Cyclic Triples

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Abstract

Let $\mathbb{C}$ denote the complex field and let d be a positive integer. We essentially determine all the triples A, B, C of $(d+1)\times(d+1)$ matrices over $\mathbb{C}$ that satisfy

$$A^{d+1} = \alpha I,\ B^{d+1} = \beta I,\ C^{d+1} = \gamma I,\ BA = qAB,\ CB = qBC,\ AC = qCA$$

for some nonzero complex numbers α, β, γ, and a primitive root q of unity of order $d+1$.

Contents

1 Introduction

Let $\mathbb{C}$ denote the complex field and let $Mat_{d+1}(\mathbb{C})$ denote the set of $(d+1)\times(d+1)$ matrices over $\mathbb{C}$ with the index set $\{0,1,\ldots,d\}$.

Definition 1.1. Let A denote a matrix in $Mat_{d+1}(\mathbb{C})$. We say A is *left-cyclic* whenever each of the entries $A_{i,i-1}$ and A_{0d} is nonzero for $i = 1, 2, ..., d$ and all other entries of A are zero ; or A is *right-cyclic* whenever its transpose is left-cyclic. We say a square matrix is *cyclic* whenever it is left-cyclic or right-cyclic.

Definition 1.2. Let $\mathbf{V}$ denote a vector space over $\mathbb{C}$ with finite dimension. Let $A : \mathbf{V} \longrightarrow \mathbf{V}, B : \mathbf{V} \longrightarrow \mathbf{V}$, and $C : \mathbf{V} \longrightarrow \mathbf{V}$ denote linear transformations which satisfy $(i)-(iii)$ below.

(i) There exists a basis for $\mathbf{V}$ with respect to which the matrix representing A is left-cyclic, the matrix representing B is diagonal, and the matrix representing C is right-cyclic.

(ii) There exists a basis for $\mathbf{V}$ with respect to which the matrix representing A is right-cyclic, the matrix representing B is left-cyclic, and the matrix representing C is diagonal.

(iii) There exists a basis for $\mathbf{V}$ with respect to which the matrix representing A is diagonal, the matrix representing B is right-cyclic, and the matrix representing C is left-cyclic.

We call such a triple (A, B, C) a *cyclic triple* on $\mathbf{V}$.

The following is our main result.

Theorem 1.3. *Let $\mathbf{V}$ denote a vector space over $\mathbb{C}$ with dimension $d+1$. Let $A : \mathbf{V} \longrightarrow \mathbf{V}, B : \mathbf{V} \longrightarrow \mathbf{V}$, and $C : \mathbf{V} \longrightarrow \mathbf{V}$ denote linear transformations. We prove the following are equivalent.*

(i) (A, B, C) is a cyclic triple on $\mathbf{V}$.

(ii) There exist three nonzero complex numbers α, β, γ and a primitive root q of unity of order $d+1$ such that

$$A^{d+1} = \alpha I,\ B^{d+1} = \beta I,\ C^{d+1} = \gamma I,\ BA = qAB,\ CB = qBC,\ AC = qCA.$$

(iii) There exists a basis $v_0, v_1, \ldots, v_d$ for $\mathbf{V}$ with respect to which the matrices representing A (resp. B, C) is left-cyclic (resp. diagonal, right-cyclic) with the following forms,

$$A : \eta \begin{pmatrix} 0 & & & & 1 \\ q^{-2} & 0 & & & \\ & q^{-4} & \ddots & & \\ & & \ddots & 0 & \\ 0 & & & q^{-2d} & 0 \end{pmatrix},$$

$$(resp.\ B : \xi \begin{pmatrix} 1 & & & & 0 \\ & q & & & \\ & & \ddots & & \\ & & & q^{d-1} & \\ 0 & & & & q^d \end{pmatrix}, C : \zeta \begin{pmatrix} 0 & q & & & 0 \\ & 0 & q^2 & & \\ & & \ddots & \ddots & \\ & & & 0 & q^d \\ 1 & & & & 0 \end{pmatrix})$$

for some nonzero complex numbers η, ξ, ζ, *and a primitive root* q *of unity of order* $d+1$.

