國 立 交 通 大 學

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碩 士 論 文

圖的拉普拉斯特徵值 1 的重數

The multiplicity of Laplacian eigenvalue one

研 究 生: 陳 巧 玲

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摘　要

我們對於部份的樹的拉普拉斯特徵值 1 的重數給予一個演算法。令 T 是一個有點 u 和 u 的點集 $w_1, w_2, w_3, \ldots, w_k, u_1, u_2, \ldots, u_s$ 其中 $\deg(u_j)=2$ 且 $\deg(w_i)=1$。對於 T 的剩餘部分，T_j 是一個有獨一的點 t_j 的樹並且與點 u_j 相鄰，$1 \leq j \leq s$。則我們有以下的結果

$$m_T(1) = (k-1) + \sum_{i=1}^{s} m_{T_i}(1)$$

。

除此之外，我們在論文的最後一章節對 caterpillar 使用我們的演算法來計算拉普拉斯特徵值 1 的重數。

The multiplicity of Laplacian eigenvalue one

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Abstract

We give a tree algorithm of the multiplicity $m_T(1)$ of Laplacian eigenvalue 1. Let T be the tree with a vertex u, and the vertices $w_1, w_2, w_3, \ldots, w_k, u_1, u_2, \ldots, u_s$ are all neighbors of u with deg(u_j)=2 and deg(w_i)=1. For the remaining parts of T, T_j is a tree with unique vertex t_j in T_j adjacent to u_j, $1 \leq j \leq s$. Then

$$m_T(1) = (k-1) + \sum_{i=1}^{s} m_{T_i}(1)$$

In addition, we apply our algorithm to some special trees called caterpillar in our last section.

誌謝

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Contents

0.1 Introduction

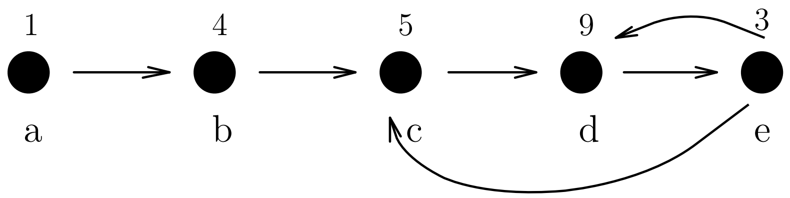
The Laplacian matrix of a graph and its eigenvalues can be used in several areas of mathematical research and have physical interpretation of various physical and chemical theories. The adjacency matrix of a graph and its eigenvalues were much more investigated in the past than the Laplacian matrix [1]. However, according to the Interlacing theorem [2], the eigenvalues of Laplacian matrix represent more interlacing behavior than the eigenvalues of adjacency matrix. Regarding the interlacing behavior, the adjacency matrix only removes vertices, but the Laplacian matrix removes not only vertices but also edges. Moreover, the Perron-Frobenius theory only shows that the largest eigenvalue of a connected graph goes down when one removes an edge or a vertex. But in the Interlacing theorem, it also tells us what happens with the other eigenvalues. For example, in [3] and [4] the Interlacing theorem can be applied to show that in some connected graphs, the largest eigenvalues are exactly 2. In the recent research, Ji-Ming Guo [5] gives an upper bound of the kth Laplacian eigenvalue of a tree, and A.E.Brouwer, W.H. Haemers [6] give a lower bound for the Laplacian eigenvalues of a graph. In their paper, they give us some information between eigenvalues and the degree of vertices. However in this paper, we want to find the multiplicity of 1 of some trees. Note that if the multiplicity of Laplacian eigenvalue one is k then the $(n-k+1)$-th Laplacian eigenvalue λ_{n-k+1} is bound above by 1u. We construct a labeled digraph and give four operations in the digraph. Moreover, we present an algorithm of a tree to find the multiplicity of 1. Also, we give some applications of the algorithm.

0.2 Preliminary

An ordered pair $G = (V(G), E(G))$ is a **graph** if $V(G)$ is a finite set and $E(G)$ is a subset of $V(G) \times V(G) \setminus \{ \ (a, a) \ | \ a \in V(G) \ \}$ such that $(u, v) \in E(G)$ iff $(v, u) \in E(G)$ for $u, v \in V(G)$. The elements in $V(G)$ are called **vertices**, and elements in $E(G)$ are called **edges** of G. The **order** of a graph is the cardinality of $V(G)$. Let $G = (V(G), E(G))$ be a graph. For $(u, v) \in E(G)$, we say that u and v are **adjacent**. The **degree** of u is the number deg(u) of vertices that are adjacent to u. The graph is **connected** if for each pair of vertices $x, y \in V(G)$, there exists a sequence of vertices $x = u_0, u_1, u_2, \ldots, u_t = y$ such that u_i and u_{i+1} are adjacent for $0 \le i \le t-1$. The **components** of the graph are its maximal connected subgraphs. $G - u$ is the graph with vertex set $V(G-u) = V(G) \setminus \{u\}$ and edge set $E(G-u) = E(G) \setminus \{ (u,a),(a,u) \mid a \in V(G) \}$.

