國　立　清　華　大　學

碩　士　論　文

Colin de Verdière圖不變量
在一些圖運算下的行為
Behaviors of the Colin de Verdière graph invariant under some graph operations

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Abstract

Kotlov, Lovász and Vempala in [1] offered a reformulation for the Colin de Verdière graph invariant μ by introducing another graph invariant ν defined via graph labellings. These two parameters are related by the equality $\mu(G) + \nu(\overline{G}) = |G| - 1$ for $G \neq \overline{K_2}$. In this paper we examine how these two invariants μ and ν vary under some well-known graph operations, such as Cartesian products, disjoint unions and graph joins.

First, we introduce "almost one-directional" labelling to derive that for the disjoint union of graphs $\{G_i\}$, $\max \nu(G_i) \leq \nu(\bigcup G_i) \leq \max \nu(G_i) + 1$. Also we show a sufficient condition for the first equality to hold. This nearly characterizes the behavior of ν under disjoint unions. As an application, we are able to compute the exact value μ for complete multipartite graphs. The inequality also provides us with some necessary conditions for a disconnected graph being ν-minimal. Therefore, this also motivates us to look into how μ-maximal graphs with separating cliques could be built up by smaller ones via clique sums. Using the characterization of μ under clique sum proved by van der Holst, Lovász and Schrijver in [4], we derive a criterion in judging whether a clique sum of two μ-maximal graphs is μ-maximal. Lastly, we show that the growth rate of ν under Cartesian products has a linear upper bound in the number of graphs while that of μ has a exponential lower bound in the number of graphs.

中文摘要

Kotlov、Lovász 和 Vempala [1] 以 graph labellings 定義了一個圖不變量 ν 並以此給出 Colin de Verdière 圖不變量 μ 的另一個等價定義。當 $G \neq \overline{K_2}$ 時，這兩個圖不變量滿足等式 $\mu(G) + \nu(\overline{G}) = |G| - 1$。這篇論文旨在探討 μ 和 ν 在一些為人熟知的圖運算下的行為，如 Cartesian products、disjoint unions 和 graph joins。

首先我們引進 “almost one-directional” labelling 來證明對於一系列的圖 $\{G_i\}$ 的 disjoint union，有不等式 $\max \nu(G_i) \leq \nu(\bigcup G_i) \leq \max \nu(G_i) + 1$。而且我們提供了一個充分條件使得第一個等式成立。這近乎刻劃 ν 在 disjoint unions 下的行為。作為一個簡單的應用，我們能夠計算完全多部圖的 μ 值。除此之外，這個不等式也提供一些判斷不連通圖是否為 ν-minimal graph 的必要條件。這也促使我們去探討，要如何用小的 μ-maximal graphs 來生成一個給定的有 separating cliques 的 μ-maximal graph。藉由 van der Holst、Lovász 和 Schrijver 三人於 [4] 中所刻劃的 μ 在 clique sum 下的行為，我們給出一個判斷 μ-maximal graphs 的 clique sum 是否仍是 μ-maximal 的準則。最後我們證明 ν 在 Cartesian products 下的成長速率有一個以圖的片數為變數的線性成長的上界，而 μ 則有一個指數成長的下界。

Contents

1 Introduction

Throughout the text, all graphs $G = (V, E)$ are finite, undirected, simple and loopless. All the matrices considered are over real numbers.

Given a connected graph G of order n, let A be the adjacency matrix of G. By Perron-Frobenius Thoerem, we know the largest eigenvalue of A is of multiplicity 1. Therefore, we are interested in the multiplicity of the second large eigenvalue of A. Indeed, we consider the generalized adjacency matrices (with no constraints on the diagonal):

$$S_G := \{A \in \mathbb{R}^{(n)} \mid A_{ij} > 0 \text{ if } ij \in E \text{ and } A_{ij} = 0 \text{ if } ij \notin E, \forall i \neq j\},$$

where $\mathbb{R}^{(n)}$ denotes the collection of all real symmetric matrices of order n. Again by Perron-Frobenius Theorem, for a matrix $A \in S_G$, the multiplicity of its largest eigenvalue is 1. Thus, we want to know how large could the multiplicity of its second large eigenvalue λ_2 could be, i.e. to understand the value

