國 立 交 通 大 學

應用數學系

博 士 論 文

無三角形且含五邊形之距離正則圖

Triangle-free Distance-regular Graphs with Pentagons

博 士 生：潘業忠

指導教授：翁志文 教授

中 華 民 國 九 十 七 年 六 月

無三角形且含五邊形之距離正則圖

Triangle-free Distance-regular Graphs with Pentagons

博 士 生：潘業忠　　Student : Yeh-Jong Pan

指導教授：翁志文　　Advisor : Chih-Wen Weng

國 立 交 通 大 學

應用數學系

博 士 論 文

A Thesis

Submitted to Department of Applied Mathematics

College of Science

National Chiao Tung University

In Partial Fulfillment of the Requirements

For the Degree of Doctor of Philosophy

In Applied Mathematics

June 2008

Hsinchu, Taiwan, Republic of China

中 華 民 國 九 十 七 年 六 月

誌　謝

在職進修原本就是一件辛苦的事，對我而言，這條路尤其漫長而艱辛，別的不說，光這幾年所走過的路，豈止萬里而已，估計大約可以環繞台灣 200 圈了，現在總算可以稍事休息，待養精蓄銳後，再往下一個目標前進。

取得博士學位是我現階段的目標，而完成博士論文是這個階段的終點，但這卻只是通往研究之路的起點，後面還有很長的一段路要走，回首這些日子的點點滴滴，對於週遭曾幫助我的人，心中始終存著感激，僅以此文表達我誠摯的謝意。

首先要感謝的是我的指導教授翁志文教授，這幾年來，除了在研究課題上的指引之外，其它如研究方法與研究態度、論文的寫作、投稿及對審稿者的回應等方面，亦多所教導，而其間不管是投稿的論文、演講稿或博士論文，每每修改十幾、二十幾次，他總是耐心的和我討論，給我很好的建議，我能夠順利的取得博士學位，完全要歸功於翁老師的耐心指導。此外，我也非常感謝 **Terwilliger** 教授，他提供了很多研究上的寶貴意見，讓我獲益良多。

其次要感謝的是黃大原教授，他是我讀碩士班時的指導教授。當初，我跟他說要來報考博士班，他就自行影印歷屆考古題寄給我，並針對我的讀書計劃給我很好的意見，甚至後來推薦我去找翁老師當指導教授，這些都對我有很大的幫助，老師對我的支持與愛護，令我銘記於心。

傅恆霖教授一直是相當受學生歡迎的老師，從早期便是我非常敬佩與學習的對象，我在教書的過程中，不管是當導師或者和學生互動，很多是受傅老師身教的影響，雖然他不是我的指導教授，但卻是人生的導師，能夠兩度受教，真是幸運。陳秋媛教授是很棒的老師，雖然我只旁聽過一學期的課，但從她課前的準備、上課的認真詳細及對學生的關心，不難看出她也很受歡迎，而從這門課當中，我除了獲得演算法的知識以外，也學到一些教學經驗及得到一些啟發。

此外，還要感謝很多幫助過我的朋友，例如 **DavidGuo**、飛黃、賓賓、貴弘、嘉文、**Taller**、小培、**Robin**、亮銓、維展、明欣、國元、經凱、泰峰、惠蘭、祐寧、貓頭等，可能還有一些遺漏掉的朋友，在此一併致謝。

最後，要謝謝我的母親、家人及親友的支持與鼓勵，感謝大家。

無三角形且含五邊形之距離正則圖

博士生：潘業忠　　　　　　指導教授：翁志文

國 立 交 通 大 學

應 用 數 學 系

摘　　要

考慮一個具有 Q-多項式性質的距離正則圖 Γ，假設 Γ 的直徑 D 至少為 3 且其相交參數 $a_1=0$ 且 $a_2\neq 0$，我們將證明下列(i)-(iii)是等價的：

(i) Γ 具有 Q-多項式性質且不含長度為 3 的平行四邊形。

(ii) Γ 具有 Q-多項式性質且不含任何長度為 i 的平行四邊形，其中 $3\leq i\leq D$。

(iii) Γ 具有古典參數(D, b, α, β)，其中 b, α, β 是實數，且 $b<-1$。

而當條件(i)-(iii) 成立時，我們證得 Γ 具有 3-bounded 性質。利用這個性質，我們可以證明其相交參數 c_2 等於 1 或 2；且如果 $c_2=1$，則 $(b, \alpha, \beta) = (-2, -2, \frac{(-2)^{D+1}-1}{3})$。

Triangle-free Distance-regular Graphs with Pentagons

Student : Yeh-Jong Pan

Department of Applied Mathematics
National Chiao Tung University
Hsinchu, Taiwan

Advisor : Chih-Wen Weng

Department of Applied Mathematics
National Chiao Tung University
Hsinchu, Taiwan

To Jean, Peggy, and Penny.

Abstract

Let Γ denote a distance-regular graph with Q-polynomial property. Assume the diameter D of Γ is at least 3 and the intersection numbers $a_1 = 0$ and $a_2 \neq 0$. We show the following (i)-(iii) are equivalent.

(i) Γ is Q-polynomial and contains no parallelograms of length 3.

(ii) Γ is Q-polynomial and contains no parallelograms of any length i for $3 \leq i \leq D$.

(iii) Γ has classical parameters (D, b, α, β) for some real constants b, α, β with $b < -1$.

When (i)-(iii) hold, we show that Γ has 3-bounded property. Using this property we prove that the intersection number c_2 is either 1 or 2, and if $c_2 = 1$ then $(b, \alpha, \beta) = (-2, -2, ((-2)^{D+1} - 1)/3)$.

Table of Contents

Chapter 1

Introduction

Distance-regular graphs were introduced by Biggs as a combinatorial generalization of distance-transitive graphs in 1970. They became a popular topic after that Desarte studied P-polynomial schemes [5], which are exactly the distance-regular graphs, motivated by problems of coding theory in his thesis. After that, Leonard proved that the dual eigenvalues of a Q-polynomial distance-regular graph satisfy a recurrence relation and derived explicit formulae of the intersection numbers [12]. With these formulae it sheds light on the classification of Q-polynomial distance-regular graphs, as also stated in the book of Eiichi Bannai and Tatsuro Ito on Algebraic Combinatorics I : Association Schemes [1].

Brouwer, Cohen, and Neumaier found that the intersection numbers of most known families of distance-regular graphs could be described in terms of four parameters (D, b, α, β) [3, p. ix, p193]. They invented the term *classical* to describe such graphs. The class of distance-regular graphs which have classical parameters is a special case of distance-regular graphs with the Q-polynomial property [3, Corollary 8.4.2]. Note that the converse is not true, since an ordinary n-gon has the Q-polynomial property, but does not have classical parameters [3, Table 6.6]. Many authors proved the converse under various additional assumptions. Let Γ denote a distance-regular graph with diameter $D \geq 3$ (See Chapter 2 for formal definitions.). Indeed assume Γ is Q-polynomial. Then Brouwer, Cohen, Neumaier in [3, Theorem 8.5.1] show that if Γ is a near polygon, with the intersection number $a_1 \neq 0$, then Γ has classical parameters. Weng generalizes this result with a weaker assumption, without kites of length 2 or 3 in Γ, to replace the near polygon assumption [23, Lemma 2.4]. For the complement case

$a_1 = 0$, Weng shows that Γ has classical parameters if (i) Γ contains no parallelograms of length 3 and no parallelograms of length 4; (ii) Γ has the intersection number $a_2 \neq 0$; and (iii) Γ has diameter $d \geq 4$ [25, Theorem 2.11]. We improve the above result by showing Theorem 3.2.1 in chapter 3.

Many authors study distance-regular graph Γ with $a_1 = 0$ and other additional assumptions. For example, Miklavič assumes Γ is Q-polynomial and shows Γ is 1-homogeneous [13]; Koolen and Moulton assume Γ has degree 8, 9 or 10 and show that there are finitely many such graphs [11]; Jurišić, Koolen and Miklavič assume Γ has an eigenvalue with multiplicity equal to the valency, $a_2 \neq 0$, and the diameter $d \geq 4$ to show $a_4 = 0$ and Γ is 1-homogeneous [10].

In this thesis we aim at distance-regular graphs which have classical parameters (D, b, α, β) and intersection numbers $a_1 = 0$ and $a_2 \neq 0$. Since $b < -1$ [14], our work is a part of the classification of classical distance-regular graphs of negative type [27]. It worths to mention that all classical distance-regular graphs with $b = 1$ are classified by Y. Egawa, A. Neumaier and P. Terwilliger independently (See [3, p195] for details). Let Γ be a distance-regular graph which has classical parameters (D, b, α, β) and $a_1 = 0$, $a_2 \neq 0$, and $D \geq 3$. It was previously known that Γ has 2-bounded property [26, 19]. By applying this to a strongly regular subgraph of Γ, we find an upper bound of c_2 in terms of an expression of b in chapter 4. After that we prove the 3-bounded property of Γ in chapter 5. Finally we use the 3-bounded property to conclude that $c_2 = 1$ or 2.

The following preprints and papers are included in this thesis:

1. Y. Pan, M. Lu, and C. Weng, Triangle-free distance-regular graphs, *J. Algebr. Comb.*, 27(2008), 23-34.

2. Y. Pan and C. Weng, 3-bounded Property in a Triangle-free Distance-regular Graph, *European Journal of Combinatorics*, 29(2008), 1634-1642.

3. Y. Pan and C. Weng, A note on triangle-free distance-regular graphs with $a_2 \neq 0$, preprint (2007), submitted to *Journal of Combinatorial Theory, Series B*.

This thesis is organized as follows.

In Chapter 2 we introduce definitions, terminologies and some results concerning distance-regular graphs and block designs.

In Chapter 3 we discuss a combinatorial property of distance-regular graphs which have classical parameters.

In Chapter 4 we work on distance-regular graphs with classical parameters and use the multiplicity technique to find an upper bound of c_2.

In Chapter 5 we prove the 3-bounded property of the distance-regular graphs.

In Chapter 6 we use the 3-bounded property and Fisher's inequality to show the upper bound $c_2 \leq 2$ of c_2. This upper bound rules out almost all the graphs of our target in the classification. Also we find that if $c_2 = 1$, then $(b, \alpha, \beta) = (-2, -2, \frac{(-2)^{D+1}-1}{3})$.

Chapter 2

Preliminaries

In this chapter we review some definitions, basic concepts and some previous results concerning distance-regular graphs and block designs. See Bannai and Ito [1] or Terwilliger [20] for more background information of distance-regular graphs and van Lint and Wilson [22] for block designs.

Let Γ=(X, R) denote a finite undirected, connected graph without loops or multiple edges with vertex set X, edge set R, distance function ∂, and diameter D:=max$\{\,\partial(x, y) \mid x, y \in X\}$. By a *pentagon*, we mean a 5-tuple $x_1x_2x_3x_4x_5$ consisting of vertices of Γ such that $\partial(x_i, x_{i+1}) = 1$ for $1 \leq i \leq 4$, $\partial(x_5, x_1) = 1$ and no other edges between two distinct vertices.

For a vertex $x \in X$ and an integer $0 \leq i \leq D$, set $\Gamma_i(x) := \{\, z \in X \mid \partial(x, z) = i\}$. The *valency* $k(x)$ of a vertex $x \in X$ is the cardinality of $\Gamma_1(x)$. The graph Γ is called *regular* (with *valency* k) if each vertex in X has valency k.

An *incidence structure* is a triple $(\mathbf{P}, \mathfrak{B}, \mathbb{I})$, where $\mathbf{P}$ and $\mathfrak{B}$ are two sets and $\mathbb{I} \subseteq \mathbf{P} \times \mathfrak{B}$. The elements of $\mathbf{P}$ and $\mathfrak{B}$ are called *points* and *blocks* respectively. If $(p, B)$$\in \mathbb{I}$, then we say point p and block B are incident.