A triple $G^* = (V(G^*), E(G^*), f_{G^*})$ is a **labeled digraph** if $V(G^*)$ is a finite set, $E(G^*)$ is a subset of $V(G^*) \times V(G^*) \setminus \{ (a,a) \mid a \in V(G^*) \}$ and $f_{G^*} : V(G^*) \to \mathbb{N} \cup \{0\}$ is a function. The **indegree** of u is $deg^-_{G^*}(u) = \mid \{ b \mid (b,u) \in E(G^*) \} \mid$. The **outdegree** of u is $deg^+_{G^*}(u) = \mid \{ c \mid (u,c) \in E(G^*) \} \mid$.

Example.



The labeled digraph G^*

$V(G^*) = \{a,b,c,d,e\}$, $E(G^*) = \{(a,b),(b,c),(c,d),(d,e),(e,d),(e,c)\}$

$f_{G^*}(c) = 5$, $deg^-_{G^*}(c) = 2$, $deg^+_{G^*}(c) = 1$

0.3 Laplacian of a simple graph

In this section, let $G = (V(G), E(G))$ be a graph of order n. The matrices considered in this section are $n \times n$ matrices with rows and columns indexed by $V(G)$. Set $D(G)$ to be a diagonal matrix such that $D(G)_{xx} = deg(x)$, and

$A(G)$ to be a matrix with

$$(A(G))_{xy} = \begin{cases} 1 & \text{if } (x, y) \in E(G), \\ \\ 0 & \text{else.} \end{cases}$$

$A(G)$ is referred to the **adjacency matrix** of G. Let $L(G) = D(G) - A(G)$, $L(G)$ is called the **Laplacian matrix** (or simply **Laplacian**) of G, and the eigenvalues of $L(G)$ are called the **Laplacian eigenvalues** of G. Since $L(G)$ is a symmetric matrix, it is diagonalizable. For an eigenvalue λ of $L(G)$, let $m_G(\lambda)$ be the multiplicity of λ. Denoted by $m_G(\lambda) = 0$ if λ is not an eigenvalue of $L(G)$.

0.4 Labeled digraph representing a matrix

Recall that in a graph G, the Laplacian matrix $L(G)$ has nonnegative integers on the diagonal and values $0, -1$ off diagonal. It is natural to give a name for such a matrix.

Definition 0.4.1. An $n \times n$ matrix M has **Laplacian type** if $M_{xx} \in \mathbb{N} \cup \{0\}$ and $M_{xy} \in \{0, -1\}$ for $x \neq y$, $x, y \in \{1, 2, \ldots, n\}$.

In particular, the Laplacian matrix of a graph G has Laplacian type. Note that a matrix with Laplacian type in general needs not to be symmetric. Let M be a Laplacian type with rows and columns indexed by a finite set V. The labeled digraph $G^*_M = (V(G^*_M), E(G^*_M), f_{G^*_M})$ **associated with** M, if $V(G^*_M) = V$, $E(G^*_M) = \{ (x, y) \mid M_{xy} = -1 \}$ and $f_{G^*_M}(x) = M_{xx}$. On the other hand, for each labeled digraph $F^* = (V(F^*), E(F^*), f_{F^*})$ the matrix M_{F^*} with rows and columns indexed by $V(F^*)$ such that

$$(M_{F^*})_{xy} = \begin{cases} f_{F^*}(x) & \text{if } x = y, \\ \\ -1 & \text{if } (x, y) \in E(F^*), \\ \\ 0 & \text{else} \end{cases}$$

for $x, y \in V(F^*)$, is called the **characteristic matrix** of F^*. Besides, in $n \times n$ matrix N, the **rank(N)** is the maximal number of its linearly independent columns, and the **nullity(N)** is $n-$rank(N).

0.5 Four operations

Let H be a connected simple graph , and we build up the vertex labeled digraph $H^* = (V(H^*), E(H^*), f_{H^*})$ associated with $L(H) - I$ corresponding to H. And in H^* we can find the multiplicity $m_H(1)$ of Laplacian eigenvalue 1 of H directly.

We consider the following four operations σ_p, τ_p, $\rho_{w,t}$, $\gamma_{w,t}$ on H^*.