$$\max_{A \in S_G} \text{multiplictiy of } \lambda_2(A).$$

For a given $A \in S_G$, if we consider the matrix $M := -A + \lambda_2 I$, then the original problem is turned into understanding the corank of M. That is, if we let

$$\mathscr{O}_G := \{M \in \mathbb{R}^{(n)} \mid M_{ij} < 0 \text{ if } ij \in E \text{ and } M_{ij} = 0 \text{ if } ij \notin E, \forall i \neq j\},$$

then the original problem is transferred to understanding $\max\limits_{M} \operatorname{corank}(M)$ over all matrices $M \in \mathscr{O}_G$ with exactly one negative eigenvalue.

In 1990, Colin de Verdière [11] introduced an interesting graph invariant μ, considering the maximum corank of matrices in $\mathscr{O}_G$ with exactly one negative eigenvalue, subject to a nondegeneracy condition called the Strong Arnold Property. It turns out that the invariant μ not only nicely describes topological properties of graphs, but also links up with geometric graph representations. Before speaking more about μ, we would like to provide its definition first. Our definition follows from the matrix reformulation given by van der Holst, Lovász and Schrijver in [5].

Let G be a graph, not necessarily connected. For matrices A, B of the same size, let $A \circ B$ denote their Schur product, where the (i, j)th entry of $A \circ B$ is $A_{ij} B_{ij}$.

Definition 1.1. ([2]) Let $M, X \in \mathbb{R}^{(n)}$. Then X is said to *fully annihilate* M if

$$MX = M \circ X = I \circ X = O.$$

We say M possesses the *Strong Arnold Property (SAP)* if the only symmetric matrix that fully annihilates M is the zero matrix.

Definition 1.2. A *Colin de Verdière matrix* for G is a matrix $M \in \mathscr{O}_G$ which satisfies the following conditions:

(M1) M has exactly one negative eigenvalue (counting multiplicity).

(M2) M possesses SAP.

The *Colin de Verdière graph invariant* $\mu(G)$ is the maximum corank over all Colin de Verdière matrices for G. If a Colin de Verdière matrix for G has corank $\mu(G)$, then it is called a *μ-optimal matrix* for G.

Clearly, by (M1) for all graphs, we have $\mu(G)$ is less than $|G|$, the order of G.

Example 1.3. Let K_n denote the complete graph on n vertices. Observe that any $M \in \mathscr{O}_{K_n}$ automatically satisfies SAP and $-J_n \in \mathscr{O}_{K_n}$ has corank $n-1$, where J_n is the n by n matrix with all entries equal to 1. Consequently, $-J_n$ is a μ-optimal matrix for K_n and $\mu(K_n) = n-1$. For $n \geq 2$, SAP forces a μ-optimal matrix for $\overline{K_n}$ to have exactly one zero in the diagonal and thus $\mu(\overline{K_n}) = 1$.

In general, SAP yields that for a non-edgeless graph G, $\mu(G) = \max \mu(C_i)$ over all connected components C_i of G. Indeed, suppose a matrix $M \in \mathbb{R}^{(n)}$ can be written as a direct sum of two matrices, say $M = M_1 \oplus M_2$, both M_1 and M_2 singular. For $i = 1, 2$, pick a nonzero vector s_i in the kernel of M_i. Let $s_1' = s_1 \oplus \mathbf{0}$ and $s_2' = \mathbf{0} \oplus s_2$. Then $s_1' s_2'^T + s_2' s_1'^T \in \mathbb{R}^{(n)}$ is nonzero and fully annihilates M. Conversely, if M_2 is invertible and M_1 has SAP, then M has SAP. Suppose $X \in \mathbb{R}^{(n)}$ fully annihilates $M_1 \oplus M_2$. Write $X = \begin{pmatrix} X_1 & X_2^T \\ X_2 & X_3 \end{pmatrix}$ corresponding to the order of M_1 and M_2. The nonsingularity of M_2 forces X_2 and X_3 to be zero. Consequently, X_1 is zero by the SAP of M_1. Thus X is zero and M has SAP. Also note that a symmetric matrix has rank 1 if and only if it is of the form $\pm uu^T$ for some vector u of appropriate size. Therefore, by Perron-Frobenius Theorem, if a connected graph G of order n is not complete, then all matrices in $\mathscr{O}_G$ have corank at most $n-2$. Together with the property of μ under disjoint union and Example 1.3, we have that $\mu(G) \leq |G| - 2$ unless G is complete or $G = \overline{K_2}$. Moreover, one can see that for graphs other than K_1, due to Perron-Frobenius Theorem and (M2), the condition (M1) can be replaced with