A t-(v, κ, λ) design is an incidence structure $(\mathbf{P}, \mathfrak{B}, \mathbb{I})$, where $|\mathbf{P}| = v$, satisfying the following conditions:

- For each block $B \in \mathfrak{B}$, there are exactly κ points incident with B.

- For two distinct blocks B and B', there exists a point p incident with B, but p

is not incident with B'.

- For any set T of t points, there are exactly λ blocks incident with all points of T.

It is easy to prove that the number of blocks incident with any fixed point p of $\mathbf{P}$ is the same [22, Theorem 19.3] and is called the *replication number* of the design. Actually the number is $\lambda\binom{v-1}{t-1}/\binom{k-1}{t-1}$.

2.1 Distance-regular Graphs

A graph $\Gamma = (X, R)$ is said to be *distance-regular* whenever for all integers $0 \leq h, i, j \leq D$, and all vertices $x, y \in X$ with $\partial(x, y) = h$, the number

$$p^h_{ij} = |\Gamma_i(x) \cap \Gamma_j(y)|$$

is independent of x, y. The constants p^h_{ij} are known as the *intersection numbers* of Γ.

Let Γ=(X, R) be a distance-regular graph. For two vertices $x, y \in X$ with $\partial(x, y) = i$, set

$$\begin{aligned} B(x, y) &:= \Gamma_1(x) \cap \Gamma_{i+1}(y), \\ C(x, y) &:= \Gamma_1(x) \cap \Gamma_{i-1}(y), \\ A(x, y) &:= \Gamma_1(x) \cap \Gamma_i(y). \end{aligned}$$

Note that

$$\begin{aligned} |B(x, y)| &= p^i_{1\,i+1}, \\ |C(x, y)| &= p^i_{1\,i-1}, \\ |A(x, y)| &= p^i_{1\,i} \end{aligned}$$

are independent of x, y.

For convenience, set $c_i := p^i_{1\,i-1}$ for $1 \leq i \leq D$, $a_i := p^i_{1\,i}$ for $0 \leq i \leq D$, $b_i := p^i_{1\,i+1}$ for $0 \leq i \leq D-1$, $k_i := p^0_{i\,i}$ for $0 \leq i \leq D$, and set $b_D := 0$, $c_0 := 0$, $k := b_0$. Note that k is the valency of Γ. It follows immediately from the definition of p^h_{ij} that $b_i \neq 0$ for $0 \leq i \leq D-1$ and $c_i \neq 0$ for $1 \leq i \leq D$. Moreover

$$k = a_i + b_i + c_i \quad \text{for} \quad 0 \leq i \leq D, \tag{2.1.1}$$

and

$$k_i = \frac{b_0 \cdots b_{i-1}}{c_1 \cdots c_i} \quad \text{for} \quad 1 \leq i \leq D. \tag{2.1.2}$$

A *strongly regular graph* is a distance-regular graph with diameter 2. We quote a couple of Lemmas about strongly regular graphs which will be used in Chapter 4 and Chapter 6.

Lemma 2.1.1. *[22, Theorem 21.1] Suppose Ω is a strongly regular graph with intersection numbers a_i, b_i, c_i, where $0 \leq i \leq 2$. Let $v = |\Omega|$ and $k = b_0$. Suppose that $r \geq s$ are the eigenvalues other than k. Let f and g be the multiplicities of r and s respectively. Then*

$$f = \frac{1}{2}(v - 1 + \frac{(v-1)(c_2 - a_1) - 2k}{\sqrt{(c_2 - a_1)^2 + 4(k - c_2)}}) \tag{2.1.3}$$

and

$$g = \frac{1}{2}(v - 1 - \frac{(v-1)(c_2 - a_1) - 2k}{\sqrt{(c_2 - a_1)^2 + 4(k - c_2)}}) \tag{2.1.4}$$

are nonnegative integers.

Proof. Let A be the adjacency matrix of Ω, J be the v by v all-one matrix, and j be the v by 1 all-one vector. We have $AJ = kJ$, $Aj = kj$, and $A^2 = kI + a_1A + c_2(J - I - A)$ by direct computation. Note that k is an eigenvalue of A with eigenvector j whose multiplicity is one since Ω is connected. Suppose that x is an eigenvalue with eigenvector orthogonal to j. Then

$$x^2 + (c_2 - a_1)x + (c_2 - k) = 0. \tag{2.1.5}$$

Equation (2.1.5) has two solutions

$$r, s = \frac{1}{2}(a_1 - c_2 \pm \sqrt{(a_1 - c_2)^2 + 4(k - c_2)}\ ). \tag{2.1.6}$$

Since f and g are multiplicities of r and s respectively, we have the following two equations.

$$1 + f + g = v \tag{2.1.7}$$

and

$$0 = tr(A) = k + fr + gs. \tag{2.1.8}$$

Solving (2.1.7) and (2.1.8) for f, g by (2.1.6), we have (2.1.3) and (2.1.4). It is obvious that f and g are nonnegative integers. □

Lemma 2.1.2. *[2, p. 276, Theorem 19] Let Ω be a strongly regular graph with valency $b_0 = k$, $a_1 = 0$, and $c_2 = 1$. Then $k \in \{2, 3, 7, 57\}$.* □

Proof. Note that $c_1 = 1$ and $b_1 = k - a_1 - c_1 = k - 1$. Then $v := |\Omega| = 1 + k_1 + k_2 = 1 + k^2$. Substituting v, c_2 and a_1 into (2.1.3) we have

$$f = \frac{1}{2}\left(k^2 + \frac{k^2 - 2k}{\sqrt{4k-3}} \right). \tag{2.1.9}$$

Equation (2.1.9) implies $k^2 - 2k = 0$ or $4k - 3 = s^2$ for some integer s since f is a nonnegative integer. If $k^2 - 2k = 0$ then $k = 2$. Suppose $4k - 3 = s^2$, then

$$k = \frac{s^2 + 3}{4}. \tag{2.1.10}$$

Substituting (2.1.10) into (2.1.9) yields

$$s^5 + s^4 + 6s^3 - 2s^2 + (9 - 32f)s = 15. \tag{2.1.11}$$

Hence s is a factor of 15. The result follows from substituting s into k and deleting the case $k = 1$. □

Example 2.1.3. The Petersen graph shown in Figure 2.1 is a strongly regular graph with intersection numbers $a_1 = 0$, $a_2 = 2$, $c_1 = c_2 = 1$, $b_0 = 3$, $b_1 = 2$.

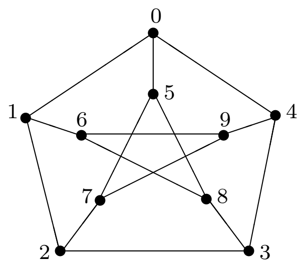

Figure 2.1: Petersen graph.

Example 2.1.4. [3, p. 285](Hermitian forms graph $Her_2(D)$) Let U denote a finite vector space of dimension D over the field $GF(4)$. Let H denote the D^2-dimensional vector space over $GF(2)$ consisting of the Hermitian forms on U. Thus $f \in H$ if and only if $f(u, v)$ is linear in v, and $f(v, u) = \overline{f(u, v)}$ for all $u, v \in U$. Pick $f \in H$. We define

$$\mathrm{rk}(f) = \dim(U \setminus \mathrm{Rad}(f)),$$

where

$$\mathrm{Rad}(f) = \{u \in U \mid f(u, v) = 0 \text{ for all } v \in U\}.$$

Set $X = H$, and $xy \in R$ if and only if $\mathrm{rk}(x - y) = 1$ for all $x, y \in X$. Then $\Gamma = (X, R)$ is a distance-regular graph with diameter D and intersection numbers

$$c_i = \frac{2^{i-1}(2^i - (-1)^i)}{3} \qquad (1 \le i \le D), \tag{2.1.12}$$

$$b_i = \frac{2^{2D} - 2^{2i}}{3} \qquad (0 \le i \le D). \tag{2.1.13}$$

By (2.1.1), (2.1.12) and (2.1.13) we have

$$a_i = \frac{2^{2i-1} + (-1)^i 2^{i-1} - 1}{3} \qquad (1 \le i \le D). \tag{2.1.14}$$

Note that $a_1 = 0$ and $a_2 = 3$. It was shown in [9] that Γ is the unique distance-regular graph with intersection numbers satisfying (2.1.12) and (2.1.13).

Example 2.1.5. [3, p. 372](Gewirtz graph) Suppose $(\mathbf{P}, \mathfrak{B}, \mathbb{I})$ is a 3-$(22, 6, 1)$ design, where $\mathbb{I} = \{(p, B) \mid p \in \mathbf{P}, B \in \mathfrak{B}, \text{ and } p \in B\}$. Fix an element p of $\mathbf{P}$. Let $X = \{B \in \mathfrak{B} \mid p \notin B\}$ and $R = \{B_1B_2 \mid B_1, B_2 \in X \text{ and } B_1 \cap B_2 = \emptyset\}$. Then $\Gamma = (X, R)$ is a distance-regular graph which is known as *Gewirtz* graph. It is a strongly regular graph with intersection numbers $a_1 = 0$, $a_2 = 8$, $c_1 = 1$, $c_2 = 2$, $b_0 = 10$, and $b_1 = 9$. It was shown in [6] and [7] that Γ is the unique strongly regular graph with intersection numbers satisfying $b_0 = 10$, $b_1 = 9$, $c_1 = 1$, and $c_2 = 2$.

Example 2.1.6. [3, Theorem 11.4.2](Witt graph M_{23}) Suppose $(\mathbf{P}, \mathfrak{B}, \mathbb{I})$ is a 5-$(24, 8, 1)$ design where $\mathbb{I} = \{(p, B) \mid p \in \mathbf{P}, B \in \mathfrak{B}, \text{ and } p \in B\}$. Fix a point $\sigma \in \mathbf{P}$, and let $\mathfrak{B}'$ be the collection of 506 blocks in $\mathfrak{B}$ missing σ. Then $(\mathbf{P} \setminus \{\sigma\}, \mathfrak{B}')$ is a 4-$(23, 8, 4)$ design. Let $X = \mathfrak{B}'$ and $R = \{B_1B_2 \mid B_1 \cap B_2 = \emptyset \text{ for distinct } B_1, B_2 \in X\}$. Then $\Gamma = (X, R)$ is a distance-regular graph which is known as *Witt* graph M_{23}. It has diameter $D = 3$ and intersection numbers $a_1 = 0$, $a_2 = 2$, $a_3 = 6$, $c_1 = c_2 = 1$, $c_3 = 9$, $b_0 = 15$, $b_1 = 14$

and $b_2 = 12$. It was shown in [3, Theorem 11.4.2] that Γ is the unique distance-regular graph of diameter 3 with intersection numbers satisfying $b_0 = 15$, $b_1 = 14$, $b_2 = 12$, $c_1 = c_2 = 1$, and $c_3 = 9$.

Throughout this chapter we assume $\Gamma=(X,\ R)$ is a distance-regular graph.

Definition 2.1.7. Pick an integer $2 \le i \le D$. By a *parallelogram* of length i in Γ, we mean a 4-tuple $xyzw$ of vertices of X such that

$$\partial(x, y) = \partial(z, w) = 1, \quad \partial(x, z) = i,$$

$$\partial(x, w) = \partial(y, w) = \partial(y, z) = i - 1.$$

For a parallelogram of length i, see Figure 2.2.

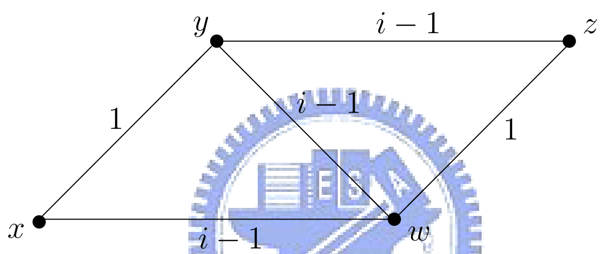

Figure 2.2: A parallelogram of length i.

