in this section.

Definition 3.1. Let V denote a vector space over $\mathbb{C}$ with finite positive dimension. By a cyclic pair on V we mean an ordered pair of linear transformations $A : V \to V$ and $B : V \to V$ that satisfy conditions (i), (ii) below.

(i) There exists a basis for V with respect to which the matrix representing A is diagonal and the matrix representing B is cyclic.

(ii) There exists a basis for V with respect to which the matrix representing B is diagonal and the matrix representing A is cyclic.

Lemma 3.2. *Cyclic matrices are diagonalizable with nonzero eigenvalues.*

Proof. For any cyclic matrix

$$A = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & a_0 \\ a_1 & 0 & 0 & \dots & 0 & 0 \\ 0 & a_2 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a_d & 0 \end{bmatrix},$$

the characteristic polynomial of A is

$$f(x) = x^{d+1} - \prod_{i=0}^{d} a_i. \tag{3.1}$$

Since $a_1, \cdots, a_d$ are not zeros, $f(x)$ has $d+1$ distinct roots. Hence A has $d+1$ distinct eigenvalues. This implies A is diagonalizable. □

Lemma 3.3. *Suppose*

$$A = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & \alpha \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \end{bmatrix} (\alpha \neq 0)$$

and θ is an eigenvalue of A. Let u be an eigenvector corresponding to θ. Then $\theta^{d+1} = \alpha$ and

$$u = \begin{bmatrix} u_0 \\ u_0\theta^{-1} \\ u_0\theta^{-2} \\ \vdots \\ u_0\theta^{-d} \end{bmatrix}$$

for some scalar $u_0 \in \mathbb{C}$.

Proof. Suppose

$$u = \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \\ u_d \end{bmatrix}$$

and $Au = \theta u$ for $u_0, u_1, \cdots, u_d \in \mathbb{C}$. Then

$$Au = 0 \begin{bmatrix} \alpha u_d \\ u_0 \\ u_1 \\ \vdots \\ u_{d-1} \end{bmatrix} = \theta \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \\ u_d \end{bmatrix}.$$

Hence $u_i = \theta u_{i+1}$ $(0 \leq i \leq d-1)$ and $u_d = \dfrac{\theta}{\alpha} u_0$. Then $u_0 = \theta^d u_d = \dfrac{\theta^{d+1}}{\alpha} u_0$. Note that $u_0 \neq 0$ since $u \neq 0$ and $\theta \neq 0$. Hence $\theta^{d+1} = \alpha$ and $u_i = \theta^{-i} u_0$ $(0 \leq i \leq d)$. $\square$

Theorem 3.4. *Let V denote a vector space over $\mathbb{C}$ with dimension $d+1$. Let $A : V \to V$ and $B : V \to V$ denote linear transformations. Then the following (i)-(iii) are equivalent.*

(i) (A, B) *is a cyclic pair on* V.

(ii) There exists a basis $v_0, v_1, ..., v_d$ for V with respect to which the matrices representing A and B have the following forms,

$$A : \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & \alpha \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \end{bmatrix}, \quad B : \begin{bmatrix} \beta & 0 & 0 & \dots & 0 \\ 0 & \beta q & 0 & \dots & 0 \\ 0 & 0 & \beta q^2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \beta q^d \end{bmatrix},$$

where $\alpha, \beta \in \mathbb{C}$ are nonzero scalars and $q \in \mathbb{C}$ is a primitive root of unity of order $d+1$.

(iii) There exists two nonzero complex numbers α, β such that $A^{d+1} = \alpha I$, $B^{d+1} = \beta^{d+1} I$, $BA = qAB$, where q is a primitive root of unity of order $d+1$.

Proof. (i)$\rightarrow$(ii) Suppose that (A, B) is a cyclic pair. Find a basis $u_0, u_1, ..., u_d$ such that the matrices representing A and B are as follows.

$$A : \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & a_0 \\ a_1 & 0 & 0 & \dots & 0 & 0 \\ 0 & a_2 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a_d & 0 \end{bmatrix}, \quad B : \begin{bmatrix} b_0 & 0 & 0 & \dots & 0 \\ 0 & b_1 & 0 & \dots & 0 \\ 0 & 0 & b_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & b_d \end{bmatrix}. \tag{3.2}$$