(M1′) M has at most one negative eigenvalue (counting multiplicity).

Colin de Verdière proved in [11] that μ is minor-monotone, that is if G' is a minor of G then $\mu(G') \leq \mu(G)$. The proof is rather nontrivial, in which SAP plays an important role. In view of Example 1.3, we have

$$\mu(G) + 1 \geq \eta(G) \geq \omega(G),$$

where $\omega(G)$ is the order of a maximum clique contained in G, and the Hadwiger number $\eta(G)$ is the order of a maximum clique minor in G. On the other hand, by Graph Minor Theorem([6]), we know that there are only finitely many forbidden minors for graphs satisfying $\mu \leq k$ for each nonnegative integer k. Denote the collection by $\mathscr{F}_k$. The amazing property of μ is that it is able to interpret topological properties of graphs via linear algebraic formulations. We list some results below. For a complete proof, we refer one to the survey [5] for an overview.

(i) $\mu(G) = 0$ iff $G = K_1$.

(ii) $\mu(G) \leq 1$ iff G is a disjoint union of paths or equivalently $\mathscr{F}_1 = \{K_{1,3}, K_3\}$.

(iii) $\mu(G) \leq 2$ iff G is outplanar or equivalently $\mathscr{F}_2 = \{K_{2,3}, K_4\}$.

(iv) $\mu(G) \leq 3$ iff G is planar or equivalently $\mathscr{F}_3 = \{K_{3,3}, K_5\}$.

(v) $\mu(G) \leq 4$ iff G is linkless embeddable or equivalently $\mathscr{F}_4$ = Petersen family.

In [1], Kotlov, Lovász and Vempala reformulated the definition of μ in terms of positive semidefinite matrix.

Theorem 1.4. *([5]) For $G \neq \overline{K_2}$, the maximum corank among all $A \in \mathbb{R}^{(n)}$ with properties below is $\mu(G) + 1$:*

(A1) for all $i \neq j$, $A_{ij} < 1$ if $ij \in E(G)$, and $A_{ij} = 1$ if $ij \notin E(G)$;

(A2) A is positive semidefinite;

(A3) A has the SAP with respect to G: if $X \in \mathbb{R}^{(n)}$ such that $X_{ij} = 0$ for $i = j$ or $ij \in E$ and $AX = O$, then $X = O$.

In view of interests in large value of $\mu(G)$, they also defined a dual invariant of $\mu(G)$ via graph vector representations. Consider a matrix A satisfying (A1)-(A3). There exists an orthogonal matrix Q such that $A = Q^T DQ$ for some diagonal matrix D with diagonal nonnegative and decreasing. Writing $D = D^{1/2}D^{1/2}$, then we have $A = U^T U$, where $U = D^{1/2}Q$. Therefore, we can view A as a gram matrix of vectors in dimension $d = \operatorname{rank}(A)$ with $A_{ij} = u_i^T u_j$, where $u_i \in \mathbb{R}^d$. Thus it leads to the following definition.

Definition 1.5. ([1]) Let $\nu(G)$ be the smallest integer d such that a labelling $i \mapsto u_i \in \mathbb{R}^d$ with the following properties exists :

(U1) for all $i \neq j$, $u_i^T u_j < 1$ if $ij \notin E(G)$, and $u_i^T u_j = 1$ if $ij \in E(G)$;

(U2) if $X \in \mathbb{R}^{(n)}$ satisfies $X_{ij} = 0$ for $i = j$ or $ij \notin E(G)$ and $\sum_j X_{ji} u_j = 0$ for each i, then $X = O$.