2.2 D-bounded Distance-regular Graphs

Assume $\Gamma = (X, R)$ is distance-regular with diameter $D \ge 3$. Recall that a sequence x, y, z of vertices of Γ is *geodetic* whenever

$$\partial(x, y) + \partial(y, z) = \partial(x, z).$$

Definition 2.2.1. A sequence x, y, z of vertices of Γ is *weak-geodetic* whenever

$$\partial(x, y) + \partial(y, z) \le \partial(x, z) + 1.$$

Definition 2.2.2. A subset $\Omega \subseteq X$ is *weak-geodetically closed* if for any weak-geodetic sequence x, y, z of Γ,

$$x,\ z \in \Omega \Longrightarrow y \in \Omega.$$

Weak-geodetically closed subgraphs are called *strongly closed subgraphs* in [18]. We refer the readers to [17, 4, 9, 19, 26, 8] for information on weak-geodetically closed subgraphs.

We make one more definition which will be used later.

Definition 2.2.3. Let Ω be a subset of X, and pick any vertex $x \in \Omega$. Ω is said to be *weak-geodetically closed with respect to* x, whenever for all $z \in \Omega$ and for all $y \in X$,

$$x, y, z \text{ are weak-geodetic} \quad \Longrightarrow \quad y \in \Omega. \tag{2.2.1}$$

Note that Ω is weak-geodetically closed with respect to a vertex $x \in \Omega$ if and only if

$$C(z,x) \subseteq \Omega \ \text{ and } \ A(z,x) \subseteq \Omega \qquad \text{for all } \ z \in \Omega$$

[26, Lemma 2.3]. Also Ω is weak-geodetically closed if and only if for any vertex $x \in \Omega$, Ω is weak-geodetically closed with respect to x. The following theorems will be used later in this thesis.

Theorem 2.2.4. *[26, Theorem 4.6] Let Γ be a distance-regular graph with diameter $D \geq 3$. Let Ω be a regular subgraph of Γ with valency γ and set $d := \min\{i \mid \gamma \leq c_i + a_i\}$. Then the following (i),(ii) are equivalent.*

(i) Ω is weak-geodetically closed with respect to at least one vertex $x \in \Omega$.

(ii) Ω is weak-geodetically closed with diameter d.

In this case $\gamma = c_d + a_d$.

Suppose (i) and (ii) hold. Then Ω is distance-regular, with diameter d, and intersection numbers

$$c_i(\Omega) = c_i(\Gamma), \tag{2.2.2}$$

$$a_i(\Omega) = a_i(\Gamma) \tag{2.2.3}$$

for $0 \leq i \leq d$.

Lemma 2.2.5. *([19, Lemma 2.6]) Let* Γ *be a distance-regular graph with diameter* 2*, and let* x *be a vertex of* Γ*. Suppose* $a_2 \neq 0$*. Then the subgraph induced on* $\Gamma_2(x)$ *is connected of diameter at most 3.* □

Definition 2.2.6. Γ is said to be *i-bounded* whenever for all $x, y \in X$ with $\partial(x, y) \leq i$, there is a regular weak-geodetically closed subgraph of diameter $\partial(x, y)$ containing x, y.

The properties of D-bounded distance-regular graphs were studied in [24], and these properties were used in the classification of classical distance-regular graphs of negative type [27].

Theorem 2.2.7. *([26, Proposition 6.7],[19, Theorem 1.1]) Let* Γ *be a distance-regular graph with diameter* $D \geq 3$*. Suppose* $a_1 = 0$*,* $a_2 \neq 0$ *and* Γ *contains no parallelograms of length* 3*. Then* Γ *is* 2*-bounded.* □

Theorem 2.2.8. *([26, Lemma 6.9],[19, Lemma 4.1]) Let* Γ *be a distance-regular graph with diameter* $D \geq 3$*. Suppose* $a_1 = 0$*,* $a_2 \neq 0$ *and* Γ *contains no parallelograms of any length. Let* x *be a vertex of* Γ*, and let* Ω *be a weak-geodetically closed subgraph of* Γ *with diameter* 2*. Suppose there exists an integer* i *and a vertex* $u \in \Omega \cap \Gamma_{i-1}(x)$*, and suppose* $\Omega \cap \Gamma_{i+1}(x) \neq \emptyset$*. Then for all* $t \in \Omega$*, we have* $\partial(x, t) = i - 1 + \partial(u, t)$*.* □

Theorem 2.2.9. *([24, Corollary 2.2]) Let* $\Gamma = (X, R)$ *denote a distance-regular graph with diameter* D*. Suppose that* Γ *is* D*-bounded. For two distinct vertices* $x, y \in X$*, there exists a unique regular weak-geodetically closed subgraph* $\Delta(x, y)$ *containing* x *and* y *with diameter* $\partial(x, y)$*. Furthermore,* $\Delta(x, y)$ *is a distance-regular graph.* □

Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter D. Suppose that Γ is D-bounded. For two distinct vertices $x, y \in X$, we use $\Delta(x, y)$ to denote the unique weak-geodetically closed subgraph containing x and y with diameter $\partial(x, y)$.

Theorem 2.2.10. *([24, Lemma 2.6]) Let Γ denote a distance-regular graph with diameter D. Suppose that Γ is D-bounded. Then*

$$b_i > b_{i+1} \quad (0 \leq i \leq D-1). \tag{2.2.4}$$

Proof. For $0 \leq i \leq D-1$, pick x, y with $\partial(x,y) = i+1$. Then $\Delta(x,y)$ is a distance-regular graph with diameter $i+1$ by Theorem 2.2.9. Note that $b_i(\Delta(x,y)) = b_i - b_{i+1} \neq 0$. The result follows immediately. □

2.3 Q-polynomial Property

Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $D \geq 3$. Let $\mathbb{R}$ denote the real number field. Let $\mathrm{Mat}_X(\mathbb{R})$ denote the algebra of all the matrices over $\mathbb{R}$ with the rows and columns indexed by the elements of X. For $0 \leq i \leq D$ let A_i denote the matrix in $\mathrm{Mat}_X(\mathbb{R})$, defined by the rule

$$(A_i)_{xy} = \begin{cases} 1, & \text{if } \partial(x,y) = i; \\ 0, & \text{if } \partial(x,y) \neq i \end{cases} \qquad \text{for } \ x, y \in X.$$

We call A_i the *distance matrices* of Γ. We have

$$A_0 = I, \tag{2.3.1}$$

$$A_0 + A_1 + \cdots + A_D = J \qquad (J = \text{all } 1'\text{s matrix}), \tag{2.3.2}$$

$$A_i^t = A_i \quad \text{for } 0 \leq i \leq D \qquad (A_i^t \text{ means the transpose of } A_i), \tag{2.3.3}$$

$$A_i A_j = \sum_{h=0}^{D} p_{ij}^h A_h \quad \text{for } \ 0 \leq i, j \leq D, \tag{2.3.4}$$

$$A_i A_j = A_j A_i \quad \text{for } \ 0 \leq i, j \leq D. \tag{2.3.5}$$

Let M denote the subspace of $\mathrm{Mat}_X(\mathbb{R})$ spanned by $A_0, A_1, \ldots, A_D$. Then M is a commutative subalgebra of $\mathrm{Mat}_X(\mathbb{R})$, and is known as the *Bose-Mesner algebra* of Γ.

By [3, p. 59, 64], M has a second basis $E_0, E_1, \ldots, E_D$ such that

$$E_0 = |X|^{-1}J, \tag{2.3.6}$$

$$E_iE_j = \delta_{ij}E_i \quad \text{for} \;\; 0 \le i, j \le D, \tag{2.3.7}$$

$$E_0 + E_1 + \cdots + E_D = I, \tag{2.3.8}$$

$$E_i^t = E_i \quad \text{for} \;\; 0 \le i \le D. \tag{2.3.9}$$

The $E_0, E_1, \ldots, E_D$ are known as the *primitive idempotents* of Γ, and E_0 is known as the *trivial* idempotent. Let E denote any primitive idempotent of Γ. Then we have

$$E = |X|^{-1}\sum_{i=0}^{D}\theta_i^* A_i \tag{2.3.10}$$

for some $\theta_0^*, \theta_1^*, \ldots, \theta_D^* \in \mathbb{R}$, called the *dual eigenvalues* associated with E.

Set $V = \mathbb{R}^{|X|}$ (column vectors), and view the coordinates of V as being indexed by X. Then the Bose-Mesner algebra M acts on V by left multiplication. We call V the *standard module* of Γ. For each vertex $x \in X$, set

$$\hat{x} = (0, 0, \ldots, 0, 1, 0, \ldots, 0)^t, \tag{2.3.11}$$

where the 1 is in coordinate x. Also, let $\langle\, , \rangle$ denote the dot product

$$\langle u, v\rangle = u^t v \quad \text{for} \;\; u, v \in V. \tag{2.3.12}$$

Then referring to the primitive idempotent E in (2.3.10), we compute from (2.3.9)-(2.3.12) that for x, $y \in X$,

$$\langle E\hat{x}, E\hat{y}\rangle = |X|^{-1}\theta_i^*, \tag{2.3.13}$$

where $i = \partial(x, y)$.

Let $\circ$ denote the entry-wise multiplication in $\mathrm{Mat}_X(\mathbb{R})$. Then

$$A_i \circ A_j = \delta_{ij}A_i \quad \text{for} \;\; 0 \le i, j \le D,$$

so M is closed under $\circ$. Thus there exists $q_{ij}^k \in \mathbb{R}$ for $0 \le i, j, k \le D$ such that

$$E_i \circ E_j = |X|^{-1}\sum_{k=0}^{D} q_{ij}^k E_k \quad \text{for} \;\; 0 \le i, j \le D.$$

Γ is said to be *Q-polynomial* with respect to the given ordering E_0, E_1,…, E_D of the primitive idempotents, if for all integers $0 \leq h, i, j \leq D$, $q^h_{ij} = 0$ (resp. $q^h_{ij} \neq 0$) whenever one of h, i, j is greater than (resp. equal to) the sum of the other two. Let E denote any primitive idempotent of Γ. Then Γ is said to be Q-polynomial with respect to E whenever there exists an ordering E_0, $E_1 = E$,…, E_D of the primitive idempotents of Γ, with respect to which Γ is Q-polynomial. If Γ is Q-polynomial with respect to E, then the associated dual eigenvalues are distinct [20, p. 384].

The following theorem about the Q-polynomial property will be used in this thesis.

