國立陽明交通大學
應用數學系
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Department of Applied Mathematics
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折疊超立方體圖的特威利格代數之研究

The Terwilliger Algebra of the Folded n-Cube and Its Applications

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中華民國一百一十一年六月
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碩士論文

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折疊超立方體圖的特威利格代數之研究

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摘　　要

由 Junie T. Go 所撰寫的論文 [1] 中，包含了許多與超立方體圖 Q_n 的特威利格代數 $\mathcal{T}(Q_n)$ 相關的有趣結果。我們受到這篇論文的啟發，推導出一些有關折疊超立方體圖 $\square_n$ 的特威利格代數 $\mathcal{T}(\square_n)$ 應用。我們可以分別利用超立方體圖 Q_n 或是 Q_{n-1} 以兩種不同的方法（對蹠商、對蹠合併）建造出折疊超立方體圖 $\square_n$。我們使用不可約的超立方體圖特威利格代數模 $\mathcal{T}(Q_n)$-模（或是 $\mathcal{T}(Q_{n-1})$-模）來建構不可約的折疊超立方體圖特威利格代數模 $\mathcal{T}(\square_n)$-模。我們發現給定直徑為 D 的折疊超立方體圖 $\square_n$ 及端點 r，將其鄰接矩陣 A 與對偶鄰接矩陣 A^* 限制在不可約的折疊超立方體圖特威利格代數模 $\mathcal{T}(\square_n)$-模下 A, A^* 會滿足以下關係式（係數 $a_{n,r}$ 根據 n, r 而變）：

$$A^2A^* - 2AA^*A + A^*A^2 = 4A^2 + 16A^* - a_{n,r}1,$$
$$A^{*2}A - 2A^*AA^* + AA^{*2} = 4(AA^* + A^*A) + 4(n-1)A.$$

對於非負整數 m，給定初始條件 $a_{1,0} = 4$ 以及 $a_{4m+2,2m+1} = -16(2m+1)$，可得係數 $a_{n,r}$ 滿足以下遞迴式：

$$a_{n,r} = \begin{cases} 4n^2, & r = 0, \\ a_{n-2,r-1} - 16 & 1 \le r \le \lfloor \frac{n}{2} \rfloor. \end{cases}$$

我們得到了由 A, A^* 根據以上關係式所生成代數之中心元素。

關鍵字：特威利格代數、折疊超立方體圖、阿斯基-威爾森關係

The Terwilliger Algebra of the Folded n-Cube and Its Applications

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Abstract

The paper [1], written by Junie T. Go, contains lots of interseting results of the Terwilliger algebra $\mathcal{T}(Q_n)$ of the hypercube Q_n. Inspired from the paper, we derive some results related to the Terwilliger algebra $\mathcal{T}(\square_n)$ of the folded n-cube $\square_n$. To construct the folded n-cube $\square_n$, we may either use Q_n by quotient method or use Q_{n-1} by merge method. We use the irreducible $\mathcal{T}(Q_n)$-module (or $\mathcal{T}(Q_{n-1})$-module) of the Terwilliger Algebra of hypercube to construct the irreducible $\mathcal{T}(\square_n)$-module of the Terwilliger Algebra of the folded n-cube. We discover that the adjacency matrix A and the dual adjacency matrix A^* of the irreducible $\mathcal{T}(\square_n)$-module with the diameter D of $\square_n$ and endpoint r of the Terwilliger Algebra of the folded n-cube satisfy the relations

$$A^2A^* - 2AA^*A + A^*A^2 = 4A^2 + 16A^* - a_{n,r}1,$$
$$A^{*2}A - 2A^*AA^* + AA^{*2} = 4(AA^* + A^*A) + 4(n-1)A,$$

for some scalar $a_{n,r}$. Suppose that $a_{1,0} = 4$ and $a_{4m+2,2m+1} = -16(2m+1)$ for $m \geq 0$. Then

$$a_{n,r} = \begin{cases} 4n^2, & r = 0, \\ a_{n-2,r-1} - 16 & 1 \leq r \leq \lfloor \frac{n}{2} \rfloor. \end{cases}$$

We obtain the center of the algebra generated by symbol A, A^* subject to the above two equations.

Keywords: Terwilliger algebra, folded n-cube, Askey-Wilson relation

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黃伯丞　謹誌於

國立陽明交通大學應用數學系

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1. Introduction

Throughout the thesis, Γ denotes a finite simple connected graph with nonempty vertex set X. Let ∂_Γ denote the path-length distance function of Γ. The *diameter* D of Γ is defined by

$$D = \max_{x,\, y \in X} \partial_\Gamma(x, y).$$

Given any $x \in X$ and let

$$\Gamma_i(x) = \{y \in X \mid \partial_\Gamma(x, y) = i\} \quad \text{for } i = 0, 1, \ldots, D.$$

For short, $\Gamma(x) = \Gamma_1(x)$. We call Γ *distance-regular* whenever for all $i \in \{0, 1, \ldots, D\}$ and all $x, y \in X$ with $\partial_\Gamma(x, y) = i$, the numbers

$$|\Gamma_i(x) \cap \Gamma(y)|, \quad |\Gamma_{i+1}(x) \cap \Gamma(y)|, \quad |\Gamma_{i-1}(x) \cap \Gamma(y)|$$

are independent of x and y.

For all $i = 0, 1, \ldots, D$, the *i-th distance matrix* $A_i \in \mathrm{Mat}_X(\mathbb{C})$ is defined by

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } \partial_\Gamma(x, y) = i, \\ 0 & \text{if } \partial_\Gamma(x, y) \neq i \end{cases}$$

for all $x, y \in X$. We abbreviate $A = A_1$ and A is called the *adjaceny matrix* of Γ.

Let n be a positive integer. The *hypercube* Q_n of dimension n, i.e. *n-cube*, is a graph with vertex set VQ_n and edge set EQ_n as follows

$$\begin{aligned} VQ_n &= \{a_1 a_2 \cdots a_n : \; a_i \in \mathbb{F}_2\}, \\ EQ_n &= \{xy : \; x, y \in VQ_n \text{ differ at exactly one position}\}, \end{aligned}$$

where $\mathbb{F}_2 = \{0, 1\}$ is the finite field of two elements. For $a \in \mathbb{F}_2$, let $\overline{a}$ denote $1 - a$, and for $x = a_1 a_2 \cdots a_n \in VQ_n$, let $\overline{x}$ denote $\overline{a}_1 \overline{a}_2 \cdots \overline{a}_n$.

The *folded n-cube* $\square_n$ is the graph obtained form Q_n with vertex set $V\square_n$ and edge set $E\square_n$ as follows

$$\begin{aligned} V\square_n =&\{\{a_1a_2\cdots a_{n-1}a_n, \overline{a}_1\overline{a}_2\cdots\overline{a}_{n-1}\overline{a}_n\} : a_i \in \mathbb{F}_2\},\\ E\square_n =&\{\{x,\overline{x}\}\{y,\overline{y}\} : x, y \in VQ_n,\\ &\quad x \text{ and } y \text{ (or } x \text{ and } \overline{y}) \text{ differ at exactly one position}\}. \end{aligned}$$

The method of constructing $\square_n$ above is called *quotient method*.

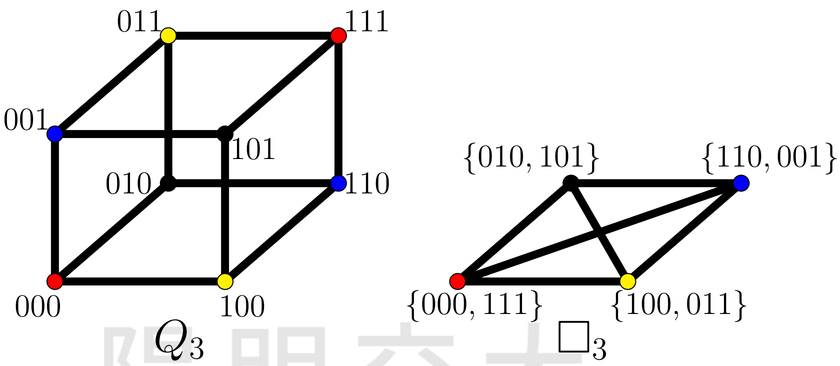

Figure 1.1: Using quotient method to gets $\square_3$ from Q_3

For $x = \{a_1a_2\cdots a_{n-1}a_n, \overline{a}_1\overline{a}_2\cdots\overline{a}_{n-1}\overline{a}_n\} \in V\square_n$, we might assume $a_n = 0$ and use $x' = a_1a_2\cdots a_{n-1}$ to represent x. Then for $x, y \in V\square_n$,

$$xy \in E\square_n \Leftrightarrow x'y' \in EQ_{n-1} \text{ or } y' = \overline{x'}.$$

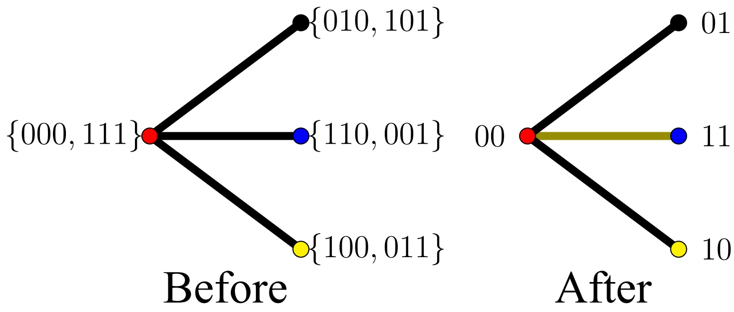

Figure 1.2: Neighbors of $\{000, 111\}$ in $\square_3$

Hence we can use another method to obtain $\square_n$ form Q_{n-1} by adding an edge between every pair of vertices at distance $n-1$, i.e.

$$
\begin{aligned}
V\square_n =& \{a_1 a_2 \cdots a_{n-1} : \; a_i \in \mathbb{F}_2\} = VQ_{n-1}, \\
E\square_n =& \{xy : \; x, y \in VQ_{n-1} \text{ differ at exactly one position}\} \\
& \cup \{x\overline{x} : \; x \in VQ_{n-1}\}.
\end{aligned}
$$

The method is called *merge method*. By using the merge method, the adjacency matrix of $\square_n$ could be written as

$$
A(\square_n) = A(Q_{n-1}) + A_{n-1}(Q_{n-1}).
$$

So we use either quotient method or merge method to obtain a folded n-cube from a hypercube.

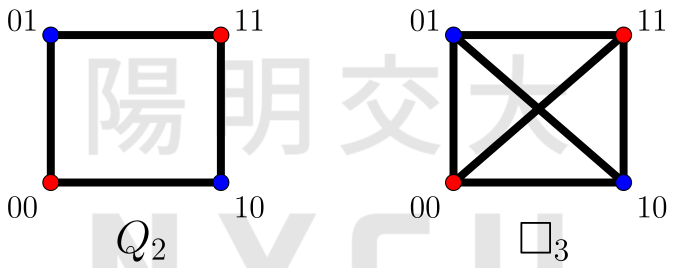

Figure 1.3: Using merge method to gets $\square_3$ from Q_2

In the thesis, every algebra is a *unital associative algebra*. Given a nonempty finite set Y, let $\mathrm{Mat}_Y(\mathbb{C})$ be the algebra consisting of the complex square matrices indexed by Y. The goal of this thesis is to decompose the Twilliger algebra of folded n-cube with the matrix representation of adjacency matrix and dual adjacency matrix.