The mapping $i \mapsto u_i$ with property (U1) is called a *gram labelling* of G. A gram labelling with property (U2) is said to be *nondegenerate*.

Remark 1.6. Let U be the matrix with u_i being the ith column. Then the condition of (U2) can be reformulated as

(U2′) if $X \in \mathbb{R}^{(n)}$ satisfies $X_{ij} = 0$ for $i = j$ or $ij \notin E(G)$ and $UX = O$, then $X = O$.

Let $\overline{G}$ denote the complement graph of G. As a consequence of Theorem 1.4, we have

Theorem 1.7. *([1]) For $G \neq K_2$, $\nu(G) = n - \mu(\overline{G}) - 1$ and $\nu(K_2) = 1$.*

It follows from the minor-monotonicity of μ and the above identity that ν is decreasing under taking subgraphs. However, ν is far from being minor-monotone as it was shown in [1] that every graph G has a subdivision G' such that $\nu(G') \leq 4$. Nevertheless, there is an unexpected connection between μ and ν.

Theorem 1.8. *([1]) The inequality $\nu(G) \leq \mu(G) + 1$ holds for all planar graphs G.*

They further proposed the conjecture that the inequality shall hold in general, known as the Graph Complement Conjecture, which remains unsolved. Also, they asked whether $\mu = \nu$ for graphs with nice properties, such as being vertex-transitive and twin-free. We will show that the answer is negative for ν large. Indeed, we prove the following

Proposition 1.9. *Let $\{G_i\}_{i\in\mathbb{N}}$ be a family of non-edgeless graphs with bounded orders. Then*

$$\lim_{r\to\infty}\frac{\mu(\square_{i=1}^{r}G_i)}{\nu(\square_{i=1}^{r}G_i)}=\infty$$

In understanding the topological characterizations and obstructions for μ, we also consider the following graphs.

Definition 1.10. We say a graph G is *μ-maximal* if G is a complete graph or $\mu(G+e) > \mu(G)$ for any edge $e \in E(\overline{G})$. We say G is *ν-minimal* if for any subgraph H of G, $\nu(H) < \nu(G)$ if $H \neq G$.

Clearly, K_2 is the only ν-minimal graph for $\nu = 1$. It is also easy to verify that ν-minimal graphs for $\nu = 2$ are $3K_2$ and P_4 (path on 4 vertices). In [1], they fully characterize graphs for $\nu = 2$. Consequently, ν-minimal graphs for $\nu = 3$ are known; the disconnected ones are $C_k \cup K_2$, $k \geq 5$, the disjoint union of a cycle on k vertices and an edge. It seems that K_2 plays an important role in understanding the behavior of ν under disjoint union. This observation leads to one of our main results proved in section 3 that describes the behaviour of ν under disjoint union.

Theorem 1.11. *Let $G, G_1, \ldots, G_r$ be graphs such that $\nu(G) \geq \nu(G_i)$ for each i. Then*

$$\nu(G) \leq \nu(G \cup \bigcup_{i=1}^{r} G_i) \leq \nu(G) + 1.$$

Moreover, if $\nu(G \cup K_2) = \nu(G) > 1$ and for each G_i there exists a vertex v_i such that $\nu(G_i - v_i) < \nu(G)$, then the first equality holds.

As a direct consequence of Theorem 1.11, we can compute the exact value of μ for all complete multipartite graphs. It has been shown in [5] that for $p \geq q$,

$$\mu(K_{p,q}) = \begin{cases} q & \text{if } p < 3; \\ q+1 & \text{if } p \geq 3; \end{cases}$$

One shall see later in section 3 that the values of μ for complete multipartite graphs are almost only dependent on the order of a maximum coclique. Moreover, Theorem 1.11 provides some necessary conditions for disconnected graphs being ν-minimal graphs. Thus, it motivates us to analyze those μ-maximal graphs that have separating cliques, since their complements are ν-minimal and connected. In section 4, we will show that all μ-maximal graphs can be built up from those with no separating cliques via clique sum under certain criterion. In section 5, we investigate some inequalities for μ and ν under Cartesian products and prove Proposition 1.9.