2 Cyclic pairs

To prove Theorem 1.3 we need some previous results in [1, 3]. For the thesis to be self-contained, these results are stated in this section and the proofs are given in slightly different ways.

Lemma 2.1. *Cyclic matrices are diagonalizable with distinct nonzero eigenvalues.*

Proof. For any left-cyclic matrix

$$A = \begin{pmatrix} 0 & & & & a_0 \\ a_1 & 0 & & & \\ & a_2 & \ddots & & \\ & & \ddots & 0 & \\ 0 & & & a_d & 0 \end{pmatrix}$$

the characteristic polynomial of A is

$$f(x) = x^{d+1} - \prod_{i=0}^{d} a_i.$$

Since $a_0, a_1, \ldots, a_d$ are not zeros, $f(x)$ has $d+1$ distinct roots. Hence A has $d+1$ distinct eigenvalues. This implies A is diagonalizable with nonzero eigenvalues. For any right-cyclic matrix A, since A^T is left-cyclic and A have the same characteristic polynomial with A^T, A is also diagonalizable with nonzero eigenvalues. We complete the proof. □

Definition 2.2. Let $\mathbf{V}$ denote a vector space over $\mathbb{C}$ with finite positive dimension. By a *cyclic pair* on $\mathbf{V}$ we mean an ordered pair of linear transformations $A : \mathbf{V} \longrightarrow \mathbf{V}$ and $B : \mathbf{V} \longrightarrow \mathbf{V}$ that satisfy conditions (1), (2) below.

(i) There exists a basis for $\mathbf{V}$ with respect to which the matrix representing A is diagonal and the matrix representing B is cyclic.

(ii) There exists a basis for $\mathbf{V}$ with respect to which the matrix representing B is diagonal and the matrix representing A is cyclic.

Lemma 2.3. *Suppose*

$$A = \begin{pmatrix} 0 & & & & \alpha \\ 1 & 0 & & & \\ 0 & 1 & \ddots & & \\ & & \ddots & 0 & \\ & & & 1 & 0 \end{pmatrix} \quad (\alpha \neq 0)$$

is a left-cyclic matrix and $\theta \neq 0$ is an eigenvalue of A. Let u be an eigenvector corresponding to θ. Then

$$\theta^{d+1} = \alpha \text{ and } u = \begin{pmatrix} u_0 \\ u_0\theta^{-1} \\ u_0\theta^{-2} \\ \vdots \\ u_0\theta^{-d} \end{pmatrix}$$

for some nonzero scalar $u_0 \in \mathbb{C}$.

Proof. Since the characteristic polynomial of A is $x^{d+1} - \alpha$, it is obvious that

$$\theta^{d+1} = \alpha.$$

Suppose

$$u = \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \\ u_d \end{pmatrix}.$$

Observe

$$Au = \begin{pmatrix} 0 & & & & \alpha \\ 1 & 0 & & & \\ 0 & 1 & \ddots & & \\ & & \ddots & 0 & \\ & & & 1 & 0 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \\ u_d \end{pmatrix} = \begin{pmatrix} \alpha u_d \\ u_0 \\ u_1 \\ \vdots \\ u_{d-1} \end{pmatrix} = \begin{pmatrix} \theta u_0 \\ \theta u_1 \\ \theta u_2 \\ \vdots \\ \theta u_d \end{pmatrix},$$

since $Au = \theta u$. Hence $u_i = \theta u_{i+1}$ for $i = 0, 1, \ldots, d-1$ and $u_d = (\theta/\alpha)u_0 = \theta^{-d}u_0$. Then $u_i = u_0\theta^{-i} \quad (1 \leq i \leq d)$. Note that $u_0 \neq 0$ since $u \neq 0$ and $\theta \neq 0$. Hence the proof is completed. □

Theorem 2.4. *Let $\mathbf{V}$ denote a vector space over $\mathbb{C}$ with dimension $d+1$. Let $A : \mathbf{V} \to \mathbf{V}$ and $B : \mathbf{V} \to \mathbf{V}$ denote linear transformations. Then the following (i)-(iii) are equivalent.*

(i) (A, B) *is a cyclic pair on* $\mathbf{V}$.