The thesis is organized as follows: In Section 2, we introduce some background about Terwilliger algebra and facts of the hypercubes related to the folded n-cubes. In Section 3, we will apply tensor product and equtiable quotient to decompose the Terwilliger algebra of $\square_n$. In Section 4, we discover that the adjacency matrix A and the dual adjacency matrix A^* of the irreducible $\mathcal{T}(\square_n)$-module with the diameter D of $\square_n$ and endpoint r of the Terwilliger Algebra of the folded n-cube satisfy the relations

$$A^2A^* - 2AA^*A + A^*A^2 = 4A^2 + 16A^* - a_{D,r,n}1,$$
$$A^{*2}A - 2A^*AA^* + AA^{*2} = 4(AA^* + A^*A) + 4(n-1)A$$

for some scalar $a_{D,r,n}$. We obtain the center of the algebra generated by symbol A, A^* subject to the above two equations.

2. Preliminaries

In this section, we will introduce the definition of Terwilliger algebra [2]–[4] and the decomposition of the hypercube [1].

Definition 2.1. If Γ is distance-regular with diameter D, then for all $0 \leq i \leq D$, the numbers a_i, b_i, c_i defined by

$$a_i = |\Gamma_i(x) \cap \Gamma(y)|, \quad b_i = |\Gamma_{i+1}(x) \cap \Gamma(y)|, \quad c_i = |\Gamma_{i-1}(x) \cap \Gamma(y)|$$

for any $x, y \in X$ with $\partial_\Gamma(x, y) = i$ are called the *intersection numbers* of Γ.

If Γ is distance-regular, we observe that

$$AA_i = b_{i-1}A_{i-1} + a_iA_i + c_{i+1}A_{i+1} \tag{2.1}$$

for all $0 \leq i \leq D$, where $b_{-1}A_{-1}$ and $c_{D+1}A_{D+1}$ are interpreted as zero matrices in $\mathrm{Mat}_X(\mathbb{C})$.

Definition 2.2. The *Bose-Mesner algebra* $\mathcal{M}$ of Γ is the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by A_i for all $0 \leq i \leq D$.

Indeed,

$$\mathcal{M} = \mathrm{Span}_{\mathbb{C}}\{A_0, A_1, \ldots, A_{D-1}, A_D\},$$

where we denote $\mathrm{Span}_{\mathbb{C}}$ the algebra span under $\mathbb{C}$.
Inspired from the recurrence relation (2.1) of distance matrices with intersection numbers, we use it to obtain any i-th distance matrices by the adjacency matrix and the lemma below follows.

Lemma 2.3. *If Γ is distance-regular, then $\{A^0, A^1, \ldots, A^D\}$ is a basis for $\mathcal{M}$.*

From now on, we suppose that Γ is distance-regular. Since A is real symmetric and $\dim \mathcal{M} = D + 1$, it follows that A has $D + 1$ mutually distinct real eigenvalues $\theta_0, \theta_1, \ldots, \theta_D$. Consider the linear system:

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \theta_0 & \theta_1 & \cdots & \theta_D \\ \vdots & \vdots & & \vdots \\ \theta_0^D & \theta_1^D & \cdots & \theta_D^D \end{pmatrix} \begin{pmatrix} E_0 \\ E_1 \\ \vdots \\ E_D \end{pmatrix} = \begin{pmatrix} I \\ A \\ \vdots \\ A^D \end{pmatrix}.$$

Since the Vandermonde matrix of $\{\theta_i\}_{i=0}^D$ is invertible, there exist unique $E_0, E_1, \ldots, E_D \in \mathcal{M}$ such that $\{E_i\}_{i=0}^D$ form another basis for $\mathcal{M}$ [5, Section 4.1]. The matrix E_i is called the i^{th} *primitive idempotent* of Γ associated with θ_i for $0 \leq i \leq D$. Clearly,

$$\sum_{i=0}^{D} E_i = I.$$

Besides, from the linear system and [5, Section 4.1] we know that

$$\sum_{i=0}^{D} AE_i = A\Big(\sum_{i=0}^{D} E_i\Big) = A = \sum_{i=0}^{D} \theta_i E_i$$

and

$$AE_i = \theta_i E_i \quad \text{for all } 0 \leq i \leq D.$$

Definition 2.4. Let B and C be $n \times m$ matrices. Then we define $B \odot C$ to be the $n \times m$ matrix given by

$$(B \odot C)_{ij} = B_{ij} C_{ij}$$

for all $1 \leq i \leq n$ and $1 \leq j \leq m$.

A graph Γ is said to be *Q-polynomial* with respect to the ordering $\{E_i\}_{i=0}^D$ if there are scalars a_i^*, b_i^*, c_i^* such that $b_D^* = c_0^* = 0$, $b_{i-1}^* c_i^* \neq 0$ for all $0 \leq i \leq D$ and

$$E_1 \odot E_i = \frac{1}{|X|}(b_{i-1}^* E_{i-1} + a_i^* E_i + c_{i+1}^* E_{i+1})$$

for $0 \leq i \leq D$, where $b_{-1}^* E_{-1},\ c_{D+1}^* E_{D+1}$ are interpreted as zero matrices in $\mathrm{Mat}_X(\mathbb{C})$.
Fix $x \in X$. For all $0 \leq i \leq D$, let $E_i^* = E_i^*(x)$ denote the diagonal matrix in $\mathrm{Mat}_X(\mathbb{C})$ defined by

$$(E_i^*)_{yy} = \begin{cases} 1 & \text{if } \partial_\Gamma(x,y) = i, \\ 0 & \text{if } \partial_\Gamma(x,y) \neq i \end{cases}$$

for all $y \in X$. Since $E_i^* E_j^* = \delta_{ij} E_i^*$, the matrices $\{E_i^*\}_{i=0}^D$ are called the i^{th} *dual primitive idempotent* of Γ with respect to x.

Definition 2.5. The *dual Bose-Mesner algebra* $\mathcal{M}^*$ of Γ is the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by $E_i{}^*$

for all $0 \le i \le D$.

Indeed,

$$\mathcal{M}^* = \text{Span}_{\mathbb{C}}\{E_0^*, E_1^*, \ldots, E_D^*\}.$$

For all $0 \le i \le D$, the i^{th} *dual distance matrix* $A_i{}^* = A_i{}^*(x)$ is the diagonal matrix in $\text{Mat}_X(\mathbb{C})$ defined by

$$(A_i^*)_{yy} = |X|(E_i)_{xy} \quad \text{for all } y \in X.$$

The matrices $\{A_i^*\}_{i=0}^D$ form another basis for $\mathcal{M}^*$ [6, Section 3.1]. We abbreviate $A^* = A_1^*$ and A^* is called the *dual adjacency matrix* of Γ with respect to $x \in X$. Like Lemma 2.3, we use the recurrance relation of primitive idempotents $\{E_i\}_{i=0}^D$ to obtain any dual distance matrices by the dual adjacency matrix A^*.

Lemma 2.6. *([2, Lemma 3.11]) If Γ has Q-polynomial property, then A^* generates $\mathcal{M}^*$.*

After defining the Bose-Mesner algebra and dual Bose-Mesner algebra of Γ, we use them to construct the Terwilliger algebra of Γ and there is a useful theorem about it.

Definition 2.7. The *Terwilliger algebra* $\mathcal{T}(\Gamma)$ of Γ with respect to x is the subalgebra of $\text{Mat}_X(\mathbb{C})$ generated by $\mathcal{M}$ and $\mathcal{M}^*$.

Theorem 2.8. *If Γ is distance-regular with Q-polynomial property, then $\mathcal{T}(\Gamma) = \langle A, A^* \rangle$.*

2.1 The adjacency matrix and dual adjacency matrix of the hypercube

Let V denote the vector space consisting of all column vectors over $\mathbb{C}$ indexed by vertex set X where $|X| = 2^n$. We will use the isomorphism

$$\mathbb{C}^{2^n} \cong \mathbb{C}^{2^{\otimes n}} := \underbrace{\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{n \text{ times}}$$

with standard basis $\{e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_{n-1}} \otimes e_{i_n} \,:\, i_j \in \mathbb{F}_2\}$ of $(\mathbb{C}^2)^{\otimes n}$ to represent the matrices of hypercubes Q_n, folded $(n+1)$-cube $\square_{n+1}$ of merge method and $\{e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_{n-1}} \otimes e_{i_n} \,:\, i_j \in$

$\mathbb{F}_2, \{i_1 i_2 \cdots i_n 0, \bar{i}_1 \bar{i}_2 \cdots \bar{i}_n 1\} \in V\square_{n+1}\}$ of $(\mathbb{C}^2)^{\otimes n}$ to represent the matrices of folded $(n+1)$-cube $\square_{n+1}$ of quotient method. For the Terwilliger algebra $\mathcal{T}(\Gamma)$ of Γ, the vector space V has a natural $\mathcal{T}(\Gamma)$-module structure and it is called the standard $\mathcal{T}(\Gamma)$-module. If Γ is distance-regular with Q-polynomial property, then it is sufficient to use $A(\Gamma)$ and $A^*(\Gamma)$ to represent $\mathcal{T}(\Gamma)$. In this subsection, we show some lemmas related to the construction of some matrices of Q_n by induction on n [6, Section 3.2]. We will construct the related matrices of $\square_n$ in the next section based on these results.

Lemma 2.9. *Let $A_n(Q_n)$ be the n-th distance matrix of Q_n. Then for $n \geq 2$, we have*

$$A_n(Q_n) = A_{n-1}(Q_{n-1}) \otimes A(Q_1).$$

Lemma 2.10. *Let $A(Q_n)$ be the adjacency matrix of Q_n. Then $A(Q_1) = J_2 - I_2$ and for $n \geq 2$,*

$$A(Q_n) = A(Q_{n-1}) \otimes I_2 + I_{2^{n-1}} \otimes A(Q_1).$$

Lemma 2.11. *The eigenvalues of $A(Q_n)$ are*

$$\theta_i(Q_n) = n - 2i$$

with corresponding multiplicity $m_i = \binom{n}{i}$ for $i = 0, 1, \ldots, n$.

Lemma 2.12. *Let $E_i(Q_n)$ be the i^{th} primitive idempotent of Q_n associated with θ_i for $i = 0, 1, \ldots, n$, where θ_i is same as the eigenvalue of $A(Q_n)$ in Lemma 2.11. Then $E_0(Q_1) = \frac{1}{2}J_2, E_1(Q_1) = I_2 - \frac{1}{2}J_2$ and for $i = 0, 1, \ldots, n$,*

$$E_i(Q_n) = E_i(Q_{n-1}) \otimes E_0(Q_1) + E_{i-1}(Q_{n-1}) \otimes E_1(Q_1). \tag{2.2}$$

Note that $E_{-1}(Q_{n-1})$ and $E_n(Q_{n-1})$ are interpreted as the zero maps.