(a) Type I operation σ_p:

Suppose that $f_{H^*}(p) = 0$, $deg^+(p) = 1$ and $(p, q) \in E(H^*)$. Then we have a new labeled digraph

$$\sigma_p(H^*) = (\sigma_p(V(H^*)), \sigma_p(E(H^*)), \sigma_p(f_{H^*})),$$

where

$$\sigma_p(V(H^*)) = V(H^*),$$

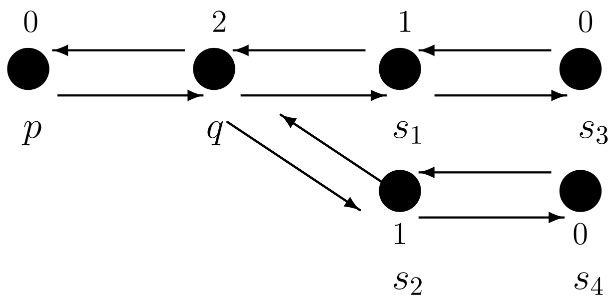
$$\sigma_p(E(H^*)) = E(H^*) - \{\, (a, q) \,|\, (a, q) \in E(H^*), a \neq p \,\},$$

and

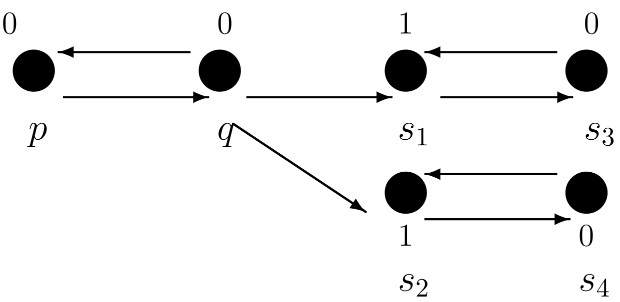
$$\sigma_p(f_{H^*}(u)) = \begin{cases} f_{H^*}(u) & \text{if } u \neq q, \\ 0 & \text{if } u = q. \end{cases}$$

The new labeled digraph $\sigma_p(H^*) = (\sigma_p(V(H^*)), \sigma_p(E(H^*)), \sigma_p(f_{H^*}))$ associated with matrix $M_{\sigma_p(H^*)}$, where

$$(M_{\sigma_p(H^*)})_{st} = \begin{cases} 0 & \text{if } s = t = q, \\ 0 & \text{if } t = q,\ (s,q) \in E(H^*),\ s \neq p, \\ (L(H) - I)_{st} & \text{otherwise.} \end{cases}$$



The labeled digraph H^*



The new labeled digraph $\sigma_p(H^*)$

(b) Type II operation τ_p:

Suppose that $f_{H^*}(p) = 0$, $deg^-(p) = 1$ and $(q, p) \in E(H^*)$. Then we have a new labeled digraph

$$\tau_p(H^*) = (\tau_p(V(H^*)), \tau_p(E(H^*)), \tau_p(f_{H^*})),$$

where

$$\tau_p(V(H^*)) = V(H^*),$$

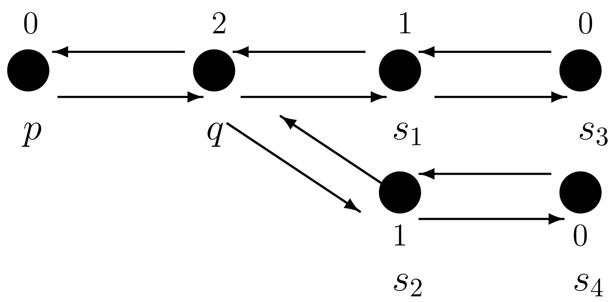
$$\tau_p(E(H^*)) = E(H^*) - \{\,(q, a)\,|\,(q, a) \in E(H^*), a \neq p\,\},$$

and

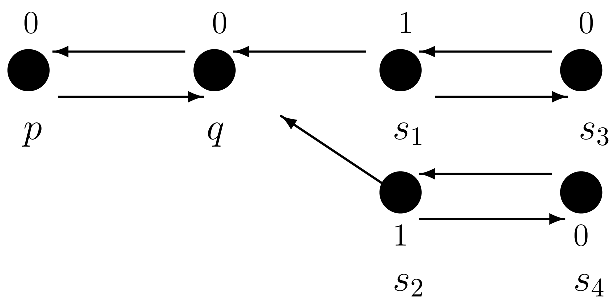
$$\tau_p(f_{H^*}(u)) = \begin{cases} f_{H^*}(u) & \text{if } u \neq q, \\ 0 & \text{if } u = q. \end{cases}$$