(ii) There exist two nonzero complex numbers α *and* β *such that*

$$A^{d+1} = \alpha I, \quad B^{d+1} = \beta I, \quad BA = qAB,$$

where q *is a primitive root of unity of order* $d+1$.

(iii) There exists a basis $v_0, v_1, \ldots, v_d$ *for* $\mathbf{V}$ *with respect to which the matrices representing* A *and* B *have the following forms,*

$$A : \begin{pmatrix} 0 & & & & \alpha \\ 1 & 0 & & & \\ 0 & 1 & \ddots & & \\ & & \ddots & 0 & \\ & & & 1 & 0 \end{pmatrix}, \quad B : \begin{pmatrix} \xi & & & & 0 \\ & \xi q & & & \\ & & \xi q^2 & & \\ & & & \ddots & \\ 0 & & & & \xi q^d \end{pmatrix},$$

where $\alpha, \xi \in \mathbb{C}$ *are nonzero scalars and* $q \in \mathbb{C}$ *is a primitive root of unity of order* $d+1$.

Proof. ((*iii*) $\Longrightarrow$ (*ii*)) By direct computation

$$A^{d+1} = \begin{pmatrix} \alpha & & & & 0 \\ & \alpha & & & \\ & & \alpha & & \\ & & & \ddots & \\ 0 & & & & \alpha \end{pmatrix} = \alpha I,$$

$$B^{d+1} = \begin{pmatrix} \xi^{d+1} & & & & 0 \\ & \xi^{d+1} & & & \\ & & \xi^{d+1} & & \\ & & & \ddots & \\ 0 & & & & \xi^{d+1} \end{pmatrix} = \beta I,$$

$$BA = \begin{pmatrix} 0 & & & & & \alpha\xi \\ \xi q & 0 & & & & \\ & \xi q^2 & 0 & & & \\ & & \xi q^3 & & & \\ & & & \ddots & \ddots & \\ 0 & & & & \xi q^d & 0 \end{pmatrix},$$

and

$$AB = \begin{pmatrix} 0 & & & & & \alpha\xi q^d \\ \xi & 0 & & & & \\ & \xi q & 0 & & & \\ & & \xi q^2 & & & \\ & & & \ddots & \ddots & \\ 0 & & & & \xi q^{d-1} & 0 \end{pmatrix}.$$

Therefore $A^{d+1} = \alpha I, B^{d+1} = \beta I$, and $BA = qAB$, where $\beta = \xi^{d+1}$.

($(ii) \Longrightarrow (i)$) Since $\mathbf{V}$ is over the complex field $\mathbb{C}$, there exists an eigenvalue ξ for B. Let v_0 be an eigenvector of B with respect to eigenvalue ξ, that is, $Bv_0 = \xi v_0$ with $v_0 \neq 0$. Consider vectors $v_0, Av_0, A^2v_0, \ldots, A^dv_0$.

Claim. $\{v_0,\ Av_0,\ A^2v_0,\ \ldots,\ A^dv_0\}$ is a basis of eigenvectors of B.

Set $u_i = A^iv_0$ for $i = 0, 1, \ldots, d$. Note that $u_i \neq 0$ since A is invertible. Observe $Bu_i = BA^iv_0 = q^iA^iBv_0 = \xi q^iA^iv_0 = \xi q^iu_i$, since $BA = qAB$. Hence u_i are distinct eigenvectors of B with respect to distinct eigenvalues ξq^i $(0 \leq i \leq d)$, and $\{u_0, u_1, u_2, \ldots, u_d\}$ is a basis of eigenvectors of B. This proves the claim.