Lemma 2.13. *Let $A^*(Q_n)$ be the dual adjacency matrix of Q_n with respect to a vertex $x \in Q_n$. Then $A^*(Q_1) = \mathrm{diag}(1, -1)$ and for $n \geq 2$,*

$$A^*(Q_n) = A^*(Q_{n-1}) \otimes I_2 + I_{2^{n-1}} \otimes A^*(Q_1).$$

Lemma 2.14. *The eigenvalues of $A^*(Q_n)$ are*

$$\theta_i^*(Q_n) = n - 2i$$

with corresponding multiplicity $m_i = \binom{n}{i}$ for $i = 0, 1, \ldots, n$.

2.2 The Terwilliger algebra of the hypercube

The hypercube Q_n has vertex set $VQ_n = F_2^n$ and for $x, y \in VQ_n$, there is an edge between x and y if x and y differ at exactly one position. Hence, each vertex of Q_n has n neighbors. Note that the intersection numbers of Q_n are

$$a_i(Q_n) = 0, \quad b_i(Q_n) = n - i, \quad c_i(Q_n) = i$$

for all $i = 0, 1, \ldots, n$. Recall the following results in [1], we have all the irreducible modules of the Terwilliger algebra $\mathcal{T}(Q_n)$ of Q_n and know the dimension of the Terwilliger algebra of the hypercube.

Theorem 2.15. *For each $0 \le r \le \lfloor \frac{n}{2} \rfloor$, up to isomorphism there exists a unique irreducible $\mathcal{T}(Q_n)$-submodule W_r of the standard module $V := \mathbb{C}^{VQ_n}$ such that $W_r \cap E_r^*(Q_n)V \neq 0$ and $W_r \cap E_i^*(Q_n)V = 0$ for $i < r$. Moreover W_r has a basis $\beta = (w_0, w_1, \ldots, w_{n-2r})$, where $w_i \in E_{r+i}^*(Q_n)W_r$, such that*

$$[A(Q_n)]_\beta = \begin{pmatrix} 0 & n-2r & & & 0 \\ 1 & 0 & n-2r-1 & & \\ & 2 & 0 & \ddots & \\ & & \ddots & \ddots & 1 \\ 0 & & & n-2r & 0 \end{pmatrix},$$

$$[A^*(Q_n)]_\beta = \text{diag}(n-2r, n-2r-2, \ldots, 2r-n).$$

Theorem 2.16. *For any positive integer n,*

$$\dim \mathcal{T}(Q_n) = \binom{n+3}{3}.$$

3. The Terwilliger algebra of the folded n-cube

From [4, p.196-197], we know that the folded n-cube is a distance-regular graph with Q-polynomial property. We want to decompose the Terwilliger algebra of the folded n-cube in this section. Before we construct the adjacency matrix and dual adjacency matrix of $\square_n$, there are some properties of $\square_n$ worth to mention. Let $\{x, \overline{x}\}, \{y, \overline{y}\}$ be vertices in $\square_n$, where $x, y \in V(Q_n)$. Then either $\partial_{Q_n}(x, y) \leq \frac{n}{2}$ or $\partial_{Q_n}(x, \overline{y}) \leq \frac{n}{2}$. Hence, $\partial_{\square_n}(\{x, \overline{x}\}, \{y, \overline{y}\}) \leq \frac{n}{2}$. If n is even with $\partial_{Q_n}(x, y) = \frac{n}{2}$, we have $\partial_{Q_n}(x, \overline{y}) = \frac{n}{2}$, so the diameter of $\square_n$ is $\frac{n}{2}$. If n is odd with $\partial_{Q_n}(x, y) = \frac{n-1}{2}$, we have $\partial_{Q_n}(x, \overline{y}) = \frac{n+1}{2}$, $\partial_{Q_n}(\overline{x}, y) = \frac{n+1}{2}$, $\partial_{Q_n}(\overline{x}, \overline{y}) = \frac{n-1}{2}$, so the diameter D of $\square_n$ is $\frac{n-1}{2}$. In summary, $D(\square_n) = \lfloor \frac{n}{2} \rfloor$. A similar argument shows that

$$a_i(\square_n) = 0, \quad b_i(\square_n) = n - i, \quad c_i(\square_n) = i,$$

same as those of Q_n for all $i \leq D$, with one exception in the case that n is odd and $i = D = \frac{n-1}{2}$, $a_D(\square_n) = D + 1$, $b_D(\square_n) = 0$ and $c_D(\square_n) = D$.

3.1 The adjacency matrix and dual adjacency matrix of the folded n-cube

In this subsection, we construct the adjacency matrix and dual adjacency matrix of the folded n-cube $\square_n$ from the hypercubes Q_m for $m \leq n$.

Remark 3.1. $\square_2$ is isomorphic to Q_1.

By Remark 3.1, we have $A(\square_2) = J_2 - I_2$ and $E(\square_2) = I_2 - \frac{1}{2}J_2$. Next, we want to find recurrence relations for $A(\square_n)$ and $A^*(\square_n)$.

Lemma 3.2. *Let $A(\square_n)$ be the adjacency matrix of $\square_n$. For $n \geq 3$, we have*

$$\begin{aligned} A(\square_n) &= A(Q_{n-1}) + A_{n-1}(Q_{n-1}) \\ &= A(Q_{n-2}) \otimes I_2 + (I_{2^{n-2}} + A_{n-2}(Q_{n-2})) \otimes A(\square_2). \end{aligned}$$

Proof. Immediately from Lemma 2.9 and Lemma 2.10. $\square$

Remark 3.3. For $n \geq 4$, we have

$$A(\square_n) = A(\square_{n-1}) \otimes I_2 + I_{2^{n-2}} \otimes A(\square_2) - A_{n-2}(Q_{n-2}) \otimes 2E_1(\square_2).$$

Lemma 3.4. *For $n \geq 3$, the eigenvalues of $A(\square_n)$ are*

$$\theta_i(\square_n) = n - 4i$$

with corresponding multiplicity $m_i = \binom{n}{2i}$ for $i = 0, 1, \ldots, D = \lfloor \frac{n}{2} \rfloor$.

Proof. When $n = 3$, since $A(\square_3) = J_4 - I_4$, the eigenvalues of $A(\square_3)$ are $3, -1$ with corresponding multiplicity $1, 3$, respectively. By Lemma 2.11, we know the eigenvalues of $A(Q_{n-1})$. Let v be an eigenvector of $A(Q_{n-1})$ corresponding to the eigenvalue $\theta_j(Q_{n-1}) = n-1-2j$, where $0 \leq j \leq n-1$. Then

$$\begin{aligned} &(A(Q_{n-1}) + A_{n-1}(Q_{n-1}))v \\ =&A(Q_{n-1})v + A_{n-1}(Q_{n-1})v \\ =&(n-1-2j)v + A_1^{\otimes n-1}(Q_1)(v_1 \otimes v_2 \otimes \cdots v_{n-1}) \\ =&(n-1-2j)v + (-1)^j v \\ =&\begin{cases} (n-2j)v, & \text{for even } j, \\ (n-2j-2)v, & \text{for odd } j, \end{cases} \\ =&(n-4i)v, \end{aligned}$$

where $i = \lceil \frac{j}{2} \rceil$. So $\theta_i(\square_n) = n - 4i$ with corresponding multiplicity $\binom{n-1}{2i-1} + \binom{n-1}{2i} = \binom{n}{2i}$ for $i = 0, 1, \ldots, \lfloor \frac{n}{2} \rfloor$. $\square$

Lemma 3.5. *Let $E_i(\square_n)$ be the i^{th} primitive idempotent of $\square_n$ associated with $\theta_i(\square_n)$, same as the eigenvalue of $A(\square_n)$ in Lemma 3.2, for $i = 0, 1, \ldots, D = \lfloor \frac{n}{2} \rfloor$. For $n \geq 3$ and $0 \leq i \leq D$, we have*

$$E_i(\square_n) = E_{2i-1}(Q_{n-1}) + E_{2i}(Q_{n-1}),$$

where $E_j(Q_m)$ with $j < 0$ or $j > m$ are zero map for $m \geq 1$.

Proof. By Lemma 2.12, since

$$\begin{aligned}
&A(\square_n)(E_{2i-1}(Q_{n-1}) + E_{2i}(Q_{n-1})) \\
=&(A(Q_{n-1}) + A_{n-1}(Q_{n-1}))(E_{2i-1}(Q_{n-1}) + E_{2i}(Q_{n-1})) \\
=&(n-4i+1)E_{2i-1}(Q_{n-1}) + (n-4i-1)E_{2i}(Q_{n-1}) - E_{2i-1}(Q_{n-1}) + E_{2i}(Q_{n-1}) \\
=&(n-4i)(E_{2i-1}(Q_{n-1}) + E_{2i}(Q_{n-1})) \\
=&\theta_i(\square_n)(E_{2i-1}(Q_{n-1}) + E_{2i}(Q_{n-1}))
\end{aligned}$$

for $0 \leq i \leq \lfloor \frac{n}{2} \rfloor$ and

$$\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor}(E_{2i-1}(Q_{n-1}) + E_{2i}(Q_{n-1})) = \sum_{i=0}^{n-1} E_i(Q_{n-1}) = I_{2^{n-1}},$$

the proof follows. □

Remark 3.6. Besides, for $n \geq 3$ and $0 \leq i \leq D$, we have

$$\begin{aligned}
E_i(\square_n) =&(E_{2i-1}(Q_{n-2}) + E_{2i}(Q_{n-2})) \otimes E_0(\square_2) + (E_{2i-2}(Q_{n-2}) + E_{2i-1}(Q_{n-2})) \otimes E_1(\square_2) \\
=&E_i(\square_{n-1}) \otimes E_0(\square_2) + (E_{2i-2}(Q_{n-2}) + E_{2i-1}(Q_{n-2})) \otimes E_1(\square_2),
\end{aligned}$$

where $E_j(Q_m)$ with $j < 0$ or $j > m$ are zero map for $m \geq 1$.