The new labeled digraph $\tau_p(H^*) = (\tau_p(V(H^*)), \tau_p(E(H^*)), \tau_p(f_{H^*}))$ associated with matrix $M_{\tau_p(H^*)}$, where

$$(M_{\tau_p(H^*)})_{st} = \begin{cases} 0 & \text{if } t = s = q, \\ 0 & \text{if } s = q,\ (q,t) \in E(H^*),\ t \neq p, \\ (L(H) - I)_{st} & \text{otherwise.} \end{cases}$$



The labeled digraph H^*



The labeled digraph $\tau_p(H^*)$

(c) Type III operation $\rho_{w,t}$:

Suppose that $f_{H^*}(w) = 1$, $(w,t),(t,w) \in E(H^*)$. Then we have

$$\rho_{w,t}(H^*) = (\rho_{w,t}(V(H^*)), \rho_{w,t}(E(H^*)), \rho_{w,t}(f_{H^*})),$$

where

$$\rho_{w,t}(V(H^*)) = V(H^*),$$

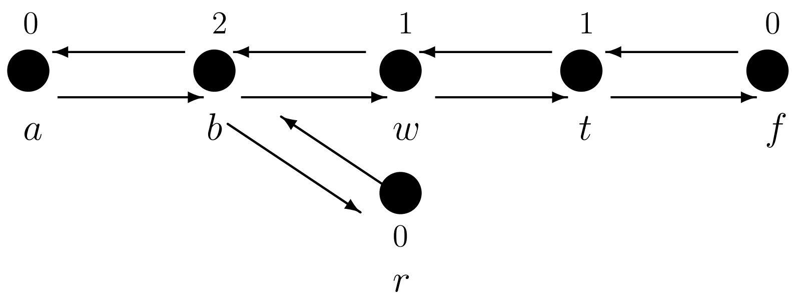
$$\rho_{w,t}(E(H^*)) = E(H^*) - \{\,(t,w)\,\}$$

and

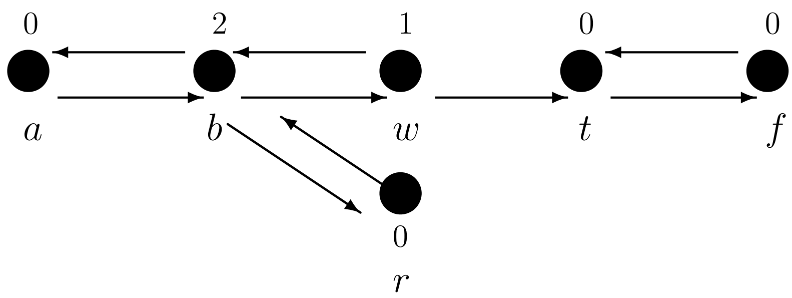
$$\rho_{w,t}(f_{H^*}(u)) = \begin{cases} f_{H^*}(u) & \text{if } u \neq t, \\ \\ f_{H^*}(u) - 1 & \text{if } u = t. \end{cases}$$

The new labeled digraph $\rho_{w,t}(H^*) = (\rho_{w,t}(V(H^*)), \rho_{w,t}(E(H^*)), \rho_{w,t}(f_{H^*}))$ associated with matrix $M_{\rho_{w,t}(H^*)}$, where

$$(M_{\rho_{w,t}(H^*)})_{mn} = \begin{cases} 0 & \text{if } m = t,\ n = w, \\ (L(H) - I)_{tt} - 1 & \text{if } m = n = t, \\ (L(H) - I)_{mn} & \text{otherwise.} \end{cases}$$



The labeled digraph H^*



The labeled digraph $\rho_{w,t}(H^*)$

(d) Type IV operation $\gamma_{w,t}$:

Suppose that $f_{H^*}(w) = 1$, $deg^+(w) = 1$ and $(w,t) \in E(H^*)$. Then we have

$$\gamma_{w,t}(H^*) = (\gamma_{w,t}(V(H^*)), \gamma_{w,t}(E(H^*)), \gamma_{w,t}(f_{H^*})),$$

where

$$\gamma_{w,t}(V(H^*)) = V(H^*),$$

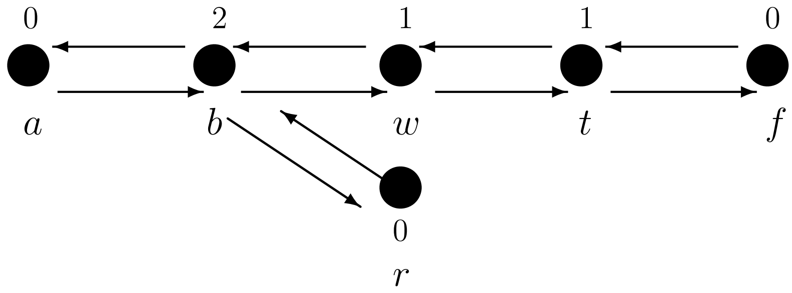
$$\gamma_{w,t}(E(H^*)) = E(H^*) - \{\,(w,t)\,\}$$