Theorem 2.3.1. *[21, Theorem 3.3] Let Γ be Q-polynomial with respect to a primitive idempotent E, and let $\theta^*_0, \ldots, \theta^*_D$ denote the corresponding dual eigenvalues. Then the following (i), (ii) hold.*

(i) For all integers $1 \leq h \leq D$, $0 \leq i, j \leq D$ and for all $x, y \in X$ such that $\partial(x, y) = h$,

$$\sum_{\substack{z\in X\\ \partial(x,z)=i\\ \partial(y,z)=j}} E\hat{z} - \sum_{\substack{z\in X\\ \partial(x,z)=j\\ \partial(y,z)=i}} E\hat{z} = p^h_{ij}\frac{\theta^*_i - \theta^*_j}{\theta^*_0 - \theta^*_h}(E\hat{x} - E\hat{y}). \tag{2.3.14}$$

(ii) For an integer $3 \leq i \leq D$,

$$\theta^*_{i-2} - \theta^*_{i-1} = \sigma(\theta^*_{i-3} - \theta^*_i) \tag{2.3.15}$$

for an appropriate $\sigma \in \mathbb{R} \setminus \{0\}$. □

2.4 Classical Parameters

A distance-regular graph Γ is said to have *classical parameters* (D, b, α, β) whenever the intersection numbers of Γ satisfy

$$c_i = \begin{bmatrix} i \\ 1 \end{bmatrix}\left(1 + \alpha\begin{bmatrix} i-1 \\ 1 \end{bmatrix}\right) \quad \text{for } 0 \leq i \leq D, \tag{2.4.1}$$

$$b_i = \left(\begin{bmatrix} D \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix}\right)\left(\beta - \alpha\begin{bmatrix} i \\ 1 \end{bmatrix}\right) \quad \text{for } 0 \leq i \leq D, \tag{2.4.2}$$

where

$$\begin{bmatrix} i \\ 1 \end{bmatrix} := 1 + b + b^2 + \cdots + b^{i-1}. \tag{2.4.3}$$

Suppose Γ has classical parameters (D, b, α, β). Combining (2.4.1)-(2.4.3) with (2.1.1), we have

$$\begin{aligned} a_i &= \begin{bmatrix} i \\ 1 \end{bmatrix}\left(\beta - 1 + \alpha\left(\begin{bmatrix} D \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix} - \begin{bmatrix} i-1 \\ 1 \end{bmatrix}\right)\right) \\ &= \begin{bmatrix} i \\ 1 \end{bmatrix}\left(a_1 + \alpha\left(1 - \begin{bmatrix} i \\ 1 \end{bmatrix} - \begin{bmatrix} i-1 \\ 1 \end{bmatrix}\right)\right) \quad \text{for } \ 0 \le i \le D. \end{aligned} \tag{2.4.4}$$

Example 2.4.1. Petersen graph shown in Figure 2.1 is a distance-regular graph which has classical parameters (D, b, α, β) with $D = 2$, $b = -2$, $\alpha = -2$ and $\beta = -3$, which satisfies $a_1 = 0$, $a_2 \neq 0$ and $1 = c_2 < b(b+1) = 2$.

Example 2.4.2. [9] Hermitian forms graph $Her_2(D)$ is a distance-regular graph with classical parameters (D, b, α, β) with $b = -2$, $\alpha = -3$ and $\beta = -((-2)^D + 1)$, which satisfies $a_1 = 0$, $a_2 \neq 0$ and $c_2 = b(b+1) = 2$.

Example 2.4.3. [22, p. 237] Gewirtz graph is a distance-regular graph which has classical parameters (D, b, α, β) with $D = 2$, $b = -3$, $\alpha = -2$, $\beta = -5$, which satisfies $a_1 = 0$, $a_2 \neq 0$ and $2 = c_2 < b(b+1) = 6$.

Example 2.4.4. [3, Table 6.1] Witt graph M_{23} is a distance-regular graph which has classical parameters (D, b, α, β) with $D = 3$, $b = -2$, $\alpha = -2$, $\beta = 5$, which satisfies $a_1 = 0$, $a_2 \neq 0$ and $1 = c_2 < b(b+1) = 2$.

We list the parameters of the above examples in the following table for summary.

name	D	b	α	β	a_1	a_2	c_2
Petersen graph	2	-2	-2	-3	0	2	1
Hermitian forms graph $Her_2(D)$	D	-2	-3	$-((-2)^D + 1)$	0	3	2
Gewirtz graph	2	-3	-2	-5	0	8	2
Witt graph M_{23}	3	-2	-2	5	0	2	1

The following theorem characterizes the distance-regular graphs with classical parameters in an algebraic way.

Theorem 2.4.5. *([21, Theorem 4.2]) Let Γ denote a distance-regular graph with diameter $D \geq 3$. Choose $b \in \mathbb{R} \setminus \{0, -1\}$. Then the following (i)-(ii) are equivalent.*

(i) Γ is Q-polynomial with associated dual eigenvalues $\theta^*_0, \theta^*_1, \ldots, \theta^*_D$ satisfying

$$\theta^*_i - \theta^*_0 = (\theta^*_1 - \theta^*_0)\begin{bmatrix} i \\ 1 \end{bmatrix} b^{1-i} \quad \text{for} \;\; 1 \leq i \leq D. \tag{2.4.5}$$

(ii) Γ has classical parameters (D, b, α, β) for some real constants α, β. □

2.5 Block Designs

In this section we introduce some results of block designs which will be used in the proof of Theorem 6.2.1.

Lemma 2.5.1. *Let $(\mathbf{P}, \mathfrak{B}, \mathbb{I})$ be a 2-(v, κ, λ) design. Suppose $|\mathfrak{B}| = b$ and r is the replication number. Then $b\kappa = vr$.*

Proof. Counting in two ways the number of pairs $(x, B) \in \mathbb{I}$, where $x \in \mathbf{P}$ and $B \in \mathfrak{B}$, the equality follows immediately. □

The following famous theorem is known as *Fisher's inequality.*

Theorem 2.5.2. *[22, Theorem 19.6] For a 2-(v, κ, λ) design with b blocks and $v > \kappa$ we have $b \geq v$.* □

Proof. Let r denote the replication number and N denote the $v \times b$ incidence matrix of the design. Then

$$NN^t = (r - \lambda)I + \lambda J, \tag{2.5.1}$$

where J is the $v \times v$ all-one matrix. Note that J has eigenvalues v and 0 with multiplicities 1 and $v - 1$ respectively. Hence the eigenvalues of NN^t are $\lambda v + (r - \lambda)$ and $r - \lambda$ with multiplicities 1 and $v - 1$ respectively. This implies

$$\det(NN^t) = (\lambda v + r - \lambda)(r - \lambda)^{v-1}, \tag{2.5.2}$$

where $\det(NN^t)$ denotes the determinant of NN^t. Observe that

$$r = \frac{\lambda(v-1)}{k-1} \; > \; \lambda. \tag{2.5.3}$$

By (2.5.2) and (2.5.3), NN^t is invertible and has rank v. Note that

$$\operatorname{rank}(NN^t) \; \leq \; \operatorname{rank}(N) \; \leq \; \min\{v, b\}.$$

The assertion of the theorem follows immediately. □

Corollary 2.5.3. *For a 2-(v, κ, λ) design with replication number r we have $r \geq \kappa$.*

Proof. This is immediate from Lemma 2.5.1 and Theorem 2.5.2. □

Chapter 3

A Combinatorial Characterization of Distance-regular Graphs with Classical Parameters

The following theorem was shown in [25, Theorem 2.11].

Theorem 3.0.4. *[25, Theorem 2.11] Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $D \geq 4$ and intersection numbers $a_1 = 0$, $a_2 \neq 0$. Suppose Γ is Q-polynomial and contains no parallelograms of length 3 and no parallelograms of length 4. Then Γ has classical parameters (D, b, α, β) with $b < -1$.*

In this chapter we show the same result holds for the case $D = 3$. Theorem 3.2.1 is the main result of this chapter.

3.1 Counting 4-vertex Configurations

To prove Theorem 3.2.1, our main theorem in this chapter, we need a couple of lemmas. The first lemma is essentially given in [13, Theorem 5.2(i)], a proof is given here for completeness.

Lemma 3.1.1. *[13, Theorem 5.2(i)] Let Γ denote a Q-polynomial distance-regular graph with diameter $D \geq 3$ and intersection number $a_1 = 0$. Fix an integer i for*

$2 \le i \le D$ and three vertices x, y, z such that

$$\partial(x,y) = 1, \quad \partial(y,z) = i-1, \quad \partial(x,z) = i.$$

Then the quantity

$$s_i(x,y,z) := |\Gamma_{i-1}(x) \cap \Gamma_{i-1}(y) \cap \Gamma_1(z)| \tag{3.1.1}$$

is equal to

$$a_{i-1} \frac{(\theta_0^* - \theta_{i-1}^*)(\theta_2^* - \theta_i^*) - (\theta_1^* - \theta_{i-1}^*)(\theta_1^* - \theta_i^*)}{(\theta_0^* - \theta_{i-1}^*)(\theta_{i-1}^* - \theta_i^*)}. \tag{3.1.2}$$

In particular (3.1.1) is independent of the choice of the vertices x, y, z.

Proof. Let $s_i(x,y,z)$ denote the expression in (3.1.1) and set

$$\ell_i(x,y,z) = |\Gamma_i(x) \cap \Gamma_{i-1}(y) \cap \Gamma_1(z)|.$$

Observe

$$s_i(x,y,z) + \ell_i(x,y,z) = a_{i-1}. \tag{3.1.3}$$

By (2.3.14) we have

$$\sum_{\substack{w \in X \\ \partial(y,w)=i-1 \\ \partial(z,w)=1}} E\hat{w} - \sum_{\substack{w \in X \\ \partial(y,w)=1 \\ \partial(z,w)=i-1}} E\hat{w} = a_{i-1} \frac{\theta_{i-1}^* - \theta_1^*}{\theta_0^* - \theta_{i-1}^*} (E\hat{y} - E\hat{z}). \tag{3.1.4}$$

Taking the inner product of (3.1.4) with $\hat{x}$ using (2.3.13) and the assumption $a_1 = 0$, we obtain

$$s_i(x,y,z)\theta_{i-1}^* + \ell_i(x,y,z)\theta_i^* - a_{i-1}\theta_2^* = a_{i-1} \frac{\theta_{i-1}^* - \theta_1^*}{\theta_0^* - \theta_{i-1}^*} (\theta_1^* - \theta_i^*). \tag{3.1.5}$$

Solving $s_i(x,y,z)$ by using (3.1.3) and (3.1.5), we get (3.1.2). □

By Lemma 3.1.1, $s_i(x,y,z)$ is a constant for any vertices x, y, z with $\partial(x,y) = 1$, $\partial(y,z) = i-1$, $\partial(x,z) = i$. Let s_i denote the expression in (3.1.1). Note that $s_i = 0$ if and only if Γ contains no parallelograms of length i.

Lemma 3.1.2. *Let Γ denote a distance-regular graph which has classical parameters (D, b, α, β). Suppose intersection numbers $a_1 = 0$ and $a_2 \neq 0$. Then $\alpha < 0$ and $b < -1$.*

Proof. Since $a_1 = 0$ and $a_2 \neq 0$, from (2.4.3) and (2.4.4) we have

$$-\alpha(b+1)^2 = a_2 - (b+1)a_1 = a_2 > 0. \tag{3.1.6}$$

Hence

$$\alpha < 0. \tag{3.1.7}$$

By direct computation from (2.4.1), we get

$$(c_2 - b)(b^2 + b + 1) = c_3 > 0. \tag{3.1.8}$$

Since

$$b^2 + b + 1 > 0,$$

(3.1.8) implies

$$c_2 > b. \tag{3.1.9}$$

Using (2.4.1) and (3.1.9), we get

$$\alpha(1+b) = c_2 - b - 1 \geq 0. \tag{3.1.10}$$

Hence $b < -1$ by (3.1.7) and $b \neq -1$. □

3.2 Combinatorial Characterization

The following theorem characterizes the distance-regular graphs with classical parameters and $a_1 = 0$, $a_2 \neq 0$ in a combinatorial way.

Theorem 3.2.1. *Let Γ denote a distance-regular graph with diameter $D \geq 3$ and intersection numbers $a_1 = 0$, $a_2 \neq 0$. Then the following (i)-(iii) are equivalent.*

(i) Γ is Q-polynomial and contains no parallelograms of length 3.

(ii) Γ is Q-polynomial and contains no parallelograms of any length i for $3 \leq i \leq D$.

(iii) Γ has classical parameters (D, b, α, β) for some real constants b, α, β with $b < -1$.

Proof. (ii)⇒(i) This is clear.