Lemma 3.7. *Let $A^*(\square_n)$ be the dual adjacency matrix of $\square_n$ with respect to a vertex $x \in \square_n$. For $n \geq 3$, we have*

$$\begin{aligned}
A^*(\square_n) &= A_1^*(Q_{n-1}) + A_2^*(Q_{n-1}) \\
&= A^*(\square_{n-1}) \otimes I_2 + (I_{2^{n-2}} + A^*(Q_{n-2})) \otimes A^*(\square_2).
\end{aligned}$$

Proof. Given $y \in \mathbb{F}_2^{n-1}$, let c_y denote the coefficient of $\hat{y}$ in $E_1(Q_{n-1})\hat{0}^{\otimes n-1}$ with respect to the basis $\{\hat{x}\}_{x\in\mathbb{F}_2^{n-1}}$ for VQ_{n-1} and d_y denote the coefficient of $\hat{y}$ in $E_1(\square_n)\hat{0}^{\otimes n-1}$ with respect to the basis $\{\hat{x}\}_{x\in\mathbb{F}_2^{n-1}}$ for $V\square_n$. We have

$$A^*(Q_{n-1})\hat{y} = 2^{n-1}c_y\hat{y},\ A^*(\square_n)\hat{y} = 2^{n-1}d_y\hat{y}.$$

Since

$$\begin{aligned}E_1(\square_n) &= E_1(Q_{n-1}) + E_2(Q_{n-1})\\ &= E_1(\square_{n-1})\otimes E_0(\square_2) + (E_0(Q_{n-2}) + E_1(Q_{n-2}))\otimes E_1(\square_2)\\ &= E_1(\square_{n-1})\otimes \frac{1}{2}J_2 + (\frac{1}{2^{n-2}}J_{2^{n-2}} + E_1(Q_{n-2}))\otimes E_1(\square_2),\end{aligned}$$

we have

$$d_y = \frac{1}{2}d_{(y_1,\ldots,y_{n-1})} + (\frac{1}{2^{n-2}} + c_{(y_1,\ldots,y_{n-1})})d_{y_n}.$$

Hence

$$\begin{aligned}A^*(\square_n)\hat{y} = 2^{n-1}d_y\hat{y} &= 2^{n-1}(\frac{1}{2}d_{(y_1,\ldots,y_{n-1})} + (\frac{1}{2^{n-2}} + c_{(y_1,\ldots,y_{n-1})})d_{y_n})\hat{y}\\ &= A^*(\square_{n-1})(\hat{y}_1\otimes\cdots\otimes\hat{y}_{n-1})\otimes\hat{y}_n + \hat{y}_1\otimes\cdots\otimes\hat{y}_{n-1}\otimes A^*(\square_2)\hat{y}_n\\ &\quad + A^*(Q_{n-2})(\hat{y}_1\otimes\cdots\otimes\hat{y}_{n-1})\otimes A^*(\square_2)\hat{y}_n.\end{aligned}$$

□

Let $\theta_0^*(\square_n) > \cdots > \theta^*_{\lfloor\frac{n}{2}\rfloor}(\square_n)$ be the eigenvalues of $A^*(\square_n)$ and $\theta_i^*(\square_n)$ have multiplicity m_i for all $0 \le i \le \lfloor\frac{n}{2}\rfloor$. By Lemma 3.7, the largest eigenvalue $\theta_0^*(\square_n)$ of $A^*(\square_n)$ is $\binom{n}{2}$ and the smallest eigenvalue $\theta^*_{\lfloor\frac{n}{2}\rfloor}(\square_n)$ of $A^*(\square_n)$ is $-\lfloor\frac{n}{2}\rfloor$. For $n \ge 2$, eigenvalues of $A^*(\square_{n+1})$ are $\theta_i^{*m_i}(\square_n) \pm (n-2i)$ for $0 \le i \le \lfloor\frac{n}{2}\rfloor$. Thus we derive Lemma 3.8.

Lemma 3.8. *The eigenvalues of* $A^*(\square_n)$ *are*

$$\theta_i^*(\square_n) = \binom{n}{2} - 2i(n-i)$$

with corresponding multiplicity $m_i = \binom{n-1}{i} + \binom{n-1}{n-i}$ *for* $i = 0, 1, \ldots, D = \lfloor\frac{n}{2}\rfloor$.

Proof. The differences of $\{\theta^*_i(\square_n)\}_{i=1}^D$ are $-2(n-1), -2(n-3), \ldots, -2(n+1-2D)$. So

$$\theta^*_i(\square_n) = \binom{n}{2} - 2(n \cdot i - i^2) = \binom{n}{2} - 2i(n-i)$$

for $i = 0, 1, \ldots, D$. The multiplicity m_i of $\theta^*_i(\square_n)$ depends on the number of vertices in distance i from a given vertex in $\square_n$, so $m_i = \binom{n-1}{i} + \binom{n-1}{n-i}$ by the merge method constrution. $\square$

n	eigenvalues with multiplicity
2	$1, -1$
3	$3, -1_3$
4	$6, 0_4, -2_3$
5	$10, 2_5, -2_{10}$
6	$15, 5_6, -1_{15}, -3_{10}$

Table 3.1: Eigenvalues of $A^*(\square_n)$ for $2 \leq n \leq 6$

We take a look on irreducible modules between the hypercube and the folded n-cube. With quotient method from Q_n to $\square_n$, for odd n, every irreducible $\mathcal{T}(Q_n)$-module W_r is of even dimension for $0 \leq r \leq \lfloor \frac{n}{2} \rfloor$. Deduce W_r to $\mathcal{T}(\square_n)$-module $\widetilde{W_r}$, $\widetilde{W_r}$ is of dimension $\lfloor \frac{n}{2} \rfloor - r + 1$. For even n, every irreducible $\mathcal{T}(Q_n)$-module W_r is of odd dimension for $0 \leq r \leq \lfloor \frac{n}{2} \rfloor$. Deduce W_r to $\mathcal{T}(\square_n)$-module $\widetilde{W_r}$, $\widetilde{W_r}$ is of dimension $\lfloor \frac{n}{2} \rfloor - r + 1$ if r is even and $\widetilde{W_r}$ is of dimension $\lfloor \frac{n}{2} \rfloor - r$ if r is odd. On the other hand, with merge method from Q_{n-1} to $\square_n$, the irreducible $\mathcal{T}(Q_{n-1})$-module W_r would be decomposed into two nonisomorphic $\mathcal{T}(\square_n)$-modules for $0 \leq r < \lfloor \frac{n}{2} \rfloor$. For odd n, W_r is decomposed into one $\mathcal{T}(\square_n)$-module of dimension $\lfloor \frac{n}{2} \rfloor - r + 1$ and one $\mathcal{T}(\square_n)$-module of dimension $\lfloor \frac{n}{2} \rfloor - r$. For even n, W_r is decomposed into one $\mathcal{T}(\square_n)$-module of dimension $\lfloor \frac{n}{2} \rfloor - r + 1$ and one $\mathcal{T}(\square_n)$-module of dimension $\lfloor \frac{n}{2} \rfloor - r - 1$ if r is even and two nonisomorphic $\mathcal{T}(\square_n)$-modules of dimension $\lfloor \frac{n}{2} \rfloor - r$ if r is odd.

By Proposition 4.1 and Proposition 4.9 in [7], we have known that the isomorphism class of irreducible $\mathcal{T}(\square_n)$-module is determined only by r. To compare with all irreducible $\mathcal{T}(Q_n)$-modules in Table 3.2 made from the results of [1], we summerize the dimension of all irreducible $\mathcal{T}(\square_n)$-modules in Table 3.3 from the results of [7].

	dimensions with multiplicity
Q_1	2
Q_2	$3, 1$
Q_3	$4, 2^2$
Q_4	$5, 3^3, 1^2$
Q_5	$6, 4^4, 2^5$
Q_6	$7, 5^5, 3^9, 1^5$

Table 3.2: The dimension and multiplicity of the nonisomorphic irreducible $\mathcal{T}(Q_n)$-module of standard $\mathcal{T}(Q_n)$-module for $1 \leq n \leq 6$

	dimensions with multiplicity
$\square_2$	2
$\square_3$	$2, 1^2$
$\square_4$	$3, 1^3, 1^2$
$\square_5$	$3, 2^4, 1^5$
$\square_6$	$4, 2^5, 2^9$
$\square_7$	$4, 3^6, 2^{14}, 1^{14}$

Table 3.3: The dimension and multiplicity of the nonisomorphic irreducible $\mathcal{T}(\square_n)$-module of standard $\mathcal{T}(\square_n)$-module for $2 \leq n \leq 7$

Consider the merge method of $\square_n$, we need to find out the matrix representation of $A_{n-1}(Q_{n-1})$ before we decompose the Terwilliger algebra of the folded n-cube.

Theorem 3.9. *For each $0 \leq r \leq \lfloor \frac{n}{2} \rfloor$, the irreducible $\mathcal{T}(Q_n)$-module W_r with basis β from Theorem 2.15, we have*

$$[A_n(Q_n)]_\beta = \begin{pmatrix} 0 & & (-1)^r \\ & \iddots & \\ (-1)^r & & 0 \end{pmatrix}.$$

Proof. By the intersection numbers of Q_n, we have the following linear system,

$$\begin{aligned} A(Q_n)^2 &= nI + 2A_2(Q_n), \\ A(Q_n)A_2(Q_n) &= (n-1)A(Q_n) + 3A_3(Q_n), \\ &\vdots \\ A(Q_n)A_{n-1}(Q_n) &= 2A_{n-2}(Q_n) + nA_n(Q_n). \end{aligned}$$

By Theorem 2.15, we know that the matrix form of $[A(Q_n)]_\beta$ with respect to a given basis β of W_r. Thus we use the recurrence relations of distance matrices to get the form of $[A_n(Q_n)]_\beta$. $\square$

Theorem 3.10. *Let D be the diameter of $\square_n$ and $\{\theta_i^*\}_{i=0}^D$ be the eigenvalues of the dual adjacency matrix $A^*(\square_n)$ with $\theta_0^* > \theta_1^* > \cdots > \theta_D^*$.*

1. *When n is odd, for each $0 \le r \le D$, up to isomorphism there exists a unique irreducible $\mathcal{T}(\square_n)$-submodule W_r of the standard module $V := \mathbb{C}^{V\square_n}$ such that $W_r \cap E_r^*(\square_n)V \neq 0$ and $W_r \cap E_i^*(\square_n)V = 0$ for $i < r$. Moreover W_r has a basis $\gamma = (w_0, w_1, \ldots, w_{D-r})$, where $w_i \in E_{r+i}^*(\square_n)W_r$, such that*

$$[A(\square_n)]_\gamma = \begin{pmatrix} 0 & n-2r & & & 0 \\ 1 & 0 & n-2r-1 & & \\ & 2 & \ddots & \ddots & \\ & & \ddots & 0 & D-r+2 \\ 0 & & & D-r & (-1)^r(D-r+1) \end{pmatrix},$$

$$[A^*(\square_n)]_\gamma = \mathrm{diag}(\theta_r^*, \theta_{r+1}^*, \ldots, \theta_D^*).$$

2. *When n is even and $0 \le r \le D$,*

   (a) *for even r, W_r is with basis $\gamma = (w_0, w_1, \ldots, w_{D-r})$, where $w_i \in E_{r+i}^*(\square_n)W_r$, such that*

$$[A(\square_n)]_\gamma = \begin{pmatrix} 0 & 2(D-r) & & & & 0 \\ 1 & 0 & 2(D-r)-1 & & & \\ & 2 & 0 & \ddots & & \\ & & \ddots & \ddots & D-r+2 & \\ & & & D-r-1 & 0 & D-r+1 \\ 0 & & & & 2(D-r) & 0 \end{pmatrix},$$

$$[A^*(\square_n)]_\gamma = \mathrm{diag}(\theta_r^*, \theta_{r+1}^*, \ldots, \theta_D^*).$$

   Especially, $[A(\square_n)]_\gamma$ is same as the matrix representation of $[A(\square_n)]_{\gamma'}$ on basis $\gamma' = (w_0, w_1, \ldots, w_{D-r})$ of irreducible $\mathcal{T}(\square_{2(D-r)})$-module $\widehat{W}_0$.