2 Notations and terminology

Let $G = (V, E)$ be a graph of order n. For simplicity, sometimes G is used to denote its vertex set if the context is clear. If vertices $i, j \in V$ are adjacent, we denote it by $i \sim j$ or $ij \in E$. The set of neighbors of $S \subset V$ in G is

$$N_G(S) := \{u \in V - S | uv \in E \text{ for some } v \in S\}.$$

For an edge subset F of $E(K_{|G|})$, $G+F := (V, E \cup F)$. If $F = \{e\}$, then we simply write $G+F$ as $G+e$. A graph $G' = (V', E')$ is a *subgraph* of G, denoted by $G' \leq G$, if $V' \subset V$ and $E' \subset E$. Moreover, we say G' is an *induced subgraph* of G if for $i, j \in V'$, $ij \in E'$ if and only if $ij \in E$. Equivalently, G' may be thought of as a graph obtained from deleting the vertex set $S = V - V'$ in G. In this case, we write G' as $G[V']$, or simply V' or $G - S$ if the context is clear. If G' is a graph obtained from G by a series of vertex deletions, edge deletions, and edge contractions, then we say G' is a *minor* of G, denoted by $G' \preceq G$. A clique in G is a (induced) complete subgraph of G; a coclique is an induced edgeless subgraph of G. A *(connected) component* of G is a maximal connected induced subgraph of G. Throughout the context, C_n may denote either a cycle on n vertices or a component of a graph. There shall be no confusion. We write J_n and I_n for the all 1's matrix and the identity matrix of order n respectively. The boldfaced $\mathbf{1}$ is used to denote the vectors with all entries equal to 1. Zero vectors are simply denoted by 0, and the zero matrices are written as O. Their orders will not be specified unless needed.

Let G_1 and G_2 be graphs. Their *union* is $G_1 \cup G_2$, where $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. Moreover, if $V(G_1) \cap V(G_2)$ is empty, then we say the union is *disjoint*. Their *Cartesian product* is $G_1 \Box G_2$, where $V(G_1 \Box G_2) = V(G_1) \times V(G_2)$ and $(i,k)(j,l) \in E(G_1 \Box G_2)$ if $i = j$ and $k \sim l$ or $i \sim j$ and $k = l$. Their *Kronecker product* is $G_1 \times G_2$, where $V(G_1 \times G_2) = V(G_1) \times V(G_2)$ and $(i,k)(j,l) \in E(G_1 \times G_2)$ if $i \sim j$ and $k \sim l$. Their *strong product* is $G_1 \boxtimes G_2 := G_1 \Box G_2 \cup G_1 \times G_2$ (identifying the vertex sets). Their *join* is $G_1 \vee G_2$, which is obtained from their disjoint union by adding all edges between G_1 and G_2.

3 Characterizations of ν under disjoint union

In this section, all unions of graphs are disjoint.

This section is devoted to proving Theorem 1.11 and deriving some of its consequences. Note that in a gram labelling, for an isolated vertex, we can always label it with the zero vector, hence we may assume that each graph has no isolated vertex from now on. The idea of the proof is rather simple. We use a disjoint edge as a detector. If the original graph can be inserted with one disjoint edge without increasing ν, then it is possible to insert arbitrarily many disjoint edges, hence those graphs with "almost one-directional" labellings without increasing ν. Now we give the precise definition of "almost one-directional" labelling.

Definition 3.1. A graph G is said to have a *central gram labelling* in $\mathbb{R}^d$ if $\exists\, u \in \mathbb{R}^d$ such that for each $\epsilon > 0$, there exists a nondegenerate gram labelling $i \mapsto u_i$ in $\mathbb{R}^d$ for G satisfying $|u_i - u| < \epsilon$ for all $i \in G$. The vector u is termed the *associated central vector*.

In the above definition, one can view each vector u_i as a function from positive real numbers to $\mathbb{R}^d$.