(iii)⇒(ii) Suppose Γ has classical parameters. Then Γ is Q-polynomial with associated dual eigenvalues $\theta^*_0, \theta^*_1, \ldots, \theta^*_D$ satisfying

$$\theta^*_i - \theta^*_0 = (\theta^*_1 - \theta^*_0) \begin{bmatrix} i \\ 1 \end{bmatrix} b^{1-i} \quad \text{for } \; 1 \le i \le D. \tag{3.2.1}$$

We need to prove $s_i = 0$ for $3 \le i \le D$. To compute s_i in (3.1.2), observe from (3.2.1) that

$$\theta^*_{i-1} - \theta^*_i = (\theta^*_0 - \theta^*_1) b^{1-i} \quad \text{for } 1 \le i \le D. \tag{3.2.2}$$

Summing (3.2.2) for consecutive i, we find

$$(\theta^*_1 - \theta^*_i) = (\theta^*_0 - \theta^*_1)(b^{-1} + b^{-2} + \cdots + b^{1-i}), \tag{3.2.3}$$

$$(\theta^*_1 - \theta^*_{i-1}) = (\theta^*_0 - \theta^*_1)(b^{-1} + b^{-2} + \cdots + b^{2-i}), \tag{3.2.4}$$

$$(\theta^*_2 - \theta^*_i) = (\theta^*_0 - \theta^*_1)(b^{-2} + b^{-3} + \cdots + b^{1-i}), \tag{3.2.5}$$

$$(\theta^*_0 - \theta^*_{i-1}) = (\theta^*_0 - \theta^*_1)(b^{0} + b^{-1} + \cdots + b^{2-i}) \tag{3.2.6}$$

for $3 \le i \le D$. Evaluating (3.1.2) by using (3.2.2)-(3.2.6), we find $s_i = 0$ for $3 \le i \le D$.

(i)⇒(iii) Observe $s_3 = 0$. Then by setting $i = 3$ in (3.1.2) and using the assumption $a_2 \ne 0$, we find

$$(\theta^*_0 - \theta^*_2)(\theta^*_2 - \theta^*_3) - (\theta^*_1 - \theta^*_2)(\theta^*_1 - \theta^*_3) = 0. \tag{3.2.7}$$

Set

$$b := \frac{\theta^*_1 - \theta^*_0}{\theta^*_2 - \theta^*_1}. \tag{3.2.8}$$

Then

$$\theta^*_2 = \theta^*_0 + \frac{(\theta^*_1 - \theta^*_0)(b+1)}{b}. \tag{3.2.9}$$

Eliminating θ^*_2, θ^*_3 in (3.2.7) using (3.2.9) and (2.3.15), we have

$$\frac{-(\theta^*_1 - \theta^*_0)^2(\sigma b^2 + \sigma b + \sigma - b)}{\sigma b^2} = 0 \tag{3.2.10}$$

for an appropriate $\sigma \in \mathbb{R} \setminus \{0\}$. Since $\theta^*_1 \ne \theta^*_0$,

$$\sigma b^2 + \sigma b + \sigma - b = 0,$$

and hence

$$\sigma^{-1} = \frac{b^2 + b + 1}{b}. \tag{3.2.11}$$

By Theorem 2.4.5, to prove that Γ has classical parameters, it suffices to prove that

$$\theta_i^* - \theta_0^* = (\theta_1^* - \theta_0^*) \begin{bmatrix} i \\ 1 \end{bmatrix} b^{1-i} \quad \text{for } \ 1 \leq i \leq D. \tag{3.2.12}$$

We prove (3.2.12) by induction on i. The case $i = 1$ is trivial and the case $i = 2$ is from (3.2.9). Now suppose $i \geq 3$. Then (2.3.15) implies

$$\theta_i^* = \sigma^{-1}(\theta_{i-1}^* - \theta_{i-2}^*) + \theta_{i-3}^* \quad \text{for } \ 3 \leq i \leq D. \tag{3.2.13}$$

Evaluating (3.2.13) using (3.2.11) and the induction hypothesis, we find that $\theta_i^* - \theta_0^*$ is as in (3.2.12). Therefore, Γ has classical parameters (D, b, α, β) for some scalars α, β. Note that $b < -1$ from Lemma 3.1.2. □

Chapter 4

An Upper Bound of c_2

In this chapter we assume that Γ has classical parameters and intersection numbers $a_1 = 0, a_2 \neq 0$ to obtain the following theorem.

Theorem 4.0.2. *Let Γ denote a distance-regular graph with diameter $D \geq 3$ and intersection numbers $a_1 = 0$, $a_2 \neq 0$. Suppose Γ has classical parameters (D, b, α, β). Then each of*

$$\frac{b(b+1)^2(b+2)}{c_2}, \quad \frac{(b-2)(b-1)b(b+1)}{2+2b-c_2} \tag{4.0.1}$$

is an integer. Moreover

$$c_2 \leq b(b+1). \tag{4.0.2}$$

Note that the bound in (4.0.2) will be improved to $c_2 \leq 2$ in Chapter 6.

4.1 Results from Simple Computations

Theorem 4.1.1. *[26, Proposition 6.7, Theorem 4.6] Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $D \geq 3$. Assume that the intersection numbers $a_1 = 0$ and $a_2 \neq 0$. Suppose that Γ contains no parallelograms of length 3. Then for each pair of vertices $v, w \in X$ at distance $\partial(v, w) = 2$, there exists a weak-geodetically closed subgraph Ω of diameter 2 in Γ containing v, w. Furthermore Ω is strongly regular with* *intersection numbers*

$$a_i(\Omega) = a_i(\Gamma), \tag{4.1.1}$$

$$c_i(\Omega) = c_i(\Gamma), \tag{4.1.2}$$

$$b_i(\Omega) = a_2(\Gamma) + c_2(\Gamma) - a_i(\Omega) - c_i(\Omega) \tag{4.1.3}$$

for $0 \leq i \leq 2$. □

Corollary 4.1.2. *Let* Γ *denote a distance-regular graph which has classical parameters* (D, b, α, β)*, where* $D \geq 3$*. Assume* Γ *has intersection numbers* $a_1 = 0$ *and* $a_2 \neq 0$*. Then there exists a weak-geodetically closed subgraph* Ω *of diameter* 2*. Furthermore the intersection numbers of* Ω *satisfy*

$$b_0(\Omega) = (1+b)(1-\alpha b), \tag{4.1.4}$$

$$b_1(\Omega) = b(1-\alpha-\alpha b), \tag{4.1.5}$$

$$c_2(\Omega) = (1+b)(1+\alpha), \tag{4.1.6}$$

$$a_2(\Omega) = -(1+b)^2\alpha, \tag{4.1.7}$$

$$|\Omega| = \frac{(1+b)(b\alpha-2)(b\alpha-1-\alpha)}{(1+\alpha)}. \tag{4.1.8}$$

Proof. Observe $b < -1$ by Lemma 3.1.2 and Γ contains no parallelograms of length 3 by Theorem 3.2.1. Hence there exists a weak-geodetically closed subgraph Ω of diameter 2 by Theorem 2.2.7. By applying (2.4.1), (2.4.2) and (2.4.4) to (4.1.1)-(4.1.3), we have (4.1.4)-(4.1.7) immediately. Observe that $|\Omega| = 1 + k(\Omega) + k(\Omega)b_1(\Omega)/c_2(\Omega)$. (4.1.8) follows from this and (4.1.4)-(4.1.6). □

Proposition 4.1.3. *[26, Proposition 3.2] Let* Γ *denote a distance-regular graph with diameter* $D \geq 3$*. Suppose there exists a weak-geodetically closed subgraph* Ω *of* Γ *with diameter* 2*. Then the intersection numbers of* Γ *satisfy the following inequality*

$$a_3 \geq a_2(c_2 - 1) + a_1. \tag{4.1.9}$$

□

Corollary 4.1.4. *Let Γ denote a distance-regular graph which has classical parameters (D, b, α, β), where $D \geq 3$. Suppose the intersection numbers $a_1 = 0$ and $a_2 \neq 0$. Then*

$$c_2 \leq b^2 + b + 2. \tag{4.1.10}$$

Proof. Applying $a_1 = 0$ in (2.4.4), we have $a_3 = -\alpha(b^2+b+1)(b+1)^2$. Then by applying (4.1.9) using Lemma 3.1.2, (4.1.1), and (4.1.7), the result follows immediately. □

4.2 Multiplicity Technique

We will improve the upper bound of c_2 in (4.1.10). We need the following lemma.

Lemma 4.2.1. *Let Γ denote a distance-regular graph which has classical parameters (D, b, α, β), where $D \geq 3$. Assume the intersection numbers $a_1 = 0$ and $a_2 \neq 0$. Let Ω be a weak-geodetically closed subgraph of diameter 2 in Γ. Let $r > s$ denote the nontrivial eigenvalues of the strongly regular graph Ω. Then the following (i), (ii) hold:*

(i) The multiplicity of r is

$$f = \frac{(b\alpha - 1)(b\alpha - 1 - \alpha)(b\alpha - 1 + \alpha)}{(\alpha - 1)(\alpha + 1)}. \tag{4.2.1}$$

(ii) The multiplicity of s is

$$g = \frac{-b(b\alpha - 1)(b\alpha - 2)}{(\alpha - 1)(\alpha + 1)}. \tag{4.2.2}$$

Proof. Let v=$|\Omega|$ and k be the valency of Ω. Note that $c_2(\Omega) = (1 + b)(1 + \alpha)$ by (2.4.1), $k(\Omega) = (1 + b)(1 - \alpha b)$ by (4.1.4), and $v = (1 + b)(b\alpha - 2)(b\alpha - 1 - \alpha)/(1 + \alpha)$ by (4.1.8). Now (4.2.1) and (4.2.2) follow from (2.1.3) and (2.1.4). □

Corollary 4.2.2. *Let Γ denote a distance-regular graph which has classical parameters (D, b, α, β), where $D \geq 3$. Assume Γ has intersection numbers $a_1 = 0$ and $a_2 \neq 0$. Then*

$$\frac{b(b + 1)^2(b + 2)}{c_2} \tag{4.2.3}$$

and

$$\frac{(b-2)(b-1)b(b+1)}{2+2b-c_2} \tag{4.2.4}$$

are both integers.

Proof. Let f and g be as (4.2.1) and (4.2.2). Set $\rho = \alpha(1+b) = c_2 - 1 - b$ being an integer. Then both

$$f + g - (1 - 3b^2 - b\rho + b^2\rho - b^3) = \frac{2b + 5b^2 + 4b^3 + b^4}{1 + b + \rho} = \frac{b(b+1)^2(b+2)}{c_2}$$

and

$$f - g - (1 - 3b^2 - b\rho + b^2\rho + b^3) = \frac{2b - b^2 - 2b^3 + b^4}{-1 - b + \rho} = \frac{(b-2)(b-1)b(b+1)}{c_2 - 2 - 2b}$$

are integers since f, g, b and ρ are integers. □

Proposition 4.2.3. *Let* Γ *denote a distance-regular graph which has classical parameters* (D, b, α, β)*, where* $D \geq 3$*. Assume* Γ *has intersection numbers* $a_1 = 0$ *and* $a_2 \neq 0$*. Then* $c_2 \leq b(b+1)$*.*

Proof. Recall $c_2 \leq b^2 + b + 2$ by (4.1.10). First, suppose

$$c_2 = b^2 + b + 2. \tag{4.2.5}$$

Then the integral condition (4.2.3) becomes

$$b^2 + 3b + \frac{-4b}{b^2 + b + 2}. \tag{4.2.6}$$

Since $0 < -4b < b^2 + b + 2$ for $b \leq -5$, we have $-4 \leq b \leq -2$. For $b = -4$ or -3, expression (4.2.6) is not an integer. The remaining case $b = -2$ implies $\alpha = -5$ by (4.1.6), $v = 28$ by (4.1.8) and $g = 6$ by (4.2.2). This contradicts to $v \leq \frac{1}{2}g(g+3)$ [22, Theorem 21.4]. Hence $c_2 \neq b^2 + b + 2$. Next suppose $c_2 = b^2 + b + 1$. Then (4.2.4) becomes

$$-b^2 + b + 1 + \frac{1}{b^2 - b - 1}. \tag{4.2.7}$$

It fails to be an integer since $b < -1$. □

Proof of Theorem 4.0.2:

The results come from Corollary 4.2.2 and Proposition 4.2.3. □

Chapter 5

3-bounded Property

Let Γ denote a distance-regular graph which has classical parameters (D, b, α, β) and $D \geq 3$. Assume the intersection numbers $a_1 = 0$ and $a_2 \neq 0$. Note that Γ contains no parallelograms of any length by Theorem 3.2.1. We have known that Γ is 2-bounded. We shall prove that Γ is 3-bounded in this chapter.