(b) For odd r, W_r *has a basis* $\beta = (w_0, w_1, \ldots, w_{D-r-1})$*, where* $w_i \in E^*_{r+i}(\square_n)W_r$*, such that*

$$[A(\square_n)]_\gamma = \begin{pmatrix} 0 & 2(D-r) & & & 0 \\ 1 & 0 & 2(D-r)-1 & & \\ & 2 & 0 & \ddots & \\ & & \ddots & \ddots & D-r+2 \\ 0 & & & D-r-1 & 0 \end{pmatrix},$$

$$[A^*(\square_n)]_\gamma = \mathrm{diag}(\theta^*_r, \theta^*_{r+1}, \ldots, \theta^*_{D-1}).$$

Especially, $[A(\square_n)]_\gamma$ *is same as the left-top* $(D-r, D-r)$ *block of* $[A(\square_n)]_{\gamma'}$ *on basis* $\gamma' = (w_0, w_1, \ldots, w_{D-r})$ *of irreducible* $\mathcal{T}(\square_{2(D-r)})$*-module* $\widehat{W_0}$*.*

(c) If D is odd, then W_{D-1} *is with basis* $\gamma = (w_0, w_1)$*, where* $w_i \in E^*_{D-1+i}(\square_n)W_r$*, such that* $[A(\square_n)]_\gamma = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$ *and* $[A^*(\square_n)]_\gamma = \mathrm{diag}(\theta^*_{D-1}, \theta^*_D)$ *but* W_D *is of dimension 0.*

(d) If D is even, then both W_{D-1} *and* W_D *are of dimension 1 and* $[A(\square_n)]_\gamma = (0)$ *with* $[A^*(\square_n)]_\gamma = (\theta^*_{D-1})$ *and* $[A^*(\square_n)]_\gamma = (\theta^*_D)$*, respectively.*

Proof. We apply algorithms to construct the basis of W_r from Theorem 2.15.

(Merge) Let $\widetilde{W_r}$ be the irreducible $\mathcal{T}(Q_{n-1})$-module at endpoint r. Let $\beta = (w_0, w_1, \ldots, w_{n-2r-1})$ be the basis of $\widetilde{W_r}$ mentioned in Theorem 2.15 and Theorem 3.9. Then we have

$$[A(Q_{n-1})]_\beta = \begin{pmatrix} 0 & n-2r-1 & & & 0 \\ 1 & 0 & n-2r-2 & & \\ & 2 & 0 & \ddots & \\ & & \ddots & \ddots & 1 \\ 0 & & & n-2r-1 & 0 \end{pmatrix},$$

$$[A_{n-1}(Q_{n-1})]_\beta = \begin{pmatrix} 0 & & (-1)^r \\ & \iddots & \\ (-1)^r & & 0 \end{pmatrix}.$$

1. When n is odd, first we need to rearrange the basis from β to

$$\widetilde{\beta} = (w_0, w_1, w_{2D-2r}, \ldots, w_{D-r}, w_{D-r+1}).$$

   Then we do equitable quotient from $[A(Q_{n-1}) + A_{n-1}(Q_{n-1})]_{\widetilde{\beta}}$ to $[X]_\gamma$ for some γ.

   (a) If r is even, then $\gamma = (w_0, w_1 + w_{2D-2r}, \ldots, w_{D-r} + w_{D-r+1})$.

   (b) If r is odd, then $\gamma = (w_0, w_1 - w_{2D-2r}, \ldots, w_{D-r} - w_{D-r+1})$.

   Restrict γ from $\mathbb{C}^{VQ_{n-1}}$ to $\mathbb{C}^{V\square_n}$. Since the dimension of Span(γ) is $D - r + 1$, Span(γ) $\cap$ $\mathrm{E}^*_\mathrm{r}(\square_\mathrm{n})\mathrm{V} \neq 0$ and Span(γ) $\cap$ $\mathrm{E}^*_\mathrm{i}(\square_\mathrm{n})\mathrm{V} = 0$ for $i < r$, $[A^*]_\gamma$ follows the dual distance matrices and we obtain the desired result.

2. When n is even, first we need to rearrange the basis from β to

$$\widetilde{\beta} = (w_0, w_1, w_{2D-2r-1}, \ldots, w_{D-r-1}, w_{D-r+1}, w_{D-r}).$$

   Then we do equitable quotient from $[A(Q_{n-1}) + A_{n-1}(Q_{n-1})]_{\widetilde{\beta}}$ to $[X]_\gamma$ for some γ.

   (a) If r is even, then $\gamma = (w_0, w_1 + w_{2D-2r-1}, \ldots, w_{D-r-1} + w_{D-r+1}, w_{D-r})$.

   (b) If r is odd, then $\gamma = (w_0, w_1 - w_{2D-2r-1}, \ldots, w_{D-r-1} - w_{D-r+1})$.

   Restrict γ from $\mathbb{C}^{VQ_{n-1}}$ to $\mathbb{C}^{V\square_n}$. Since the dimension of Span(γ) is $D - r + 1$ for even r and $D - r$ for odd r, Span(γ) $\cap$ $\mathrm{E}^*_\mathrm{r}(\square_\mathrm{n})\mathrm{V} \neq 0$ and Span(γ) $\cap$ $\mathrm{E}^*_\mathrm{i}(\square_\mathrm{n})\mathrm{V} = 0$ for $i < r$, $[A^*]_\gamma$ follows the dual distance matrices and we obtain the desired result.

(Quotient) Let $\widetilde{W_r}$ be the irreducible $\mathcal{T}(Q_n)$-module at endpoint r. Let $\beta = (w_0, w_1, \ldots, w_{n-2r})$ be the basis of $\widetilde{W_r}$ mentioned in Theorem 2.15. Then we have

$$[A(Q_n)]_\beta = \begin{pmatrix} 0 & n-2r & & & 0 \\ 1 & 0 & n-2r-1 & & \\ & 2 & 0 & \ddots & \\ & & \ddots & \ddots & 1 \\ 0 & & & n-2r & 0 \end{pmatrix}.$$

1. When n is odd, first we need to rearrange the basis from β to

$$\widetilde{\beta} = (w_0, w_{2D-2r+1}, w_1, w_{2D-2r}, \ldots, w_{D-r}, w_{D-r+1}).$$

   Then we do equitable quotient from $[A(Q_n)]_{\widetilde{\beta}}$ to $[X]_\gamma$ for some γ.

   (a) If r is even, then $\gamma = (w_0 + w_{2D-2r+1}, w_1 + w_{2D-2r}, \ldots, w_{D-r} + w_{D-r+1})$.

   (b) If r is odd, then $\gamma = (w_0 - w_{2D-2r+1}, w_1 - w_{2D-2r}, \ldots, w_{D-r} - w_{D-r+1})$.

   Restrict γ from $\mathbb{C}^{VQ_n}$ to $\mathbb{C}^{V\square_n}$ (For all vectors in γ, the values of antipodes are the same.). Since the dimension of Span(γ) is $D - r + 1$, Span(γ) $\cap$ $\mathrm{E}_\mathrm{r}^*(\square_\mathrm{n})\mathrm{V} \neq 0$ and Span(γ) $\cap$ $\mathrm{E}_\mathrm{i}^*(\square_\mathrm{n})\mathrm{V} = 0$ for $i < r$, $[A^*]_\gamma$ follows the dual distance matrices and we obtain the desired result.

2. When n is even, first we need to rearrange the basis from β to

$$\widetilde{\beta} = (w_0, w_{2D-2r}, w_1, w_{2D-2r-1}, \ldots, w_{D-r-1}, w_{D-r+1}, w_{D-r}).$$

   Then we do equitable quotient from $[A(Q_n)]_{\widetilde{\beta}}$ to $[X]_\gamma$ for some γ.

   (a) If r is even, then $\gamma = (w_0 + w_{2D-2r}, w_1 + w_{2D-2r-1}, \ldots, w_{D-r-1} + w_{D-r+1}, w_{D-r})$.

   (b) If r is odd, then $\gamma = (w_0 - w_{2D-2r}, w_1 - w_{2D-2r-1}, \ldots, w_{D-r-1} - w_{D-r+1})$.

   Restrict γ from $\mathbb{C}^{VQ_n}$ to $\mathbb{C}^{V\square_n}$ (For all vectors in γ, the values of antipodes are the same.). Since the dimension of Span(γ) is $D - r + 1$ for even r and $D - r$ for odd r, Span(γ) $\cap$ $\mathrm{E}_\mathrm{r}^*(\square_\mathrm{n})\mathrm{V} \neq 0$ and Span(γ) $\cap$ $\mathrm{E}_\mathrm{i}^*(\square_\mathrm{n})\mathrm{V} = 0$ for $i < r$, $[A^*]_\gamma$ follows the dual distance matrices and we obtain the desired result.

□

Here are some examples to construct a basis γ of an irreducible $\mathcal{T}(\square_n)$-submodule W_r of $\mathbb{C}^{V\square_n}$.

Example 3.11. We want to construct an irreducible $\mathcal{T}(\square_4)$-submodule W_0, there are two methods:

1. With merge method, let $\beta = (e_{000}, e_{100} + e_{010} + e_{001}, e_{011} + e_{101} + e_{110}, e_{111})$ be a basis of an

irreducible $\mathcal{T}(Q_3)$-submodule $\widetilde{W_0}$. Then by Theorem 2.15 and Theorem 3.9,

$$[A(Q_3)]_\beta + [A_3(Q_3)]_\beta = \begin{pmatrix} 0 & 3 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 3 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 3 & 0 & 1 \\ 1 & 0 & 3 & 0 \\ 0 & 3 & 0 & 1 \\ 1 & 0 & 3 & 0 \end{pmatrix}.$$

Changing the ordering of β to $\widetilde{\beta}$, we have

$$[A(Q_3) + A_3(Q_3)]_{\widetilde{\beta}} = \left(\begin{array}{c|cc|c} 0 & 3 & 1 & 0 \\ \hline 1 & 0 & 0 & 3 \\ 1 & 0 & 0 & 3 \\ \hline 0 & 3 & 1 & 0 \end{array}\right),$$

where $\widetilde{\beta} = (e_{000}, e_{100} + e_{010} + e_{001}, e_{111}, e_{011} + e_{101} + e_{110})$. Restrict $\gamma = (e_{000}, e_{100} + e_{010} + e_{001} + e_{111}, e_{011} + e_{101} + e_{110})$ from $\mathbb{C}^{VQ_3}$ to $\mathbb{C}^{V\square_4}$, we have

$$[A(\square_4)]_\gamma = \begin{pmatrix} 0 & 4 & 0 \\ 1 & 0 & 3 \\ 0 & 4 & 0 \end{pmatrix}.$$

2. With quotient method, let $\beta = (e_{0000}, e_{1000} + e_{0100} + e_{0010} + e_{0001}, e_{1100} + e_{1010} + e_{1001} + e_{0110} + e_{0101} + e_{0011}, e_{0111} + e_{1011} + e_{1101} + e_{1110}, e_{1111})$ be a basis of an irreducible $\mathcal{T}(Q_4)$-submodule

$\widetilde{W_0}$. Then

$$[A(Q_4)]_\beta = \begin{pmatrix} 0 & 4 & 0 & 0 & 0 \\ 1 & 0 & 3 & 0 & 0 \\ 0 & 2 & 0 & 2 & 0 \\ 0 & 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & 4 & 0 \end{pmatrix}.$$

Changing the ordering of β to $\widetilde{\beta}$, we have

$$[A(Q_4)]_{\widetilde{\beta}} = \left(\begin{array}{cc|cc|c} 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ \hline 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 3 \\ \hline 0 & 0 & 2 & 2 & 0 \end{array}\right),$$

where $\widetilde{\beta} = (e_{0000}, e_{1111}, e_{1000} + e_{0100} + e_{0010} + e_{0001}, e_{0111} + e_{1011} + e_{1101} + e_{1110}, e_{1100} + e_{1010} + e_{1001} + e_{0110} + e_{0101} + e_{0011})$. Finally, restrict $\widetilde{\gamma} = (e_{0000} + e_{1111}, e_{1000} + e_{0100} + e_{0010} + e_{0001} + e_{0111} + e_{1011} + e_{1101} + e_{1110}, e_{1100} + e_{1010} + e_{1001} + e_{0110} + e_{0101} + e_{0011})$ from $\mathbb{C}^{VQ_4}$ to $\mathbb{C}^{V\square_4}$, we have

$$[A(\square_4)]_\gamma = \begin{pmatrix} 0 & 4 & 0 \\ 1 & 0 & 3 \\ 0 & 4 & 0 \end{pmatrix},$$

where $\gamma = (e_{\{0000,1111\}}, e_{\{1000,0111\}} + e_{\{0100,1011\}} + e_{\{0010,1101\}} + e_{\{0001,1110\}}, e_{\{1100,0011\}} + e_{\{1010,0101\}} + e_{\{1001,0110\}})$.