So we know that

$$Au_i = a_{i+1}u_{i+1} \quad (0 \le i \le d-1) \tag{3.3}$$

and

$$Au_d = a_0 u_0. \tag{3.4}$$

Set

$$v_0 = u_0 \tag{3.5}$$

and

$$v_i = a_1 \cdots a_i u_i \quad (1 \le i \le d). \tag{3.6}$$

So by (3.3)—(3.6),

$$Av_i = v_{i+1} \quad (0 \le i \le d-1)$$

and

$$Av_d = a_d \cdots a_1 a_0 u_0.$$

On the other hand,

$$Bv_0 = Bu_0 = b_0 u_0 = b_0 v_0$$

and

$$Bv_i = a_1 \cdots a_i Bu_i = b_i a_1 \cdots a_i u_i = b_i v_i \quad (1 \le i \le d).$$

Hence in the basis $v_0, \cdots, v_d$, the matrices representing A, B as follows,

$$A = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & \alpha \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} b_0 & 0 & 0 & \dots & 0 \\ 0 & b_1 & 0 & \dots & 0 \\ 0 & 0 & b_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & b_d \end{bmatrix},$$

where $\alpha = a_0 \cdots a_d$. Similarly there exists a basis $w_0, w_1, \cdots, w_d$ of V, such that the matrix representing A is diagonal and the matrix representing B is

$$\begin{bmatrix} 0 & 0 & 0 & \dots & 0 & \gamma \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \end{bmatrix}, \tag{3.7}$$

for some $\gamma \in \mathbb{C}$. Note that for each i, w_i is an eigenvector of A. Let θ_i be the corresponding eigenvalue. Then by Lemma 3.3,

$$w_i = c_i \sum_{j=0}^{d} (\theta_i^{-1})^j v_j \tag{3.8}$$

for some scalar $c_i \in \mathbb{C}$. From (3.7), (3.8),

$$Bw_d = \gamma w_0 = \gamma c_0 \sum_{j=0}^{d} (\theta_0^{-1})^j v_j. \tag{3.9}$$

On the other hand, by (3.2), (3.8),

$$Bw_d = c_d \sum_{j=0}^{d} (\theta_d^{-1})^j Bv_j = c_d \sum_{j=0}^{d} (\theta_d^{-1})^j b_j v_j. \tag{3.10}$$

Comparing coefficients in (3.9)—(3.10),

$$b_j = \gamma \frac{c_0}{c_d} (\frac{\theta_d}{\theta_0})^j. \tag{3.11}$$

Note that $b_0, \cdots, b_d$ is a geometric sequence with common ratio $q = \frac{\theta_d}{\theta_0}$. Hence $b_j = \beta q^j$ where $\beta = b_0$. Observe $q^{d+1} = 1$ by Lemma 3.3 and q is primitive since $b_0, \cdots, b_d$ are distinct by Lemma 3.2.

$(ii) \rightarrow (iii)$ This is clear by direct computation.

$(iii) \rightarrow (i)$ Let $v \neq 0$ be an eigenvector to B with corresponding eigenvalue θ. Note that $\theta \neq 0$. Let $v_i = A^i v$. Suppose for some $c_0, \cdots, c_d \in \mathbb{C}$,

$$c_0 v_0 + c_1 v_1 + c_2 v_2 + \cdots + c_d v_d = 0. \tag{3.12}$$

Then

$$\begin{aligned} 0 &= \sum_{i=0}^{d} c_i v_i \\ &= \sum_{i=0}^{d} c_i A^i v. \end{aligned}$$

Applying B and using the assumption $BA = qAB$, we obtain

$$\begin{aligned} 0 &= B \sum_{i=0}^{d} c_i A^i v \\ &= \sum_{i=0}^{d} c_i q^i A^i B v \\ &= (\sum_{i=0}^{d} c_i q^i A^i) \theta v. \end{aligned}$$

Hence

$$\sum_{i=0}^{d} c_i q^i A^i = 0. \tag{3.13}$$

Observe $x^{d+1} - \alpha$ is the minimal polynomial of A, since $\alpha \neq 0$. Hence $c_0 = c_1 = \cdots = c_d = 0$. We have shown $v_0, \cdots, v_d$ is a basis of V. Observe $Av_i = v_{i+1}$, $i < d$, and $Av_d = AA^d v = \alpha I v = \alpha v_0$. On the other hand, $Bv_i = BA^i v = q^i A^i B v = \theta q^i A^i v = \theta q^i v_i$. Hence with respect to the basis $v_0, \cdots, v_d$, the matrices representing A, B has the following forms,

$$A = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & \alpha \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} \theta & 0 & 0 & \dots & 0 \\ 0 & \theta q & 0 & \dots & 0 \\ 0 & 0 & \theta q^2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \theta q^d \end{bmatrix}$$

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