5.1 Weak-geodetically Closed with respect to a Vertex

First we give a definition.

Definition 5.1.1. For any vertex $x \in X$ and any subset $C \subseteq X$, define

$$[x, C] := \{v \in X \mid \text{there exists } z \in C, \text{ such that } \partial(x, v) + \partial(v, z) = \partial(x, z)\}.$$

Throughout this section, fix two vertices $x, y \in X$ with $\partial(x, y) = 3$. Set

$$C := \{z \in \Gamma_3(x) \mid B(x, y) = B(x, z)\}$$

and

$$\Delta = [x, C]. \tag{5.1.1}$$

We shall prove Δ is a regular weak-geodetically closed subgraph of diameter 3. Note that the diameter of Δ is at least 3. If $D = 3$ then $C = \Gamma_3(x)$ and $\Delta = \Gamma$ is clearly a regular weak-geodetically closed graph. Thereafter we assume $D \geq 4$. By referring to Theorem 2.2.4, we shall prove Δ is weak-geodetically closed with respect to x, and the subgraph induced on Δ is regular with valency $a_3 + c_3$.

Lemma 5.1.2. *For adjacent vertices $z, z' \in \Gamma_i(x)$, where $i \leq D$, we have $B(x,z) = B(x,z')$.*

Proof. By symmetry, it suffices to show $B(x,z) \subseteq B(x,z')$. Suppose contradictory there exists $w \in B(x,z) \setminus B(x,z')$. Then $\partial(w,z') \neq i+1$. Note that $\partial(w,z') \leq \partial(w,x)+\partial(x,z') = 1+i$ and $\partial(w,z') \geq \partial(w,z)-\partial(z,z') = i$. This implies $\partial(w,z') = i$ and $wxz'z$ forms a parallelogram of length $i+1$, a contradiction. □

It is known that Γ is 2-bound by Theorem 2.2.7. For two vertices z, s in Γ with $\partial(z,s) = 2$, let $\Omega(z,s)$ denote the regular weak-geodetically closed subgraph containing z, s of diameter 2.

Lemma 5.1.3. *Suppose stuzw is a pentagon in Γ, where $s, u \in \Gamma_3(x)$ and $z \in \Gamma_2(x)$. Pick $v \in B(x,u)$. Then $\partial(v,s) \neq 2$.*

Proof. Suppose contradictory $\partial(v,s) = 2$. Note $\partial(z,s) \neq 1$, since $a_1 = 0$. Note that $z, w, s, t, u \in \Omega(z,s)$. Then $s \in \Omega(z,s) \cap \Gamma_2(v)$ and $u \in \Omega(z,s) \cap \Gamma_4(v) \neq \emptyset$. Hence $\partial(v,z) = \partial(v,s) + \partial(s,z) = 2+2 = 4$ by Theorem 2.2.8. A contradiction occurs since $\partial(v,x) = 1$ and $\partial(x,z) = 2$. □

Lemma 5.1.4. *Suppose stuzw is a pentagon in Γ, where $s, u \in \Gamma_3(x)$ and $z \in \Gamma_2(x)$. Then $B(x,s) = B(x,u)$.*

Proof. Since $|B(x,s)| = |B(x,u)| = b_3$, it suffices to show $B(x,u) \subseteq B(x,s)$.
By Lemma 5.1.3,

$$B(x,u) \subseteq \Gamma_3(s) \cup \Gamma_4(s).$$

Suppose

$$\begin{aligned} |B(x,u) \cap \Gamma_3(s)| &= m, \\ |B(x,u) \cap \Gamma_4(s)| &= n. \end{aligned}$$

Then

$$m + n = b_3. \tag{5.1.2}$$

By Theorem 2.3.1,

$$\sum_{r\in B(x,u)} E\hat{r} - \sum_{r'\in B(u,x)} E\hat{r'} = b_3\frac{\theta^*_1-\theta^*_4}{\theta^*_0-\theta^*_3}(E\hat{x}-E\hat{u}). \tag{5.1.3}$$

Observe $B(u,x)\subseteq\Gamma_3(s)$, otherwise $\Omega(u,s)\cap B(u,x)\neq\emptyset$ and this leads to $\partial(x,s)=4$ by Theorem 2.2.8, which contradicts to $\partial(x,s)=3$. Taking the inner product of s with both side of (5.1.3) and evaluating the result using (2.3.13), we have

$$m\theta^*_3+n\theta^*_4-b_3\theta^*_3 = b_3\frac{\theta^*_1-\theta^*_4}{\theta^*_0-\theta^*_3}(\theta^*_3-\theta^*_2). \tag{5.1.4}$$

Solve (5.1.2) and (5.1.4) to obtain

$$n = b_3\frac{(\theta^*_2-\theta^*_3)}{(\theta^*_3-\theta^*_4)}\frac{(\theta^*_1-\theta^*_4)}{(\theta^*_0-\theta^*_3)}. \tag{5.1.5}$$

Simplifying (5.1.5) using (2.4.5), we have $n=b_3$ and then $m=0$ by (5.1.2). This implies $B(x,u)\subseteq B(x,s)$ as required. □

Lemma 5.1.5. *Let $z,u\in\Delta$. Suppose $stuzw$ is a pentagon in Γ, where $z,w\in\Gamma_2(x)$ and $u\in\Gamma_3(x)$. Then $w\in\Delta$.*

Proof. Observe $\Omega(z,s)\cap\Gamma_1(x)=\emptyset$ and $\Omega(z,s)\cap\Gamma_4(x)=\emptyset$ by Theorem 2.2.8. Hence $s,t\in\Gamma_2(x)\cup\Gamma_3(x)$. Observe $s\in\Gamma_3(x)$, otherwise $w,s\in\Omega(x,z)$, and this implies $u\in\Omega(x,z)$, a contradiction to that the diameter of $\Omega(x,z)$ is 2. Hence $B(x,s)=B(x,u)$ by Lemma 5.1.4. Then $s\in C$ and $w\in\Delta$ by construction. □

Lemma 5.1.6. *The subgraph Δ is weak-geodetically closed with respect to x.*

Proof. Clearly $C(z,x)\subseteq\Delta$ for any $z\in\Delta$. It suffices to show $A(z,x)\subseteq\Delta$ for any $z\in\Delta$. Suppose $z\in\Delta$. We discuss case by case in the following. The case $\partial(x,z)=1$ is trivial since $a_1=0$. For the case $\partial(x,z)=3$, we have $B(x,y)=B(x,z)=B(x,w)$ for any $w\in A(z,x)$ by definition of Δ and Lemma 5.1.2. This implies $A(z,x)\subseteq\Delta$ by the construction of Δ. For the remaining case $\partial(x,z)=2$, fix $w\in A(z,x)$ and we shall prove $w\in\Delta$. There exists $u\in C$ such that $z\in C(u,x)$. Observe that $\partial(w,u)=2$ since $a_1=0$. Choose $s\in A(w,u)$ and $t\in C(u,s)$. Then $stuzw$ is a pentagon in Γ. The result comes immediately from Lemma 5.1.5. □

5.2 3-bounded Property

Theorem 5.2.1. *Let Γ denote a distance-regular graph which has classical parameters (D, b, α, β) and $D \geq 3$. Assume the intersection numbers $a_1 = 0$ and $a_2 \neq 0$. Then Γ is 3-bounded.*

Proof. By Theorem 2.2.4 and Lemma 5.1.6, it suffices to show that Δ defined in (5.1.1) is regular with valency a_3+c_3. Clearly from the construction and Lemma 5.1.6, $|\Gamma_1(z)\cap\Delta| = a_3+c_3$ for any $z \in C$. First we show $|\Gamma_1(x)\cap\Delta| = a_3+c_3$. Note that $y \in \Delta\cap\Gamma_3(x)$ by construction of Δ. For any $z \in C(x, y) \cup A(x, y)$,

$$\partial(x, z) + \partial(z, y) \leq \partial(x, y) + 1.$$

This implies $z \in \Delta$ by Definition 2.2.3 and Lemma 5.1.6. Hence $C(x, y)\cup A(x, y) \subseteq \Delta$. Suppose $B(x, y) \cap \Delta \neq \emptyset$. Choose $t \in B(x, y) \cap \Delta$. Then there exists $y' \in \Gamma_3(x) \cap \Delta$ such that $t \in C(x, y')$. Note that $B(x, y) = B(x, y')$. This leads to a contradiction to $t \in C(x, y')$. Hence $B(x, y) \cap \Delta = \emptyset$ and $\Gamma_1(x) \cap \Delta = C(x, y) \cup A(x, y)$. Then we have $|\Gamma_1(x) \cap \Delta| = a_3 + c_3$.

Since each vertex in Δ appears in a sequence of vertices $x = x_0, x_1, x_2, x_3$ in Δ, where $\partial(x, x_j) = j$ and $\partial(x_{j-1}, x_j) = 1$ for $1 \leq j \leq 3$, it suffices to show

$$|\Gamma_1(x_i) \cap \Delta| = a_3 + c_3 \tag{5.2.1}$$

for $1 \leq i \leq 2$. For each integer $0 \leq i \leq 2$, we show

$$|\Gamma_1(x_i) \setminus \Delta| \leq |\Gamma_1(x_{i+1}) \setminus \Delta|$$

by counting the number of pairs (s, z) for $s \in \Gamma_1(x_i)\setminus\Delta$, $z \in \Gamma_1(x_{i+1})\setminus\Delta$ and $\partial(s, z) = 2$ in two ways. For a fixed $z \in \Gamma_1(x_{i+1}) \setminus \Delta$, we have $\partial(x, z) = i + 2$ by Lemma 5.1.6, so $\partial(x_i, z) = 2$ and $s \in A(x_i, z)$. Hence the number of such pairs (s, z) is at most $|\Gamma_1(x_{i+1}) \setminus \Delta|a_2$.

On the other hand, we show this number is exactly $|\Gamma_1(x_i) \setminus \Delta|a_2$. Fix an $s \in \Gamma_1(x_i) \setminus \Delta$. Observe $\partial(x, s) = i + 1$ by Lemma 5.1.6. Observe $\partial(x_{i+1}, s) = 2$ since $a_1 = 0$. Pick any $z \in A(x_{i+1}, s)$. We shall prove $z \notin \Delta$. Suppose contradictory $z \in \Delta$ in the following arguments and choose any $w \in C(s, z)$.

Case 1: $i = 0$.

Observe $\partial(x, z) = 2$, $\partial(x, s) = 1$ and $\partial(x, w) = 2$. This forces $s \in \Delta$ by Lemma 5.1.6, a contradiction.

Case 2: $i = 1$.

Observe $\partial(x, z) = 3$, otherwise $z \in \Omega(x, x_2)$ and this implies $s \in \Omega(x, x_2) \subseteq \Delta$ by Lemma 2.2.5 and Lemma 5.1.6, a contradiction. This also implies $s \in \Delta$ by Definition 2.2.3 and Lemma 5.1.6, a contradiction.