Example 3.12. We want to construct an irreducible $\mathcal{T}(\square_6)$-submodule W_1, there are two methods:

1. With merge method, let $\beta = (e_{00001} - e_{00010}, e_{00101} + e_{01001} + e_{10001} - e_{00110} - e_{01010} - e_{10010}, e_{01101} + e_{10101} + e_{11001} - e_{01110} - e_{10110} - e_{11010}, e_{11101} - e_{11110})$ be a basis of an irreducible $\mathcal{T}(Q_5)$-submodule $\widetilde{W_1}$.

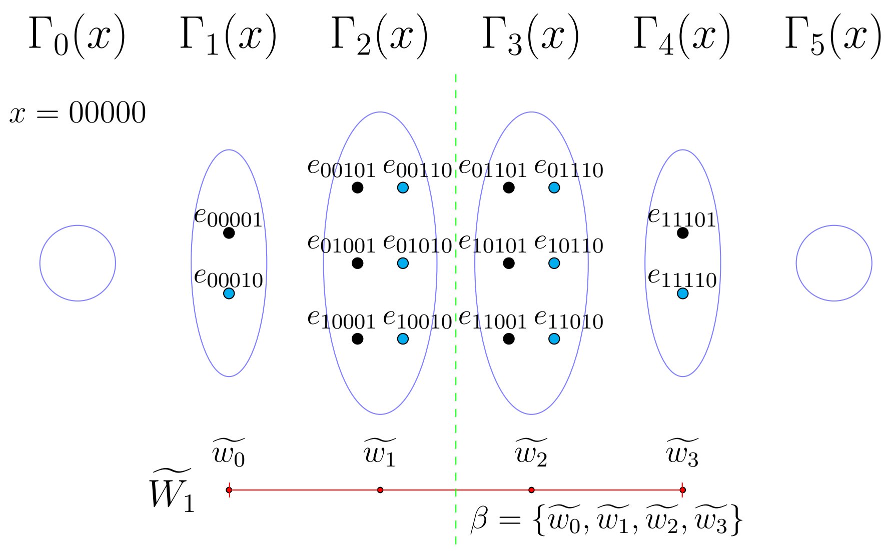

Figure 3.1: An irreducible $\mathcal{T}(Q_5)$-submodule W_1 of $\mathbb{C}^{VQ_5}$

Then by Theorem 2.15 and Theorem 3.9,

$$[A(Q_5)]_\beta + [A_5(Q_5)]_\beta = \begin{pmatrix} 0 & 3 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 3 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 3 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 3 & 0 \end{pmatrix}.$$

Changing the ordering of β to $\widetilde{\beta}$, we have

$$[A(Q_5) + A_5(Q_5)]_{\widetilde{\beta}} = \left(\begin{array}{c|cc|c} 0 & 3 & \mathbf{-1} & 0 \\ \hline 1 & 0 & 0 & 1 \\ \mathbf{-1} & 0 & 0 & \mathbf{3} \\ \hline 0 & 1 & \mathbf{1} & 0 \end{array}\right),$$

where $\widetilde{\beta} = (e_{00001} - e_{00010}, e_{00101} + e_{01001} + e_{10001} - e_{00110} - e_{01010} - e_{10010}, e_{11101} - e_{11110}, e_{01101} + e_{10101} + e_{11001} - e_{01110} - e_{10110} - e_{11010})$. Restrict $\gamma = (e_{00001} - e_{00010}, e_{00101} +$

$e_{01001} + e_{10001} - e_{00110} - e_{01010} - e_{10010} - (e_{11101} - e_{11110}))$ from $\mathbb{C}^{VQ_5}$ to $\mathbb{C}^{V\square_6}$, we have

$$[A(\square_6)]_\gamma = \begin{pmatrix} 0 & 4 \\ 1 & 0 \end{pmatrix}.$$

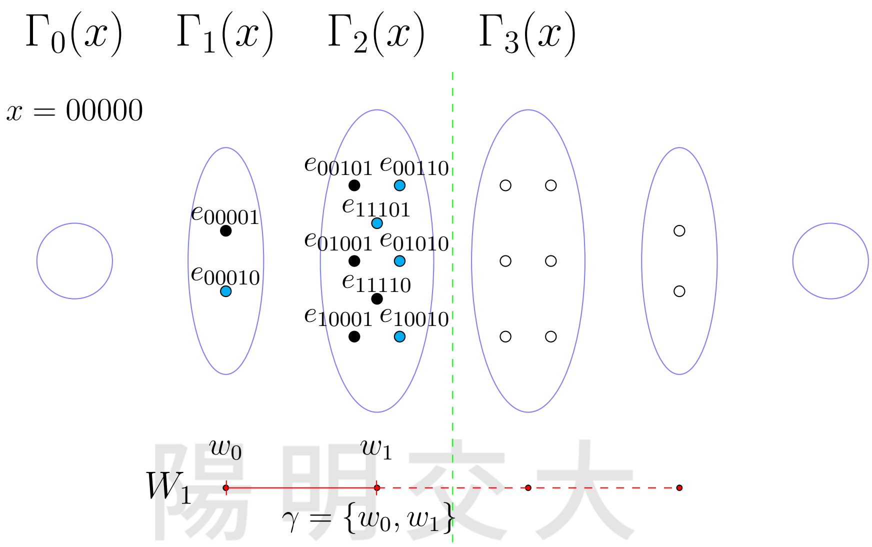

Figure 3.2: An irreducible $\mathcal{T}(\square_6)$-submodule W_1 of $\mathbb{C}^{V\square_6}$

2. With quotient method, let $\beta = (e_{000001} - e_{000010}, e_{000101} + e_{001001} + e_{010001} + e_{100001} - e_{000110} - e_{001010} - e_{010010} - e_{100010}, e_{001101} + e_{010101} + e_{011001} + e_{100101} + e_{101001} + e_{110001} - e_{001110} - e_{010110} - e_{011010} - e_{100110} - e_{101010} - e_{110010}, e_{011101} + e_{101101} + e_{110101} + e_{111001} - e_{011110} - e_{101110} - e_{110110} - e_{111010}, e_{111101} - e_{111110})$ be a basis of an irreducible $\mathcal{T}(Q_6)$-submodule $\widetilde{W_1}$.

   Then

$$[A(Q_6)]_\beta = \begin{pmatrix} 0 & 4 & 0 & 0 & 0 \\ 1 & 0 & 3 & 0 & 0 \\ 0 & 2 & 0 & 2 & 0 \\ 0 & 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & 4 & 0 \end{pmatrix}.$$

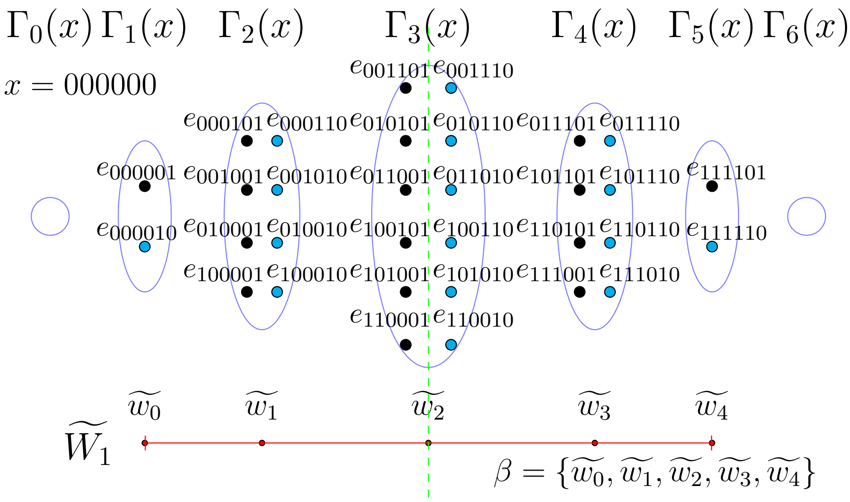

Figure 3.3: An irreducible $\mathcal{T}(Q_6)$-submodule W_1 of $\mathbb{C}^{VQ_6}$

Changing the ordering of β to $\widetilde{\beta}$, we have

$$[A(Q_6)]_{\widetilde{\beta}} = \left(\begin{array}{cc|cc|c} 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ \hline 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & \mathbf{3} \\ \hline 0 & 0 & 2 & \mathbf{2} & 0 \end{array}\right),$$

where $\widetilde{\beta} = (e_{000001} - e_{000010}, e_{111101} - e_{111110}, e_{000101} + e_{001001} + e_{010001} + e_{100001} - e_{000110} - e_{001010} - e_{010010} - e_{100010}, e_{011101} + e_{101101} + e_{110101} + e_{111001} - e_{011110} - e_{101110} - e_{110110} - e_{111010}, e_{001101} + e_{010101} + e_{011001} + e_{100101} + e_{101001} + e_{110001} - e_{001110} - e_{010110} - e_{011010} - e_{100110} - e_{101010} - e_{110010})$. Finally, restrict $\widetilde{\gamma} = (e_{000001} - e_{000010} - (e_{111101} - e_{111110}), e_{000101} + e_{001001} + e_{010001} + e_{100001} - e_{000110} - e_{001010} - e_{010010} - e_{100010} - (e_{011101} + e_{101101} + e_{110101} + e_{111001} - e_{011110} - e_{101110} - e_{110110} - e_{111010}))$ from $\mathbb{C}^{VQ_6}$ to $\mathbb{C}^{V\square_6}$, we have

$$[A(\square_4)]_{\gamma} = \begin{pmatrix} 0 & 4 \\ 1 & 0 \end{pmatrix},$$

where $\gamma = (e_{\{000001,111110\}} - e_{\{000010,111101\}}, e_{\{000101,111010\}} + e_{\{001001,110110\}} + e_{\{010001,101110\}} + e_{\{100001,011110\}} - e_{\{000110,111001\}} - e_{\{001010,110101\}} - e_{\{010010,101101\}} - e_{\{100010,011101\}})$.