and

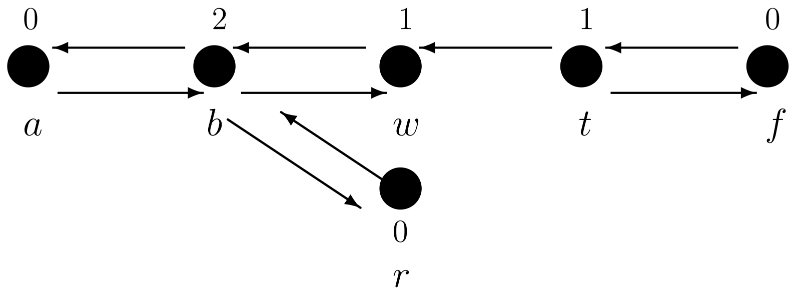
$$\gamma_{w,t}(f_{H^*}(u)) = f_{H^*}(u) \quad \forall u \in V(H^*).$$

The new labeled digraph $\gamma_{w,t}(H^*) = (\gamma_{w,t}(V(H^*)), \gamma_{w,t}(E(H^*)), \gamma_{w,t}(f_{H^*}))$ associated with matrix $M_{\gamma_{w,t}(H^*)}$, where

$$(M_{\gamma_{w,t}(H^*)})_{mn} = \begin{cases} 0 & \text{if } m = w,\ n = t, \\ (L(H) - I)_{mn} & \text{otherwise.} \end{cases}$$



The labeled digraph H^*



The labeled digraph $\gamma_{w,t}(H^*)$

These four kinds of operations are applied to the vertex labeled digraph. Consider the corresponding characteristic matrices during the processes, we can also see the operations above as operations on characteristic matrices preserving the rank. If a vertex labeled digraph associated with a matrix M of Laplacian can take use of these four operations to becomes a non-edge labeled subgraph, then the nullity of M is the number of vertices with label zero. In particular, if $M = L(G) - I$ for some graph G, we can find the multiplicity $m_G(1)$ of Laplacian eigenvalue 1 of G, where $I(G)$ is the identity matrix,

$$(I(G))_{xy} = \begin{cases} 1 & \text{if } (x = y), \\ \\ 0 & \text{else.} \end{cases}$$

.

0.6 Tree Algorithm

Definition 0.6.1. Let G be a graph. A vertex $u \in V(G)$ is called **typical** if $deg(v) \leq 2$ for any vertex v adjacent to u, and $deg(w) = 1$ for some w adjacent to u.

Theorem 0.6.2. *Let T be the tree with a typical vertex u, the vertices w_1, w_2, w_3, $\ldots$, w_k, u_1, u_2, u_3, $\ldots$, u_s are all neighbors of u with $deg(w_i) = 1$ and $deg(u_j) = 2$. For the remaining parts T_j is a tree with a unique vertex t_j in T_j adjacent to u_j for $1 \leq j \leq s$. Then*

$$m_T(1) = (k-1) + \sum_{i=1}^{s} m_{T_i}(1).$$

Proof. Let T^* be the labeled digraph associated with $L(T) - I$, where $L(T)$ is the Laplacian of T. For $f_{T^*}(w_1) = 0$, $deg^+_{T^*}(w_1) = 1$ and $(w_1, u) \in E(T^*)$, we can apply Type I operation σ_{w_1} to delete all arcs (w_j, u) and (u_i, u) for $2 \leq j \leq k$ and $1 \leq i \leq s$, and to erase the label on u, we have the new labeled digraph $\sigma_{w_1}(T^*)$. However, since $\sigma_{w_1}(f_{T^*})(w_1) = 0$, $deg^-_{\sigma_{w_1}(T^*)}(w_1) = 1$ and $(u, w_1) \in \sigma_{w_1}(T^*)$, we can apply the operation τ_{w_1}