For the basis $\{u_0, u_1, u_2, \ldots, u_d\}$,

$$Au_i = A^{i+1}v_0 = u_{i+1} \qquad (0 \leq i \leq d-1)$$

and

$$Au_d = A^{d+1}v_0 = \alpha v_0 = \alpha u_0 \qquad (A^{d+1} = \alpha I).$$

Hence with respect to the basis $\{u_0, u_1, u_2, \ldots, u_d\}$, the matrices representing A and B are

$$A : \begin{pmatrix} 0 & & & & \alpha \\ 1 & 0 & & & \\ & 1 & \ddots & & \\ & & \ddots & 0 & \\ 0 & & & 1 & 0 \end{pmatrix}, \quad B : \begin{pmatrix} \xi & & & & 0 \\ & \xi q & & & \\ & & \xi q^2 & & \\ & & & \ddots & \\ 0 & & & & \xi q^d \end{pmatrix}.$$

Similarly, there exists a basis for $\mathbf{V}$ which the matrices represent B and A as follows.

$$B : \begin{pmatrix} 0 & & & & \beta \\ 1 & 0 & & & \\ & 1 & \ddots & & \\ & & \ddots & 0 & \\ 0 & & & 1 & 0 \end{pmatrix}, \quad A : \begin{pmatrix} \eta & & & 0 \\ & \eta q^{-1} & & \\ & & \ddots & \\ 0 & & & \eta q^{-d} \end{pmatrix},$$

for some $\eta \in \mathbb{C}$, since $B^{d+1} = \beta I$ and $AB = q^{-1}BA$. Therefore, (A, B) is a cyclic pair.

($(i) \Longrightarrow (iii)$) Since (A, B) is a cyclic pair, there exists a basis $\{u_0, u_1, \ldots, u_d\}$ such that the matrices representing A is cyclic and B is diagonal. Without loss of generality, we suppose the matrix representing A, B as follows.(exchange the ordered basis to $u_d, u_{d-1}, \ldots, u_0$ as A is right-cyclic)

$$A : \begin{pmatrix} 0 & & & & a_0 \\ a_1 & 0 & & & \\ & a_2 & 0 & & \\ & & \ddots & \ddots & \\ 0 & & & a_d & 0 \end{pmatrix}, \quad B : \begin{pmatrix} b_0 & & & & 0 \\ & b_1 & & & \\ & & b_2 & & \\ & & & \ddots & \\ 0 & & & & b_d \end{pmatrix}.$$

So we know that

$$Au_i = a_{i+1}u_{i+1} \quad (0 \leq i \leq d-1) \tag{2.1}$$

and

$$Au_d = a_0 u_0. \tag{2.2}$$

Set

$$v_0 = u_0 \tag{2.3}$$

and

$$v_i = a_1 a_2 \dots a_i u_i \quad (1 \le i \le d). \tag{2.4}$$

So by (2.1) - (2.4),

$$Av_i = v_{i+1} (0 \le i \le d-1)$$

and

$$Av_d = a_d ... a_1 a_0 v_0.$$

Therefore, for the new basis $\{v_0, v_1, ... v_d\}$, the matrices represent A and B as follows,

$$A \quad : \quad \begin{pmatrix} 0 & & & & \alpha \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & \ddots & \ddots & \\ 0 & & & 1 & 0 \end{pmatrix}, \qquad (\alpha = a_0 \dots a_d)$$

$$B \quad : \quad \begin{pmatrix} b_0 & & & & 0 \\ & b_1 & & & \\ & & b_2 & & \\ & & & \ddots & \\ 0 & & & & b_d \end{pmatrix}. \qquad \text{(eigenvector invariant)}$$