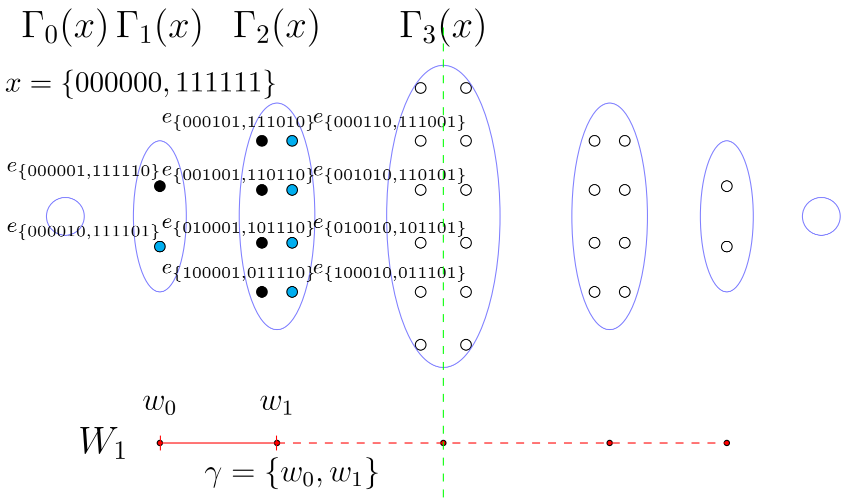

Figure 3.4: An irreducible $\mathcal{T}(\square_6)$-submodule W_1 of $\mathbb{C}^{V\square_6}$

By Theorem 3.10, we determine the dimension of the Terwilliger algebra of the folded n-cube.

Theorem 3.13. *For positive integer D, we have*

$$(i)\quad \dim\mathcal{T}(\square_{2D}) = \frac{(D+1)(D^2+2D+3)}{3},$$

$$(ii)\quad \dim\mathcal{T}(\square_{2D+1}) = \frac{(D+1)(D+2)(2D+3)}{6}.$$

Proof. (i) When $n = 2D$, if D is odd, then

$$\dim\mathcal{T}(\square_{2D}) = 2\sum_{i=1}^{\frac{D+1}{2}} (2i)^2 - (D+1)^2 = \frac{(D+1)(D^2+2D+3)}{3}.$$

If D is even, then

$$\begin{aligned}\dim\mathcal{T}(\square_{2D}) &= 2\times(1^2+3^2+\cdots+(D+1)^2) - (D+1)^2\\ &= 2\times\left(\sum_{i=1}^{D+1} i^2 - \sum_{i=1}^{\frac{D}{2}} (2i)^2\right) - (D+1)^2\\ &= \frac{(D+1)(D^2+2D+3)}{3}.\end{aligned}$$

(ii) When $n = 2D+1$,

$$\dim \mathcal{T}(\square_{2D+1}) = \sum_{i=1}^{D+1} i^2 = \frac{(D+1)(D+2)(2D+3)}{6}.$$

$\square$

Consider the matrix representation of $[A(\square_4)]_\gamma$ of the irreducible $\mathcal{T}(\square_n)$-module W_r at endpoint r for $0 \le r \le \lfloor \frac{n}{2} \rfloor$ with basis γ in Theorem 3.10. We have found the eigenvalues of each matrix.

Proposition 3.14. *(i) For $a \ge 0$, assume that*

$$B_1(a) = \begin{pmatrix} 0 & 2a+1 & & & 0 \\ 1 & 0 & 2a & & \\ & 2 & \ddots & \ddots & \\ & & \ddots & 0 & a+2 \\ 0 & & & a & a+1 \end{pmatrix}.$$

Then the eigenvalues of $B_1(a)$ are $2a+1, 2a-3, \ldots, -(2a-1)$.

(ii) For $a \ge 0$, assume that

$$B_{-1}(a) = \begin{pmatrix} 0 & 2a+1 & & & 0 \\ 1 & 0 & 2a & & \\ & 2 & \ddots & \ddots & \\ & & \ddots & 0 & a+2 \\ 0 & & & a & -(a+1) \end{pmatrix}.$$

Then the eigenvalues of $B_{-1}(a)$ are $2a-1, 2a-5, \ldots, -(2a+1)$.

(iii) For $a \geq 0$, assume that

$$B_2(a) = \begin{pmatrix} 0 & 2a & & & & 0 \\ 1 & 0 & 2a-1 & & & \\ & 2 & 0 & \ddots & & \\ & & \ddots & \ddots & a+2 & \\ & & & a-1 & 0 & a+1 \\ 0 & & & & 2a & 0 \end{pmatrix}.$$

Then the eigenvalues of $B_2(a)$ are $2a, 2a-4, \ldots, -2a$.

(iv) For $a \geq 1$, assume that

$$B_3(a) = \begin{pmatrix} 0 & 2a & & & 0 \\ 1 & 0 & 2a-1 & & \\ & 2 & 0 & \ddots & \\ & & \ddots & \ddots & a+2 \\ 0 & & & a-1 & 0 \end{pmatrix}.$$

Then the eigenvalues of $B_3(a)$ are $2a-2, 2a-6, \ldots, -2a+2$.

Proof. The following are trivial cases:

- The eigenvalue of $B_1(0) = (1)$ is 1.
- The eigenvalue of $B_{-1}(0) = (-1)$ is -1.
- The eigenvalue of $B_2(0) = B_3(1) = (0)$ is 0.

(i) Consider $\det(B_1(a) - xI)$. For $a \geq 1$,

$$
\begin{aligned}
&\det(B_1(a) - xI)\\
=&\begin{vmatrix}
-x & 2a+1 & & & 0\\
1 & -x & 2a & & \\
 & 2 & \ddots & \ddots & \\
 & & \ddots & -x & a+2\\
0 & & & a & a+1-x
\end{vmatrix}\\
=&\begin{vmatrix}
-x & 2a+1-x & \cdots & \cdots & 2a+1-x\\
1 & 1-x & \ddots & & \vdots\\
 & 2 & \ddots & \ddots & \vdots\\
 & & \ddots & a-1-x & 2a+1-x\\
0 & & & a & 2a+1-x
\end{vmatrix}\\
=&\begin{vmatrix}
-1-x & 2a & & & & 0\\
1 & -1-x & 2a-1 & & & \\
 & 2 & \ddots & \ddots & & \\
 & & \ddots & -1-x & a+2 & \\
 & & & a-1 & -1-x & 0\\
0 & & & & a & 2a+1-x
\end{vmatrix}\\
=&(2a+1-x)\det(B_3(a) - (x+1)I).
\end{aligned}
$$

Since the eigenvalues of $B_3(a)$ are $2a-2, 2a-6, \ldots, -2a+2$, we know that the eigenvalues of $B_1(a)$ are $2a+1$ and $2a-3, 2a-7, \ldots, -2a+1$.

(ii) Consider det$(B_{-1}(a) - xI)$. For $a \geq 1$,

$$
\begin{aligned}
&\det(B_{-1}(a) - xI) \\
=&\begin{vmatrix}
-x & 2a+1 & & & 0 \\
1 & -x & 2a & & \\
 & 2 & \ddots & \ddots & \\
 & & \ddots & -x & a+2 \\
0 & & & a & -(a+1)-x
\end{vmatrix} \\
=&\begin{vmatrix}
-x & 2a+1+x & -(2a+1+x) & \cdots & (-1)^{a+1}(2a+1+x) \\
1 & -1-x & 2a+1+x & & \vdots \\
 & 2 & \ddots & \ddots & \vdots \\
 & & \ddots & -(a-1)-x & 2a+1+x \\
0 & & & a & -(2a+1)-x
\end{vmatrix} \\
=&\begin{vmatrix}
1-x & 2a & & & & 0 \\
1 & 1-x & 2a-1 & & & \\
 & 2 & \ddots & \ddots & & \\
 & & \ddots & 1-x & a+2 & \\
 & & & a-1 & 1-x & 0 \\
0 & & & & a & -(2a+1)-x
\end{vmatrix} \\
=&(-2a-1-x)\det(B_3(a) - (1-x)I).
\end{aligned}
$$

Since the eigenvalues of $B_3(a)$ are $2a-2, 2a-6, \ldots, -2a+2$, we know that the eigenvalues of $B_{-1}(a)$ are $-2a-1$ and $2a-1, 2a-5, \ldots, -2a+3$.

(iii) Consider det$(B_2(a) - xI)$. For $a \geq 1$,

$$
\det(B_2(a) - xI) = \begin{vmatrix}
-x & 2a & & & & 0 \\
1 & -x & 2a-1 & & & \\
 & 2 & -x & \ddots & & \\
 & & \ddots & \ddots & a+2 & \\
 & & & a-1 & -x & a+1 \\
0 & & & & 2a & -x
\end{vmatrix}
$$

$$
= \begin{vmatrix}
-x & 2a-x & \cdots & \cdots & \cdots & 2a-x \\
1 & 1-x & 2a-x & & & \vdots \\
 & 2 & 2-x & \ddots & & \vdots \\
 & & \ddots & \ddots & 2a-x & \vdots \\
 & & & a-1 & a-1-x & 2a-x \\
0 & & & & 2a & 2a-x
\end{vmatrix}
$$

$$
= \begin{vmatrix}
-1-x & 2a & & & & 0 \\
1 & -1-x & 2a-1 & & & \\
 & 2 & \ddots & \ddots & & \\
 & & \ddots & -1-x & a+2 & \\
 & & & a-1 & -a-1-x & 0 \\
0 & & & & 2a & 2a-x
\end{vmatrix}
$$

$$
= (2a - x)\det(B_{-1}(a-1) - (x+1)I).
$$

Since the eigenvalues of $B_{-1}(a-1)$ are $2a-3, 2a-7, \ldots, -2a+1$, we know that the eigenvalues of $B_2(a)$ are $2a$ and $2a-4, 2a-5, \ldots, -2a$.

(iv) Consider $\det(B_3(a) - xI)$. For $a \geq 2$,

$$\det(B_3(a) - xI) = \begin{vmatrix} -x & 2a & & & 0 \\ 1 & -x & 2a-1 & & \\ & 2 & -x & \ddots & \\ & & \ddots & \ddots & a+2 \\ 0 & & & a-1 & -x \end{vmatrix}$$

$$= \begin{vmatrix} -x & 2a-x & \cdots & \cdots & 2a-x \\ 1 & 1-x & 2a-x & & \vdots \\ & 2 & 2-x & \ddots & \vdots \\ & & \ddots & \ddots & 2a-x \\ 0 & & & a-1 & a-1-x \end{vmatrix}$$

$$= \begin{vmatrix} -1-x & 2a-1 & & & 0 \\ 1 & -1-x & 2a-2 & & \\ & 2 & -1-x & \ddots & \\ & & \ddots & \ddots & a+1 \\ 0 & & & a-1 & a-1-x \end{vmatrix}$$

$$= \det(B_1(a-1) - (x+1)I).$$

Since the eigenvalues of $B_1(a-1)$ are $2a-1, 2a-5, \ldots, -2a+3$, we know that the eigenvalues of $B_3(a)$ are $2a-2, 2a-6, \ldots, -2a+2$. □

4. Applications

First, we define the Askey-Wilson algebra. Let $\{Y, Z\}$ denote $YZ + ZY$.