to delete all arcs (u, w_j) and (u, u_i) for $2 \le j \le k$ and $1 \le i \le s$. After that we have a new labeled digraph $\tau_{w_1}(\sigma_{w_1}(T^*))$. And in $\tau_{w_1}(\sigma_{w_1}(T^*))$, we have isolated points $w_2, w_3 \dots w_k$. And each w_i has label 0. Moreover, since $\tau_{w_1}(f_{\sigma_{w_1}(T^*)})(u_1) = 1$, and (u_1, t_1) and $(t_1, u_1) \in E(\tau_{w_1}(\sigma_{w_1}(T^*)))$, we can apply Type III operation ρ_{u_1,t_1} to delete the arc (t_1, u_1) and to decrease the label on t_1 1. Then we have a new labeled digraph $\rho_{u_1,t_1}(\tau_{w_1}\sigma_{w_1}(T^*))$. Similarly, because of $\prod_{i=1}^{p-1} \rho_{u_i,t_i}\tau_{w_1}\sigma_{w_1}(T^*)$ is a new labeled digraph and $f_{\prod_{i=1}^{p-1} \rho_{u_i,t_i}\tau_{w_1}\sigma_{w_1}(T^*)}(u_p) = 1$, (u_i, t_i) $(t_i, u_i) \in E(\prod_{i=1}^{p-1} \rho_{u_i,t_i}\tau_{w_1}\sigma_{w_1}(T^*))$, we can apply Type III ρ_{u_p,t_p} to delete the arc (t_p, u_p) and to decrease the label on t_p 1 for $2 \le p \le s$. As the results of the preceding operations, we have a new labeled digraph $\prod_{i=1}^{s} \rho_{u_i,t_i}\tau_{w_1}\sigma_{w_1}(T^*)$. Since $f_{\prod_{i=1}^{s} \rho_{u_i,t_i}\tau_{w_1}\sigma_{w_1}(T^*)}(u_1) = 1$ and $deg^+_{\prod_{i=1}^{s} \rho_{u_i,t_i}\tau_{w_1}\sigma_{w_1}(T^*)}(u_1) = 1$ and $(u_1, t_1) \in E(\prod_{i=1}^{s} \rho_{u_i,t_i}\tau_{w_1}\sigma_{w_1}(T^*))$, we can apply Type IV operation γ_{u_1,t_1} to delete arc (u_1, t_1). So, we have a new labeled digraph $\gamma_{u_1,t_1}(\prod_{i=1}^{p} \rho_{u_i,t_i}\tau_{w_1}\sigma_{w_1}(T^*))$. Furthermore, for $\prod_{j=1}^{p-1} \gamma_{u_j,t_j}$ $\prod_{i=1}^{p} \rho_{u_i,t_i}\tau_{w_1}\sigma_{w_1}(T^*)$ is a new labeled digraph , $f_{\prod_{j=1}^{p-1} \gamma_{u_j,t_j} \prod_{i=1}^{s} \rho_{u_i,t_i}\tau_{w_1}\sigma_{w_1}(T^*)}$ $(u_p) = 1$ and $deg^+_{(\prod_{j=1}^{p-1} \gamma_{u_j,t_j} \prod_{i=1}^{s} \rho_{u_i,t_i}\tau_{w_1}\sigma_{w_1}T^*)}$ $(u_p) = 1$ and $(u_p, t_p) \in E(\prod_{j=1}^{p-1}$ γ_{u_j,t_j} $\prod_{i=1}^{s} \rho_{u_i,t_i}\tau_{w_1}\sigma_{w_1}(T^*))$, we can apply Type IV operation γ_{u_p,t_p} to delete arc (u_p, t_p) for $2 \le p \le s$. Therefore, $\prod_{j=1}^{s} \gamma_{u_j,t_j} \prod_{i=1}^{s} \rho_{u_i,t_i}\tau_{w_1}\sigma_{w_1}(T^*)$ is a new labeled digraph. Note that in this new labeled digraph $\prod_{j=1}^{s} \gamma_{u_j,t_j}$

$\prod_{i=1}^{s}\rho_{u_i,t_i}\tau_{w_1}\sigma_{w_1}(T^*)$, we have several components, that are isolated points $w_2, w_3 \ldots w_k$ with label 0, $u_1, u_2 \ldots u_s$ with label 1 and T_d' corresponding to T_d, $1 \le d \le s$. Now, let's consider the characteristic matrix $M_{\prod_{j=1}^{s}\gamma_{u_j,t_j}\prod_{i=1}^{s}\rho_{u_i,t_i}\tau_{w_1}\sigma_{w_1}(T^*)}$. Since we have isolated points $w_2, w_3 \ldots w_k$ with label 0 in $\prod_{j=1}^{p}\gamma_{u_j,t_j}\prod_{i=1}^{p}\rho_{u_i,t_i}\tau_{w_1}\sigma_{w_1}(T^*)$, the row and column corresponding to each w_i are 0 in the characteristic matrix $M_{\prod_{j=1}^{s}\gamma_{u_j,t_j}\prod_{i=1}^{s}\rho_{u_i,t_i}\tau_{w_1}\sigma_{w_1}(T^*)}$. However, we have isolated points $u_1, u_2 \ldots u_s$ with label 1 in $\prod_{j=1}^{s}\gamma_{u_j,t_j}\prod_{i=1}^{s}\rho_{u_i,t_i}\tau_{w_1}\sigma_{w_1}(T^*)$, the row and column corresponding to each u_i are 1 in the characteristic matrix $M_{\prod_{j=1}^{s}\gamma_{u_j,t_j}\prod_{i=1}^{s}\rho_{u_i,t_i}\tau_{w_1}\sigma_{w_1}(T^*)}$. Moreover, each component T_d' in $\prod_{j=1}^{s}\gamma_{u_j,t_j}\prod_{i=1}^{s}\rho_{u_i,t_i}\tau_{w_1}\sigma_{w_1}(T^*)$ is a labeled digraph of T_d induced from $L(T_d) - I$. This implies