Case 3: $i = 2$.

Observe $\partial(x, z) = 2$ or 3. Suppose $\partial(x, z) = 2$. Then $B(x, x_3) = B(x, s)$ by Lemma 5.1.4 (with $x_3 = u$, $x_2 = t$). Hence $s \in \Delta$, a contradiction. So $z \in \Gamma_3(x)$. Note $\partial(x, w) \neq 2, 3$, otherwise $s \in \Delta$ by Lemma 5.1.4 and Lemma 5.1.6 respectively. Hence $\partial(x, w) = 4$. Then by applying $\Omega = \Omega(x_2, w)$ in Theorem 2.2.8 we have $\partial(x_2, z) = 1$, a contradiction to $a_1 = 0$.

From the above counting, we have

$$|\Gamma_1(x_i) \setminus \Delta| a_2 \leq |\Gamma_1(x_{i+1}) \setminus \Delta| a_2 \tag{5.2.2}$$

for $0 \leq i \leq 2$. Eliminating a_2 from (5.2.2), we find

$$|\Gamma_1(x_i) \setminus \Delta| \leq |\Gamma_1(x_{i+1}) \setminus \Delta|, \tag{5.2.3}$$

or equivalently

$$|\Gamma_1(x_i) \cap \Delta| \geq |\Gamma_1(x_{i+1}) \cap \Delta| \tag{5.2.4}$$

for $0 \leq i \leq 2$. We have known previously $|\Gamma_1(x_0) \cap \Delta| = |\Gamma_1(x_3) \cap \Delta| = a_3 + c_3$. Hence (5.2.1) follows from (5.2.4). □

Remark 5.2.2. The 3-bounded property is enough to obtain the main result of this thesis. The 4-bounded property seems to be much harder to prove.

Chapter 6

A Constant Bound of c_2

Let $\Gamma = (X, R)$ be a distance-regular graph which has classical parameters (D, b, α, β) with $D \geq 3$. Assume the intersection numbers $a_1 = 0$ and $a_2 \neq 0$. We shall show that $c_2 \leq 2$, and if $c_2 = 1$ then $(b, \alpha, \beta) = (-2, -2, ((-2)^{D+1} - 1)/3)$.

6.1 Preliminary Lemmas

Let $\Gamma = (X, R)$ be a distance-regular graph with diameter $D \geq 3$ and intersection numbers a_i, c_i, b_i for $0 \leq i \leq D$. Assume that Γ is D-bounded. By Theorem 2.2.9, for any $x, y \in X$ with $\partial(x, y) = t$, there exists a unique weak-geodetically closed subgraph $\Delta(x, y)$ containing x, y of diameter t, and $\Delta(x, y)$ is a distance-regular graph with the intersection numbers

$$a_i(\Delta(x, y)) = a_i, \quad (6.1.1)$$

$$c_i(\Delta(x, y)) = c_i, \quad (6.1.2)$$

$$b_i(\Delta(x, y)) = b_i - b_t \quad (6.1.3)$$

for $0 \leq i \leq t$ by Theorem 2.2.4 and (2.1.1). In particular, $\Delta(x, y)$ is a clique of size $1 + b_0 - b_1 = a_1 + 2$ when $t = 1$.

Lemma 6.1.1. *[27, Lemma 4.10] Let Γ denote a distance-regular graph which has classical parameters (D, b, α, β). Let Δ denote a regular weak-geodetically closed subgraph* *of Γ. Then Δ is a distance-regular graph which has classical parameters (t, b, α, β'), where t denotes the diameter of Δ, and $\beta' = \beta + \alpha(\left[{D \atop 1}\right] - \left[{t \atop 1}\right])$.* □

Proof. By Theorem 2.2.4, Δ is distance-regular with intersection numbers

$$\begin{aligned} c_i(\Delta) &= c_i = \begin{bmatrix} i \\ 1 \end{bmatrix}\left(1 + \alpha\begin{bmatrix} i-1 \\ 1 \end{bmatrix}\right), \\ a_i(\Delta) &= a_i = \begin{bmatrix} i \\ 1 \end{bmatrix}\left(a_1 + \alpha\left(1 - \begin{bmatrix} i \\ 1 \end{bmatrix} - \begin{bmatrix} i-1 \\ 1 \end{bmatrix}\right)\right), \end{aligned}$$

and

$$\begin{aligned} b_i(\Delta) = b_i - b_t &= \left(\begin{bmatrix} D \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix}\right)\left(\beta - \alpha\begin{bmatrix} i \\ 1 \end{bmatrix}\right) - \left(\begin{bmatrix} D \\ 1 \end{bmatrix} - \begin{bmatrix} t \\ 1 \end{bmatrix}\right)\left(\beta - \alpha\begin{bmatrix} t \\ 1 \end{bmatrix}\right) \\ &= \left(\begin{bmatrix} t \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix}\right)\left(\beta + \alpha\begin{bmatrix} D \\ 1 \end{bmatrix} - \alpha\begin{bmatrix} t \\ 1 \end{bmatrix} - \alpha\begin{bmatrix} i \\ 1 \end{bmatrix}\right) \end{aligned}$$

for $0 \leq i \leq t$. Hence Δ has classical parameters (t, b, α, β'), where $\beta' = \beta + \alpha\left[{D \atop 1}\right] - \alpha\left[{t \atop 1}\right]$.

□

Lemma 6.1.2. *Let $\Gamma = (X, R)$ denote a D-bounded distance-regular graph with $D \geq 3$. Let Λ be a weak-geodetically closed subgraph of Γ with diameter s, where $0 \leq s \leq D-1$. Suppose $x, y \in \Lambda$ with $\partial(x, y) = s$. Then the following (i)-(iii) hold.*

(i) For any $w \in X$, let $\mathcal{M}(w) = \{m - \{w\} \mid m \subseteq X$ is a clique of size $a_1 + 2$ containing $w\}$. Then $\mathcal{M}(w)$ is a partition of $\Gamma_1(w)$ with $|\mathcal{M}(w)| = \dfrac{b_0}{a_1 + 1}$.

(ii) If $z \in B(y, x)$, then $\Delta(x, z) \supseteq \Lambda$ and $\Delta(x, z)$ has diameter $s + 1$.

(iii) If Δ is a weak-geodetically closed subgraph of Γ with diameter $s + 1$ and contains Λ, then $\Delta = \Delta(x, z)$ for some $z \in B(y, x)$.

Proof. Note that $\Lambda = \Delta(x, y)$ by Theorem 2.2.9.

(i) The 1-bounded property implies each edge is contained in a clique of size $a_1 + 2$. Since there are b_0 edges in Γ containing a fixed vertex w, we have (i).

(ii) Note that $\Delta(x, z) \cap \Lambda$ is a weak-geodetically closed subgraph of Γ and $y \in \Delta(x, z) \cap \Lambda$ since $y \in C(z, x)$. This implies the diameter of $\Delta(x, z) \cap \Lambda$ is s and we have $\Delta(x, z) \cap \Lambda = \Lambda$ by Theorem 2.2.9. Hence $\Delta(x, z) \supseteq \Lambda$. The diameter of $\Delta(x, z)$ is $s + 1$ since $\partial(x, z) = s + 1$.

(iii) Suppose that Δ is a weak-geodetically closed subgraph of Γ with diameter $s+1$ and contains Λ. Note that $x, y \in \Delta$. Choose $z \in \Delta$ and $z \in B(y, x)$. Then $\Delta = \Delta(x, z)$ by (ii). $\square$

Lemma 6.1.3. *Let Γ denote a D-bounded distance-regular graph with $D \geq 3$. Let Λ, Λ' be two weak-geodetically closed subgraphs of Γ with diameter s, $s+3$ respectively and $\Lambda \subseteq \Lambda'$, where $0 \leq s \leq D-3$. Let $\mathbf{P}$ and $\mathfrak{B}$ be the sets of weak-geodetically closed subgraphs of Λ' which contain Λ, with diameter $s+1$ and $s+2$ respectively. Let $\mathbb{I} = \{(p, B) \mid p \in \mathbf{P}, B \in \mathfrak{B},$ and $p \subseteq B\}$. Then $(\mathbf{P}, \mathfrak{B}, \mathbb{I})$ is a 2-$(v, \kappa, 1)$ design, where*

$$\begin{aligned} v &= \frac{b_s - b_{s+3}}{b_s - b_{s+1}}, \\ \kappa &= \frac{b_s - b_{s+2}}{b_s - b_{s+1}}, \end{aligned}$$

and the replication number

$$r = \frac{b_{s+1} - b_{s+3}}{b_{s+1} - b_{s+2}}.$$

Proof. Let $x, y \in \Lambda$ with $\partial(x, y) = s$. Counting in two ways the number of pairs (ℓ, Ω), where $\ell \subseteq \Lambda'$ is a clique of size $a_1 + 2$ containing y with $\ell \not\subseteq \Lambda$, and $\Omega \in \mathbf{P}$ with $\ell \subseteq \Omega$. By Lemma 6.1.2,

$$\frac{b_s(\Lambda')}{(a_1+1)} \times 1 = |\mathbf{P}| \times \frac{b_s(\Omega)}{(a_1+1)}. \tag{6.1.4}$$

Simplifying (6.1.4) by (6.1.3) we have

$$|\mathbf{P}| = \frac{b_s(\Lambda')}{b_s(\Omega)} = \frac{b_s - b_{s+3}}{b_s - b_{s+1}}.$$

Fix $\Delta \in \mathfrak{B}$. Using the same technique as above, there are

$$\frac{b_s - b_{s+2}}{b_s - b_{s+1}}$$

distinct elements of $\mathbf{P}$ incident with Δ. Note that the number is independent of choice of Δ.

Fix any distinct $\Omega', \Omega'' \in \mathbf{P}$. Pick $z \in B(y, x) \cap \Omega'$. Then $\Omega' = \Delta(x, z)$ by Theorem 6.1.2. Pick $w \in \Omega''_1(x) - \Omega'$. Note that $w \in B(x, z)$. Then $\Delta(w, z) \in \mathfrak{B}$ containing Ω' and Ω''. Suppose that $\Delta' \in \mathfrak{B}$ is another block incident with Ω' and Ω''. Observe that $\Omega', \Omega'' \subseteq \Delta(w,z) \cap \Delta' \subseteq \Delta(w,z)$. This implies that the diameter of $\Delta(w,z) \cap \Delta'$ is $s+1$. We have $\Omega' = \Delta(w,z) \cap \Delta' = \Omega''$ by Theorem 2.2.9, which contradicts to $\Omega' \neq \Omega''$.

The replication number $r = \dfrac{b_{s+1} - b_{s+3}}{b_{s+1} - b_{s+2}}$ can be computed by the same argument of counting of $|\mathbf{P}|$. □

6.2 An Application of 3-bounded Property

Let $\Gamma = (X, R)$ be a distance-regular graph which has classical parameters (D, b, α, β) with $D \geq 3$. Suppose the intersection numbers $a_1 = 0$ and $a_2 \neq 0$. Then $\alpha < 0$ and $b < -1$ by Lemma 3.1.2. Now we are ready to prove the main theorem of this chapter.