Similarly there exists a basis $\{w_0, w_1, \dots, w_d\}$ of $\mathbf{V}$ such that the matrix representing A is diagonal and the matrix representing B as

$$\begin{pmatrix} 0 & & & & \beta \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & \ddots & \ddots & \\ 0 & & & 1 & 0 \end{pmatrix},$$

for some $\beta \in \mathbb{C}$. Note that w_0, w_1 are eigenvectors of A. Let θ_0, θ_1 be the corresponding eigenvalues. Then there exists $c_0 \in \mathbb{C}$ such that

$$w_0 \ : \ \begin{pmatrix} c_0 \\ c_0 \theta_0^{-1} \\ c_0 \theta_0^{-2} \\ \vdots \\ c_0 \theta_0^{-d} \end{pmatrix}$$

with respect to basis $v_0, v_1, \dots, v_d$ by lemma 2.3. Namely ,

$$w_0 = c_0 v_0 + c_0 \theta_0^{-1} v_1 + c_0 \theta_0^{-2} v_2 + \dots + c_0 \theta_0^{-d} v_d. \tag{2.5}$$

In the same way, there exists $c_1 \in \mathbb{C}$ such that

$$w_1 = c_1 v_0 + c_1\theta_1^{-1} v_1 + c_1\theta_1^{-2} v_2 + \ldots + c_1\theta_1^{-d} v_d. \tag{2.6}$$

By (2.5),

$$\begin{aligned} Bw_0 &= c_0 B v_0 + c_0\theta_0^{-1} B v_1 + c_0\theta_0^{-2} B v_2 + \ldots + c_0\theta_0^{-d} B v_d && (2.7)\\ &= c_0 b_0 v_0 + c_0\theta_0^{-1} b_1 v_1 + c_0\theta_0^{-2} b_2 v_2 + \ldots + c_0\theta_0^{-d} b_d v_d. && (2.8)\end{aligned}$$

Compare coefficients in (2.6) and (2.8), since $Bw_0 = w_1$, we get

$$\begin{aligned} b_0 &= \frac{c_1}{c_0},\\ b_1 &= \frac{c_1}{c_0}\frac{\theta_0}{\theta_1},\\ b_2 &= \frac{c_1}{c_0}(\frac{\theta_0}{\theta_1})^2,\\ &\vdots\\ b_d &= \frac{c_1}{c_0}(\frac{\theta_0}{\theta_1})^d. \end{aligned}$$

Note that $b_0, b_1, \ldots, b_d$ is a geometric sequence with common ratio $q = \theta_0/\theta_1$. Hence $b_j = \xi q^j$ for $i = 1, 2, \ldots, d$ with $\xi = b_0$. Observe $q^{d+1} = \theta_0^{d+1}/\theta_1^{d+1} = 1$ by lemma 2.1. Further, $q^i \neq q^j$ for $1 \leq i, j \leq d$, otherwise $b_i = b_j$, a contradiction to lemma 2.1. It implies that q is a primitive root of unity of order $d+1$. Therefore, for the basis $\{v_0, v_1, \ldots, v_d\}$, the matrices representing A and B are as follows.

$$A: \begin{pmatrix} 0 & & & & \alpha \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & \ddots & \ddots & \\ 0 & & & 1 & 0 \end{pmatrix}, B: \begin{pmatrix} \xi & & & & 0 \\ & \xi q & & & \\ & & \xi q^2 & & \\ & & & \ddots & \\ 0 & & & & \xi q^d \end{pmatrix}.$$

□

3 Proof of Theorem 1.3

Proof. ($(ii) \Longrightarrow (i)$) It suffices to show that the condition (i) in Definition 1.2 is true, since (ii) and (iii) can be obtained similarly. Consider that $A^{d+1} = \alpha I,\ B^{d+1} = \beta I,\ BA = qAB$. According to Theorem 2.4, let v be an eigenvector of B corresponding to eigenvalue ξ and form a basis $\{v,\ Av,\ A^2v,\ \ldots,\ A^dv\}$ for $\mathbf{V}$ such that the matrix representing A (resp. B) is left-cyclic (resp. diagonal) as follows

$$A: \begin{pmatrix} 0 & & & & \alpha \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & \ddots & \ddots & \\ 0 & & & 1 & 0 \end{pmatrix},\ B: \begin{pmatrix} \xi & & & & 0 \\ & \xi q & & & \\ & & \xi q^2 & & \\ & & & \ddots & \\ 0 & & & & \xi q^d \end{pmatrix}.$$