$$\begin{aligned}&\text{nullity}(M_{\prod_{j=1}^{s}\gamma_{u_j,t_j}\prod_{i=1}^{s}\rho_{u_i,t_i}\tau_{w_1}\sigma_{w_1}(T^*)})\\ &= (k-1) + \sum_{d=1}^{s}\text{nullity}(M_{T_d'}).\end{aligned}$$

Thus

$$m_T(1) = (k-1) + \sum_{i=1}^{s} m_{T_i}(1).$$

□

0.7 Applications

We need the following lemma about Laplacian eigenvalues of a path P_n of n vertices in our study. Let P_n be the path with vertex set $V(P_n) = \{ u_i \,|\, i = 1, 2, ..., n \}$ and edge set $E(P_n) = \{ \{u_i, u_{i+1}\} \,|\, i = 1, 2, ..., n-1 \}$.

Lemma 0.7.1. *[7] P_n has eigenvalues $\lambda_i(L(P_n)) = 2 - 2\cos(\pi(n-i)/n)$ for $i \in \{1, 2, \cdots, n\}$.*

By this Lemma, we know that $m_{P_n}(\lambda) = 1$ for each eigenvalue λ.

Corollary 0.7.2. *P_n has eigenvalue 1 if and only if 3 divides n.*

Proof. Since $\lambda_i(L(P_n)) = 2 - 2\cos(\pi(n-i)/n) = 1$ for $i \in \{1, 2, \cdots, n\}$, $\cos(\pi(n-i)/n) = 1/2$. Moreover, for each eigenvalue λ, $m_{P_n}(\lambda) = 1$. So, $\cos(\pi/3) = \cos(\pi(n-i)/n)$. Then $\pi/3 = \pi(n-i)/n$. This implies $n = 3i/2$. Thus 3 divides n. Let $n = 3d$, $d \in N$. If we take $i = 2d$, then we get $\lambda_i(L(P_n)) = \lambda_{2d}(L(P_{3d}) = 2 - 2\cos(\pi(3d-2d)/3d) = 2 - 2\cos(\pi/3) = 1$. Thus P_n has eigenvalue 1 . □

Definition 0.7.3. A **caterpillar** is a tree $CP(n; k_1, k_2, k_3, \ldots, k_n)$ with vertex set $V = V(P_n) \cup \bigcup_{i=1}^{n} \{u_{ij} | 1 \leq j \leq k_i\}$ and edge set $E = E(P_n) \cup$

$\bigcup_{i=1}^{n}\{\{u_i, u_{ij}|1 \leq j \leq k_i\}, k_i \geq 0.$

Theorem 0.7.4. *Let $H_1 = CP(n; k_1, k_2, k_3, \dots, k_n)$ be the graph where $k_{2i} = 0$ for all i and n is odd, then $m_{H_1}(1) = \sum_{j=0}^{(n-1)/2} k_{2j+1} - (n+1)/2$.*

Proof. Take u_1 to be the typical vertex, then by theorem 6.2 we have $m_{H_1}(1) = (k_1 - 1) + m_{CP(n-2;k_3,k_4,\dots,k_n)}(1)$. Similarly, when we take u_{2t+1} be the typical vertex in $CP(n-2t; k_{2t+1}, k_{2t+2}, \dots, k_n)$, where $t \geq 1$. Then

$$m_{H_1}(1) = (k_1 - 1) + (k_3 - 1) + \dots + CP(1; k_n)$$

$$= (k_1 - 1) + (k_3 - 1) + \dots + (k_n - 1)$$

$$= \sum_{j=0}^{(n-1)/2} k_{2j+1} - (n+1)/2.$$

□

Theorem 0.7.5. *Let $H_2 = CP(n; k_1, k_2, k_3, \dots, k_n)$ be the graph where $k_{2i} = 0$ for all i and n is even, then $m_{H_2}(1) = \sum_{j=0}^{(n-2)/2} k_{2j+1} - (n-2)/2$.*