Theorem 6.2.1. *Let Γ denote a distance-regular graph which has classical parameters (D, b, α, β) and $D \geq 3$. Assume the intersection numbers $a_1 = 0$ and $a_2 \neq 0$. Then $c_2 \leq 2$.*

Proof. It was shown in Theorem 5.2.1 that Γ is 3-bounded. Fix a vertex $x \in X$ and a weak-geodetically closed subgraph Δ containing x of diameter 3. By (6.1.1)-(6.1.3), and Lemma 6.1.1 we find $a_1(\Delta) = 0$ and Δ has classical parameters $(3, b, \alpha, \beta')$ where $\beta' = \beta + \alpha(\left[\begin{smallmatrix} D \\ 1 \end{smallmatrix}\right] - \left[\begin{smallmatrix} 3 \\ 1 \end{smallmatrix}\right])$. Note that

$$\beta' \;=\; 1 + \alpha - \alpha(\begin{bmatrix} 3 \\ 1 \end{bmatrix}) \;=\; 1 - \alpha b - \alpha b^2 \tag{6.2.1}$$

by applying $a_1(\Delta) = 0$ to (2.4.4). Let $\mathbf{P}$ denote the set of all maximal cliques containing x in Δ, and $\mathfrak{B}$ be the set of all weak-geodetically closed subgraphs of diameter 2 containing x in Δ. Let $\mathbb{I} = \{(p, B) \mid p \in \mathbf{P}, B \in \mathfrak{B},$ and $p \subseteq B\}$. Then $(\mathbf{P}, \mathfrak{B}, \mathbb{I})$ is a 2-$(v, \kappa, 1)$ design by Lemma 6.1.3, where

$$\kappa \;=\; \frac{b_0(\Delta) - b_2(\Delta)}{b_0(\Delta) - b_1(\Delta)} \;=\; (1+b)(1-\alpha b) \tag{6.2.2}$$

and the replication number

$$r \;=\; \frac{b_1(\Delta)}{b_1(\Delta) - b_2(\Delta)} \;=\; \frac{b(1+b)(1-\alpha b - \alpha b^2 - \alpha)}{b(1 - \alpha b - \alpha)} \tag{6.2.3}$$

by (2.4.2) and (6.2.1). Applying (6.2.2), (6.2.3), and Corollary 2.5.3 to the design, we have

$$\frac{(1+b)(1-\alpha b-\alpha b^2-\alpha)}{(1-\alpha b-\alpha)} \ \geq \ (1+b)(1-\alpha b). \tag{6.2.4}$$

Note that

$$(1-\alpha b-\alpha) = \frac{b_1(\Delta)-b_2(\Delta)}{b} < 0 \tag{6.2.5}$$

since $b_1(\Delta)-b_2(\Delta) > 0$ by Theorem 2.2.10 and $b < -1$. By (6.2.4), (6.2.5), and $b < -1$ we have

$$(1-\alpha b-\alpha b^2-\alpha) \ \geq \ (1-\alpha b)(1-\alpha b-\alpha). \tag{6.2.6}$$

Simplifying (6.2.6) we have

$$\alpha b(\alpha b+\alpha+b-1) \leq 0. \tag{6.2.7}$$

Observe that $\alpha b > 0$ since $\alpha < 0$ and $b < -1$. Then

$$\alpha b+\alpha+b-1 \leq 0. \tag{6.2.8}$$

Note that $\alpha b+\alpha+b-1 = c_2-2$ by (2.4.1) and hence $c_2 \leq 2$. □

For the case $c_2 = 1$, we have the following result.

Theorem 6.2.2. *Let* Γ *denote a distance-regular graph which has classical parameters* (D, b, α, β) *and* $D \geq 3$. *Assume the intersection numbers* $a_1 = 0$, $a_2 \neq 0$ *and* $c_2 = 1$. *Then* $(b, \alpha, \beta) = (-2, -2, ((-2)^{D+1}-1)/3)$.

Proof. Substituting $a_1 = 0$ and $c_2 = 1$ into (2.4.4), (2.4.1), and (2.4.3) we have

$$\alpha = \frac{-b}{1+b}, \tag{6.2.9}$$

$$\beta = \frac{b^{D+1}-1}{b^2-1}. \tag{6.2.10}$$

Let $\Omega \subset \Delta$ be two weak-geodetically closed subgraphs of Γ with diameters 2 and 3 respectively. Note that Ω is a strongly regular graph with $a_1(\Omega) = 0$, $c_2(\Omega) = 1$ by (6.1.1) and (6.1.2). Substituting this into (2.1.1) and (2.1.2) we have

$$|\Omega| = 1 + k_1(\Omega) + k_2(\Omega) = 1 + (b_0(\Omega))^2. \tag{6.2.11}$$

Hence we have

$$b_0(\Omega) = 2,\ 3,\ 7,\ 57 \tag{6.2.12}$$

by Lemma 2.1.2. Note that

$$b_0(\Omega) = b_0 - b_2 = 1 + b + b^2 \tag{6.2.13}$$

by (6.1.3), (2.1.13), (6.2.9), and (6.2.10). Solving (6.2.12) with (6.2.13) for integer $b < -1$ we have $b = -2$, -3, or -8. By (2.1.2), (6.1.2), and (6.1.3) we have

$$k_3(\Delta) = \frac{(b_0 - b_3)(b_1 - b_3)(b_2 - b_3)}{c_1 c_2 c_3}. \tag{6.2.14}$$

Evaluating (6.2.14) using (2.4.1)-(2.4.3), (6.2.9), and (6.2.10) we find

$$k_3(\Delta) = \frac{b^3(b^2+1)(b^2+b+1)(b^3+b^2+2b+1)}{1-b}. \tag{6.2.15}$$

The number $k_3(\Delta)$ is not an integer when $b = -3$ or -8. Hence $b = -2$ and $\alpha = -2$, $\beta = ((-2)^{D+1} - 1)/3$ by (6.2.9) and (6.2.10) respectively. □

Example 6.2.3. [9] Hermitian forms graphs $Her_2(D)$ are the distance-regular graphs which have classical parameters (D, b, α, β) with $b = -2$, $\alpha = -3$, and $\beta = -(-2)^D - 1$, which have $a_1 = 0$, $a_2 \neq 0$, and $c_2 = (1+\alpha)(b+1) = 2$. This is the only known class of examples that satisfies the assumptions of Theorem 6.2.1 with $c_2 = 2$.

Example 6.2.4. [22, p. 237] Gewirtz graph is the distance-regular graph with intersection numbers $a_1 = 0$, $a_2 = 8$, and $c_2 = 2$, which has classical parameters (D, b, α, β) with $D = 2$, $b = -3$, $\alpha = -2$, and $\beta = -5$. It is still open if there exists a class of distance-regular graphs which have classical parameters $(D, -3, -2, (-1-(-3)^D)/2)$ for $D \geq 3$.

Example 6.2.5. [3, Table 6.1] Witt graph M_{23} is the distance-regular graph which has classical parameters (D, b, α, β) with $D = 3$, $b = -2$, $\alpha = -2$, and $\beta = 5$, which has $a_1 = 0$, $a_2 = 2$, and $c_2 = 1$. This is the only known example that satisfies the assumptions of Theorem 6.2.1 with $c_2 = 1$.

For summary, we list the parameters in the following table.

name	a_1	a_2	c_2	D	b	α	β
Petersen graph	0	2	1	2	-2	-2	-3
Witt graph M_{23}	0	2	1	3	-2	-2	5
??	0	2	1	$D \geq 4$	-2	-2	$\frac{(-2)^{D+1}-1}{3}$
Hermitian forms graph $Her_2(D)$	0	3	2	D	-2	-3	$-((-2)^D+1)$
Gewirtz graph	0	8	2	2	-3	-2	-5
??	0	8	2	$D \geq 3$	-3	-2	$\frac{-1-(-3)^D}{2}$

We close our thesis with two conjectures.

Conjecture 6.2.6. *(With graph M_{23} does not grow.) There is no distance-regular graph which has classical parameters $(D, -2, -2, \dfrac{(-2)^{D+1}-1}{3})$ with $D \geq 4$.*

Conjecture 6.2.7. *(Gewirtz graph does not grow.) There is no distance-regular graph which has classical parameters $(D, -3, -2, -\dfrac{1+(-3)^D}{2})$ with $D \geq 3$.*

Bibliography

[1] E. Bannai and T. Ito, *Algebraic Combinatorics I: Association Schemes*, Benjamin/Cummings, Menlo Park, 1984.

[2] B. Bollobás, *Modern Graph Theory*, Springer-Verlag, New York, 1998.

[3] A.E. Brouwer, A.M. Cohen, and A. Neumaier, *Distance-Regular Graphs*, Springer-Verlag, Berlin, 1989.

[4] A.E. Brouwer and H.A. Wilbrink, The structure of near polygons with quads., *Geometriae Dedicata*, 14(1883), 145-176.

[5] P. Delsarte, An algebraic approach to the association schemes of coding theorey, *Philips Research Reports Suppl.*, 10(1973).

[6] A. Gewirtz, Graphs with maximal even girth, *Canad. J. Math.*, 21(1969), 915–934.

[7] A. Gewirtz, The uniqueness of $g(2, 2, 10, 56)$, *Trans. New York Acad. Sci.*, 31(1969), 656–675.

[8] A. Hiraki, Strongly closed subgraphs in a regular thick near polygon, *European Journal of Combinatorics*, 20(8)(1999), 789–796.

[9] A.A. Ivanov and S.V. Shpectorov, Characterization of the association schemes of Hermitian forms over $GF(2^2)$, *Geometriae Dedicata*, 30(1989), 23–33.

[10] A. Jurišić, J. Koolen, and Š. Miklavič, Triangle- and pentagon-free distance-regular graphs with an eigenvalue multiplicity equal to the valency, *Journal of Combinatorial Theory, Series B*, 94(2)(2005), 245–258.

[11] J.H. Koolen and V. Moulton, There are finitely many triangle-free distance-regular graphs with degree 8, 9 or 10, *Journal of Algebraic Combinatorics*, 19(2)(2004), 205–217.

[12] D. A. Leonard, Orthogonal polynomials, duality and association schemes, *SIAM J. Math. Anal.*, 13(1982), 656–663.

[13] Š. Miklavič, Q-polynomial distance-regular graphs with $a_1 = 0$, *European Journal of Combinatorics*, 25(7)(2004), 911–920.

[14] Y. Pan, M. Lu, and C. Weng, Triangle-free distance-regular graphs, *Journal of Algebraic Combinatorics*, 27(2008), 23–34.

[15] Y. Pan and C. Weng, 3-bounded property in a triangle-free distance-regular graph, *European Journal of Combinatorics*, 29(2008), 1634–1642.

[16] Y. Pan and C. Weng, A note on triangle-free distance-regular graphs with $a_2 \neq 0$, Preprint.

[17] E.E. Shult and A. Yanushka, Near n-gons and line systems, *Geometriae Dedicata*, 9(1980), 1–72.

[18] H. Suzuki, On strongly closed subgraphs of highly regular graphs, *European Journal of Combinatorics*, 16(1995), 197–220.

[19] H. Suzuki, Strongly closed subgraphs of a distance-regular graph with geometric girth five, *Kyushu Journal of Mathematics*, 50(2)(1996), 371–384.

[20] P. Terwilliger, The subconstituent algebra of an association scheme (Part I), *Journal of Algebraic Combinatorics*, 1(1992), 363–388.

[21] P. Terwilliger, A new inequality for distance-regular graphs, *Discrete Mathematics*, 137(1995), 319–332.

[22] J.H. van Lint and R.M. Wilson, *A Course in Combinatorics*, Cambridge University Press, Cambridge, 1992.

[23] C. Weng, Kite-free P- and Q-polynomial schemes, *Graphs and Combinatorics*, 11(1995), 201-207.

[24] C. Weng, D-bounded distance-regular graphs, *European Journal of Combinatorics*, 18(1997), 211–229.

[25] C. Weng, Parallelogram-free distance-regular graphs, *Journal of Combinatorial Theory, Series B*, 71(2)(1997), 231–243.

[26] C. Weng, Weak-geodetically closed subgraphs in distance-regular graphs, *Graphs and Combinatorics*, 14(1998), 275–304.

[27] C. Weng, Classical distance-regular graphs of negative type, *Journal of Combinatorial Theory, Series B*, 76(1999), 93–116.