Similarly, let $v,\ Cv,\ C^2v,\ \ldots,\ C^dv$ form another basis for $\mathbf{V}$ such that the matrices representing C (resp. B) is left-cyclic (resp. diagonal) as follows

$$C : \begin{pmatrix} 0 & & & & \gamma \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & \ddots & \ddots & \\ 0 & & & 1 & 0 \end{pmatrix}, \ B : \begin{pmatrix} \xi & & & & 0 \\ & \xi q^{-1} & & & \\ & & \xi q^{-2} & & \\ & & & \ddots & \\ 0 & & & & \xi q^{-d} \end{pmatrix},$$

since $B^{d+1} = \beta I, C^{d+1} = \gamma I, CB = qBC$, namely, $BC = q^{-1}CB$. Observe

$$\xi q^i = \xi q^{-(d+1-i)} \quad (1 \leq i \leq d).$$

We know that A^iv and $C^{d+1-i}v$ are the same eigenvector of B corresponding eigenvalue ξq^i. Hence

$$A^iv = c_{d+1-i}C^{d+1-i}v \quad (1 \leq i \leq d),$$

where c_i is nonzero complex number. Note that the basis

$$\{v,\ Av,\ \ldots,\ A^iv,\ \ldots,\ A^dv\}$$

is regarded as

$$\{v,\ c_dC^dv,\ \ldots,\ c_{d+1-i}C^{d+1-i}v,\ \ldots,\ c_1Cv\}.$$

Hence for the basis $\{v,\ Av,\ A^2v,\ \ldots,\ A^dv\}$, the matrix representing C is right-cyclic as follows

$$C \ : \ \begin{pmatrix} 0 & c_d\gamma & & & 0 \\ & 0 & c_{d-1}c_d^{-1} & & \\ & & 0 & \ddots & \\ & & & \ddots & c_1c_2^{-1} \\ c_1^{-1} & & & & 0 \end{pmatrix}.$$

Now we find the basis $\{v,\ Av,\ A^2v,\ \ldots,\ A^dv\}$ such that the matrices representing A (resp. B, C) is left cyclic (resp. diagonal, right-cyclic) .

Hence (A, B, C) is a cyclic triple.

($(i) \Longrightarrow (ii)$) By Theorem 2.4, it is obvious that there exists three nonzero complex numbers α, β and γ such that $A^{d+1} = \alpha I,\ B^{d+1} = \beta I$, and $C^{d+1} = \gamma I$. By the condition (i) in Definition 1.2, there exists a basis $\{u_0,\ u_1,\ \ldots,\ u_d\}$ such that the matrices representing A (resp. B, C) is left-cyclic (resp. diagonal, right-cyclic) as follows

$$A \quad : \quad \begin{pmatrix} 0 & & & & a_0 \\ a_1 & 0 & & & \\ & a_2 & \ddots & & \\ & & \ddots & 0 & \\ 0 & & & a_d & 0 \end{pmatrix},$$

$$(resp.\ B \quad : \quad \begin{pmatrix} b_0 & & & & 0 \\ & b_1 & & & \\ & & b_2 & & \\ & & & \ddots & \\ 0 & & & & b_d \end{pmatrix},$$

$$C \quad : \quad \begin{pmatrix} 0 & c_1 & & & 0 \\ & 0 & c_2 & & \\ & & 0 & \ddots & \\ & & & \ddots & c_d \\ c_0 & & & & 0 \end{pmatrix}).$$