Definition 4.1. ([8]) Fix scalars $\beta, \gamma, \gamma^*, \varrho, \varrho^*, \omega, \eta, \eta^* \in \mathbb{C}$. The *Askey-Wilson algebra* is the algebra generated by Y and Z subject to the following Askey-Wilson relations

$$Y^2Z - \beta YZY + ZY^2 - \gamma\{Y, Z\} - \varrho Z = \gamma^* Y^2 + \omega Y + \eta 1,$$
$$Z^2Y - \beta ZYZ + YZ^2 - \gamma^*\{Y, Z\} - \varrho^* Y = \gamma Z^2 + \omega Z + \eta^* 1.$$

Given a graph Γ. Choose $Y = A$ and $Z = A^*$, where A is the adjacency matrix of Γ and A^* is the dual adjacency matrix of Γ with respect to a vertex $x \in V\Gamma$. By [1, p.399], if $\Gamma = Q_n$, then A and A^* satisfy the Askey-Wilson relations with

$$\beta = 2, \omega = 4, \gamma = \gamma^* = \varrho = \varrho^* = \eta = \eta^* = 0.$$

Throughout this section, we use programming to test the cases under the folded n-cube $\square_n$ for $n \leq 10$. For general $A = A(\square_n)$ and $A^* = A^*(\square_n)$, we observe that (A, A^*) satisfies

$$A^{*2}A - 2A^*AA^* + AA^{*2} = 4\{A, A^*\} + 4(n-1)A.$$

Under this assumption, the scalars are $\beta = 2, \gamma^* = 4, \varrho^* = 4(n-1), \omega = \gamma = \eta^* = 0$. But it is not obvious to find the other equation with proper ϱ, η for large n. So we take a look to the irreducible $\mathcal{T}(\square_n)$-module W_r at endpoint r for $0 \leq r \leq \lfloor \frac{n}{2} \rfloor$. Consider the algebra generated by $[A(\square_n)]_\beta$ and $[A^*(\square_n)]_\beta$ where β is a basis of W_r in Theorem 3.10. We denote this algebra by $\mathcal{T}(\square_n)\restriction_{W_r}$.

4.1 The center of $\mathcal{T}(\square_n)\restriction_{W_r}$

In this subsection, we let $A = [A(\square_n)]_\beta$ and $A^* = [A^*(\square_n)]_\beta$. Before we give the general relation of $\mathcal{T}(\square_n)\restriction_{W_r}$, we consider the case when $r = 0$. For two generators A and A^* of $\mathcal{T}(\square_n)\restriction_{W_0}$, we have the relations

$$
\begin{aligned}
A^2A^* - 2AA^*A + A^*A^2 &= 4A^2 + 16A^* - 4n^2 1, \\
A^{*2}A - 2A^*AA^* + AA^{*2} &= 4\{A, A^*\} + 4(n-1)A.
\end{aligned}
$$

This observation gives us an idea that $\mathcal{T}(\square_n)\restriction_{W_r}$ is a special case of Askey-Wilson algebra and for the relations of $\mathcal{T}(\square_n)\restriction_{W_r}$, only the coefficient of 1 changes. More precisely, $\mathcal{T}(\square_n)\restriction_{W_r}$ is isomorphic to a special case of the Hahn algebra.

Definition 4.2. ([9]) The *Hahn algebra* has 2 generators $\widehat{K_1}, \widehat{K_2}$ subjected to the relations

$$
\begin{aligned}
[[\widehat{K_1}, \widehat{K_2}], \widehat{K_1}] &= a\widehat{K_1}^2 + b\widehat{K_1} + c_2\widehat{K_2} + d_2 1, \\
[\widehat{K_2}, [\widehat{K_1}, \widehat{K_2}]] &= a\{\widehat{K_1}, \widehat{K_2}\} + b\widehat{K_2} + c_1\widehat{K_1} + d_1 1,
\end{aligned}
$$

where $[A, B] = AB - BA$ and a, b, c_1, c_2, d_1, d_2 are structure constants.

With similar idea of $r = 0$, the following proposition gives the coefficient of 1 in the general relations of $\mathcal{T}(\square_n)\restriction_{W_r}$.

Proposition 4.3. *For A and A^* of $\mathcal{T}(\square_n)\restriction_{W_r}$, where W_r is the irreducible $\mathcal{T}(\square_n)$-module with the diameter D of $\square_n$ and endpoint r, let the coefficient of constant term be $a_{n,r}$. Then A and A^* satisfy*

$$
\begin{aligned}
A^2A^* - 2AA^*A + A^*A^2 &= 4A^2 + 16A^* - a_{n,r} 1, \\
A^{*2}A - 2A^*AA^* + AA^{*2} &= 4\{A, A^*\} + 4(n-1)A.
\end{aligned}
$$

Suppose that $a_{1,0} = 4$ and $a_{4m+2,2m+1} = -16(2m+1)$ for $m \geq 0$. Then

$$
a_{4m+2} = \begin{cases} 4n^2, & r = 0, \\ a_{n-2,r-1} - 16 & 1 \leq r \leq \lfloor \frac{n}{2} \rfloor. \end{cases}
$$

By [9, p.1531], we transform the relations of $\mathcal{T}(\square_n)\restriction_{W_r}$ to a simple form with new generator B_1, B_2.

Proposition 4.4. *For $\mathcal{T}(\square_n)\restriction_{W_r}$, where W_r is the irreducible $\mathcal{T}(\square_n)$-module at endpoint r, let $B_1 = \frac{1}{2}A$*

and $B_2 = \frac{1}{2}(A^* + \frac{n-1}{2})$*, then*

$$[[B_1, B_2], B_1] = -2B_1^2 - 4B_2 + \delta_{n,r}1,$$
$$[B_2, [B_1, B_2]] = -2\{B_1, B_2\}$$

for some $\delta_{n,r}$*.*

Lemma 4.5. *For* $n \geq 2$ *and* $0 \leq r \leq D$*, if the irreducible* $\mathcal{T}(\square_n)$*-module* W_r *exists,*

$$\delta_{n,r} = \frac{(n-2r+1)^2 - 3}{2}.$$

$\square_2$	6	$\square_3$	13,1
$\square_4$	22,6,-2	$\square_5$	33,13,1
$\square_6$	46,22,6	$\square_7$	61,33,13,1
$\square_8$	78,46,22,6,-2	$\square_9$	97,61,33,13,1

Table 4.1: $2\delta_{n,r}$ of irreducible $\mathcal{T}(\square_n)$-module W_r for $2 \leq n \leq 9$

Next, we introduce another algebra subjects to different relations with additional terms related to generators.

Definition 4.6. ([10, Definition 2.1]) The *Heun algebra* $\mathcal{H}$ of the Lie type is generated by X and W with the following relations

$$[[X, W], X] = x_0 1 + x_1 X + x_2 X^2 + x_3 W,$$
$$[W, [X, W]] = y_0 1 + y_1 X + y_2 X^2 + y_3 X^3 + x_1 W + x_2 \{X, W\},$$

where x_i and y_i are free parameters for $i = 0, 1, 2, 3$.

Since the Heun algebra of Lie type is isomorphic to the Hahn algebra, we can find the center of the Hahn algebra and the central element of $\mathcal{T}(\square_n)\restriction_{W_r}$ [10, p.3].

Theorem 4.7 ([10])**.** *In the Heun algebra* $\mathcal{H}$*, the following element*

$$\Omega = z_1 X + z_2 W + z_3\{X, W\} + z_4 XWX + z_5 X^2 +$$
$$z_6 W^2 + z_7([X, W])^2 + z_8 X^3 + z_9 X^4$$

is central if the parameters z_i are given by

$$z_1 = 2y_0 - \frac{x_3y_3}{2},\ z_2 = -x_2x_3 + 2x_0,\ z_3 = x_1,\ z_4 = 2x_2,$$
$$z_5 = y_1 - \frac{x_3y_3}{2},\ z_6 = x_3,\ z_7 = 1,\ z_8 = \frac{2y_2}{3},\ z_9 = \frac{y_3}{2}.$$

Theorem 4.8. *For $\mathcal{T}(\square_n)\restriction_{W_r}$, where W_r is the irreducible $\mathcal{T}(\square_n)$-module at endpoint r,*

$$\Omega = ((n-2r+1)^2 - 11)B_2 - 4B_1B_2B_1 - 4B_2^2 + ([B_1, B_2])^2$$

is the central element of $\mathcal{T}(\square_n)\restriction_{W_r}$.

Proof. Let $X = B_1 = \frac{1}{2}A$ and $W = B_2 = \frac{1}{2}(A^* + \frac{n-1}{2})$. Consider the relation of Heun algebra, we have

$$x_0 = \delta, x_2 = -2, x_3 = -4 \text{ and } x_1 = y_0 = y_1 = y_2 = y_3 = 0.$$

So

$$z_2 = 2\delta - 8, z_4 = z_6 = -4, z_7 = 1 \text{ and } z_1 = z_3 = z_5 = z_8 = z_9 = 0$$

by Theorem 4.7. □

5. Conclusion

In this thesis, we use two different views of folded n-cubes to find the matrix form of $A(\square_n)$ of the irreducible $\mathcal{T}(\square_n)$-module W_r and the center of $\mathcal{T}(\square_n)\lceil_{W_r}$ for all $0 \leq r \leq \lfloor \frac{n}{2} \rfloor$. The observation of this thesis is mainly done by programming. For example, it is easier to perform the recurrence relation of distance matrices for the matrix form of $A_n(Q_n)$ by programming in Theorem 3.9. To find what relations subject to the algebra $\mathcal{T}(\square_n)\lceil_{W_r}$, we quickly experiment the relations for different possible scalars and use them to find the central element of $\mathcal{T}(\square_n)\lceil_{W_r}$. For some theorems proved by programming in this thesis, we wish that the general proofs would be discovered in the future. With some variation of the hypercube, such as halved n-cube $\frac{1}{2}Q_n$, folded n-cube $\square_n$, hamming graph $H(D,q)$, etc., we have a wide view of the Terwilliger algebra related to distance-regular graphs with Q-polynomial property.

We give a conjecture related to the folded hamming graph $\widetilde{H}(D,q)$.

Conjecture 5.1. Consider the hamming graph $H(D,q)$. Let $a_1a_2\ldots a_D \in VH(D,q)$. If we collect q vertices $a_1a_2\cdots a_D, a_1a_2\cdots a_D{+}1, \ldots, a_1a_2\cdots a_D{+}q{-}1$ to make a new vertex $\{a_1a_2\cdots a_D, a_1a_2\cdots a_D+1, \ldots, a_1a_2\cdots a_D+q-1\}$ in the folded hamming graph $\widetilde{H}(D,q)$, then $\widetilde{H}(D,q)$ is distance-regular with Q-polynomial property.

Problem 5.2. How to decompose the Terwilliger algebra $\mathcal{T}(\widetilde{H}(D,q))$ of the folded hamming graph $\widetilde{H}(D,q)$?

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