Proof. Similarly to theorem 7.4, we take u_{2t+1} to be the typical vertex in $CP(n-2t; k_{2t+1}, k_{2t+2}, \dots, k_n)$, where $t \geq 0$. Then

$$m_{H_2}(1) = (k_1 - 1) + (k_3 - 1) + \dots + (k_{n-3} - 1) + CP(2; k_{n-1}, k_n)$$

$$= \sum_{j=0}^{(n-2)/2} k_{2j+1} - (n-2)/2.$$

□

Theorem 0.7.6. *Let $H_3 = CP(n; 0, 0, \ldots, k_t, 0, \ldots, 0)$ be the graph where $t \equiv 0 \ (\ mod\ 3)$. If $n \equiv 1 \ (\ mod\ 3)$, then $m_{H_3}(1) = k_t$. Otherwise, if $n \equiv 0, 2$ then $m_{H_3}(1) = k_t - 1$.*

Proof. Take u_t to be the typical vertex, then we have $m_{H_3}(1) = (k_t - 1) + CP(t-2; 0, 0, \ldots, 0) + CP(n-t-1; 0, 0, \ldots, 0)$. If $n \equiv 1 \ (\bmod 3)$, then by Corollary 7.3, $m_{CP(n-t-1;0,0,\ldots,0)}(1) = 1$, and $m_{CP(t-2;0,0,\ldots,0)}(1) = 0$. Thus, $m_{H_3}(1) = (k_t - 1) + 1 = k_t$. Otherwise, if $n \equiv 0, 2 \ (\bmod 3)$ then $m_{CP(n-t-1;0,0,\ldots,0)}(1) = 0$, and $m_{CP(t-2;0,0,\ldots,0)}(1) = 0$. Thus, $m_{H_3}(1) = k_t - 1$. □

Theorem 0.7.7. *Let $H_4 = CP(n; 0, 0, \ldots, k_t, 0, \ldots, 0)$ be the graph where $t \equiv 1 \ (\ mod\ 3)$. If $n \equiv 2 \ (\ mod\ 3)$, then $m_{H_4}(1) = k_t$. Otherwise, if $n \equiv 0, 1$ $(\ mod\ 3)$ then $m_{H_4}(1) = k_t - 1$.*

Proof. Take u_t to be the typical vertex, then we have $m_{H_4}(1) = (k_t - 1) + CP(t-2; 0, 0, \ldots, 0) + CP(n-t-1; 0, 0, \ldots, 0)$. If $n \equiv 2 \ (\bmod 3)$, then by Corollary 7.3, $m_{CP(n-t-1;0,0,\ldots,0)}(1) = 1$, and $m_{CP(t-2;0,0,\ldots,0)}(1) = 0$. Thus, $m_{H_4}(1) = (k_t - 1) + 1 = k_t$. Otherwise, if $n \equiv 0, 1 \ (\bmod 3)$ then $m_{CP(n-t-1;0,0,\ldots,0)}(1) = 0$, and $m_{CP(t-2;0,0,\ldots,0)}(1) = 0$. Thus, $m_{H_4}(1) = k_t - 1$. □

Theorem 0.7.8. *Let* $H_5 = CP(n; 0, 0, \ldots, k_t, 0, \ldots, 0)$ *be the graph where* $t \equiv 2 \pmod 3$*. If* $n \equiv 0 \pmod 3$*, then* $m_{H_5}(1) = k_t + 1$*. Otherwise, if* $n \equiv 1, 2 \pmod 3$ *then* $m_{H_5}(1) = k_t$*.*

Proof. Take u_t to be the typical vertex, then we have $m_{H_5}(1) = (k_t - 1) + CP(t-2; 0, 0, \ldots, 0) + CP(n-t-1; 0, 0, \ldots, 0)$. If $n \equiv 0 \pmod 3$, then by Corollary 7.3, $m_{CP(n-t-1;0,0,\ldots,0)}(1) = 1$, and $m_{CP(t-2;0,0,\ldots,0)}(1) = 1$. Thus, $m_{H_5}(1) = (k_t - 1) + 2 = k_t + 1$. Otherwise, if $n \equiv 1, 2 \pmod 3$ then $m_{CP(n-t-1;0,0,\ldots,0)}(1) = 0$, and $m_{CP(t-2;0,0,\ldots,0)}(1) = 1$. Thus, $m_{H_5}(1) = k_t$. □

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