Set

$$v_0 = u_0 \text{ and } v_i = a_1 a_2 \dots a_i u_i \text{ for } i = 1, 2, \dots, d.$$

For the basis $\{v_0, v_1, \dots, v_d\}$, the matrix representing C (resp. B, A) is right-cyclic (resp. left-cyclic, diagonal) as

$$C : \begin{pmatrix} 0 & x_1 & & & 0 \\ & 0 & x_2 & & \\ & & 0 & \ddots & \\ & & & \ddots & x_d \\ x_0 & & & & 0 \end{pmatrix},$$

(resp.

$$A : \begin{pmatrix} 0 & & & & \alpha \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & \ddots & \ddots & \\ 0 & & & 1 & 0 \end{pmatrix}, \quad B : \begin{pmatrix} \xi & & & & 0 \\ & \xi q & & & \\ & & \xi q^2 & & \\ & & & \ddots & \\ 0 & & & & \xi q^d \end{pmatrix}),$$

with $\alpha = a_0 a_1 \dots a_d$, $\xi \neq 0$, and q is a primitive root of unity of order $d+1$. We know that $BA = qAB$, and by direct computation

$$CB : \begin{pmatrix} 0 & x_1 \xi q & & & 0 \\ & 0 & x_2 \xi q^2 & & \\ & & 0 & \ddots & \\ & & & \ddots & x_d \xi q^d \\ x_0 \xi & & & & 0 \end{pmatrix},$$

$$BC : \begin{pmatrix} 0 & x_1\xi & & & 0 \\ & 0 & x_2\xi q & & \\ & & 0 & \ddots & \\ & & & \ddots & x_d\xi q^{d-1} \\ x_0\xi q^d & & & & 0 \end{pmatrix}.$$

Hence we have

$$BA = qAB, \ CB = qBC. \tag{3.1}$$

Similarly, by condition (ii) in Definition 1.2 we have

$$AC = q'CA, CB = q'BC, \tag{3.2}$$

where q' is a primitive root of unity of order $d+1$. By (3.1) and (3.2), $CB = qBC = q'BC$. It implies $q = q'$, so that $BA = qAB, CB = qBC, AC = qCA$.

($(iii) \implies (ii)$) By direct computation, $A^{d+1} = \alpha I, B^{d+1} = \beta I, C^{d+1} = \gamma I$, where $\alpha = \eta^{d+1}, \beta = \xi^{d+1}, \gamma = \zeta^{d+1}$, and then

$$BA : \eta\xi \begin{pmatrix} 0 & & & & 1 \\ q^{-1} & 0 & & & \\ & q^{-2} & \ddots & & \\ & & \ddots & 0 & \\ 0 & & & q^{-d} & 0 \end{pmatrix}, \quad AB : \eta\xi \begin{pmatrix} 0 & & & & q^{-1} \\ q^{-2} & 0 & & & \\ & q^{-3} & \ddots & & \\ & & \ddots & 0 & \\ 0 & & & q^{-(d+1)} & 0 \end{pmatrix},$$

$$CB : \zeta\xi \begin{pmatrix} 0 & q^2 & & & 0 \\ & 0 & q^4 & & \\ & & \ddots & \ddots & \\ & & & 0 & q^{2d} \\ 1 & & & & 0 \end{pmatrix}, \quad BC : \zeta\xi \begin{pmatrix} 0 & q & & & 0 \\ & 0 & q^3 & & \\ & & \ddots & \ddots & \\ & & & 0 & q^{2d-1} \\ q^d & & & & 0 \end{pmatrix},$$

$$AC : \eta\zeta \begin{pmatrix} 1 & & & & 0 \\ & q^{-1} & & & \\ & & \ddots & & \\ & & & q^{-d+1} & \\ 0 & & & & q^{-d} \end{pmatrix}, \quad CA : \eta\zeta \begin{pmatrix} q^{-1} & & & & 0 \\ & q^{-2} & & & \\ & & \ddots & & \\ & & & q^{-d} & \\ 0 & & & & 1 \end{pmatrix}.$$

Hence $BA = qAB, CB = qBC, AC = qCA$.

((i) and $(ii) \implies (iii)$) Let v be the eigenvector of B with corresponding eigenvalue ξ, and let η be an eigenvalue of A. Then for the basis $v, \eta^{-1}q^2Av, \eta^{-2}q^{2+4}A^2v, \ldots, \eta^{-d}q^{2+4+\ldots+2d}A^dv$, where q is the primitive root of unity of order $d+1$ that satisfies (ii), the matrices representing A (resp. B) is left-cyclic (resp. diagonal) as follows

$$A : \eta \begin{pmatrix} 0 & & & & 1 \\ q^{-2} & 0 & & & \\ & q^{-4} & \ddots & & \\ & & \ddots & 0 & \\ 0 & & & q^{-2d} & 0 \end{pmatrix} \left(rep.\ B : \xi \begin{pmatrix} 1 & & & & 0 \\ & q & & & \\ & & \ddots & & \\ & & & q^{d-1} & \\ 0 & & & & q^d \end{pmatrix}\right),$$

and the matrix representing C is right-cyclic as

$$C\ :\ \begin{pmatrix} 0 & c_1 & & & 0 \\ & 0 & c_2 & & \\ & & 0 & \ddots & \\ & & & \ddots & c_d \\ c_0 & & & & 0 \end{pmatrix}.$$

Hence

$$AC\ :\ \begin{pmatrix} c_0 & & & & 0 \\ & q^{-2}c_1 & & & \\ & & q^{-4}c_2 & & \\ & & & \ddots & \\ 0 & & & & q^{-2d}c_d \end{pmatrix},\ CA\ :\ \begin{pmatrix} q^{-2}c_1 & & & & 0 \\ & q^{-4}c_2 & & & \\ & & \ddots & & \\ & & & q^{-2d}c_d & \\ 0 & & & & c_0 \end{pmatrix}.$$

We find $c_{i+1} = qc_i$ for $i = 0, 1, \ldots, d-1$ and $c_0 = qc_d$, since $AC = qCA$. Hence the matrix representing C is as follows

$$C\ :\ \begin{pmatrix} 0 & qc_0 & & & 0 \\ & 0 & q^2c_0 & & \\ & & 0 & \ddots & \\ & & & \ddots & q^dc_0 \\ c_0 & & & & 0 \end{pmatrix} = \zeta \begin{pmatrix} 0 & q & & & 0 \\ & 0 & q^2 & & \\ & & \ddots & \ddots & \\ & & & 0 & q^d \\ 1 & & & & 0 \end{pmatrix},$$

where $\zeta = c_0$. The proof is completed. □

4 Remarks

The study of a pair or a triple of linear transformations with specified combinatorial properties was first appeared in [4] with the motivation from the study of P- and Q-polynomial schemes. Also see [5] for a survey on this topic. These are related to the representation theory of some algebra defined from relations. See [2] for reference. To finish the thesis we propose the following conjecture.

Conjecture 4.1. Let $\mathbf{V}$ denote a vector space over $\mathbb{C}$ with dimension $d+1$. Let $A: \mathbf{V} \longrightarrow \mathbf{V}$, $B: \mathbf{V} \longrightarrow \mathbf{V}$, and $C: \mathbf{V} \longrightarrow \mathbf{V}$ denote linear transformations. The following (i) and (ii) are equivalent.

(i) (A, B), (B, C), (C, A) are cyclic pairs.

(ii) (A, B, C) is a cyclic triple.

References

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[4] P. Terwilliger, Two linear transformations each tridiagonal with respect to an eigenbasis of the other, *Linear Algebra Appl.*, 330, 149-203, 1999.

[5] P. Terwilliger, Introduction to Leonard pairs, OPSFA Rome 2001, *J. Comput. Appl. Math.*, 153(2), 463-475, 2003.