Example 3.2. Fix $n \geq 3$. We've shown that $\mu(\overline{K_n}) = 1$. Consequently, Theorem 1.7 implies $\nu(K_n) = n - 2$. Let u be a unit vector in $\mathbb{R}^{n-2}$ and $\{u, u_4, \ldots, u_n\}$ be an orthonormal basis of $\mathbb{R}^{n-2}$. Take any nonzero real number α. Let v_i be u if $i \le 3$ and let v_i be $u + \alpha u_i$ if $i > 3$. Then $i \mapsto v_i$ is a nondegenerate gram labelling for K_n in $\mathbb{R}^{n-2}$. Indeed, if $X \in \mathbb{R}^{(n)}$ has zero diagonal and for each i, $\sum_j X_{ji} v_j = 0$. Then for $j > 3$, as $\{u, u_4, \dots, u_n\}$ is orthonormal, we have $X_{ji} = 0$ for all i. By the symmetry of X, we have

$$X = \begin{pmatrix} X' & O \\ O & O \end{pmatrix},$$

where $X' \in \mathbb{R}^{(3)}$ with zero diagonal. Now for $i = 1$, the condition $0 = \sum_j X_{j1} v_j = (X'_{21} + X'_{31})u$ implies $X'_{21} = -X'_{31}$. Similarly, $X'_{12} = -X'_{32}$ and $X'_{13} = -X'_{23}$. Again by symmetry of X', we have $X' = O$, hence $X = O$. Therefore, K_n has a central gram labelling in $\mathbb{R}^{n-2}$ with u being the associated central vector as $|\alpha|$ can be taken arbitrarily small.

Observe that if G has a central gram labelling in $\mathbb{R}^d$, then since G is assumed to have no isolated vertices, hence non-edgeless, the associated central vector u must have norm 1. If not, then for each distinct i,j, we have

$$\begin{aligned} |u_i^T u_j - u^T u| &= |(u_i - u)^T u_j + u^T (u_j - u)| \\ &\le |(u_i - u)||u_j| + |u||u_j - u| \\ &< \epsilon(|u| + |u_j - u| + |u|) < \epsilon(2|u| + \epsilon), \end{aligned}$$

which would yield $u_i^T u_j \neq 1$ by taking ϵ sufficiently small, hence a contradiction. Moreover, let v be any other unit vector in $\mathbb{R}^d$, we may choose an orthogonal matrix Q of order d that sends u to v. Now since Q is an isometry, the new labelling $i \mapsto Qu_i$ is also a gram labelling for G in $\mathbb{R}^d$. The nondegeneracy is preserved as one can easily see via (U2′) that $QUX = O$ if and only if $UX = O$. The associated central vector of the new labelling is v. Consequently, G possessing central gram labelling is independent of the choice of unit vectors.

We break the proof of Theorem 1.11 into three lemmas.

Lemma 3.3. *For every graph, G and $G \vee K_1$ have central gram labellings in $\mathbb{R}^{\nu(G)+1}$.*

Proof. By vertex deletion, it suffices to show for $G \vee K_1$. Let $i \mapsto u_i$ be a nondegenerate gram labelling for G in $\mathbb{R}^{\nu(G)}$. Let $M = \max_{i \in G} |u_i|$. Now given $\epsilon > 0$, take $\alpha \in (0, \pi/2)$ such that both $(1 - \cos\alpha)^2 + (M \sin\alpha)^2$ and $(1 - \sec\alpha)^2$ are smaller than ϵ^2. Let s denote the added vertex. Then label each $i \in G$ with $v_i = \cos\alpha \oplus (\sin\alpha) u_i$ and s with $v_s = \sec\alpha \oplus \mathbf{0}$. For $i \neq j$ in G, we have

$$v_i^T v_j = \cos^2\alpha + (\sin^2\alpha) u_i^T u_j = 1 + (\sin^2\alpha)(u_i^T u_j - 1),$$

which is equal to 1 if $i \sim j$ and is less than 1 otherwise. Since $\sin\alpha > 0$ and $i \mapsto u_i$ is nondegenerate, by the choice of v_s, it is easy to check that $i \mapsto v_i$ is a nondegenerate gram labelling for $G \vee K_1$. Indeed, let U and V be the matrices corresponding to the labellings $i \mapsto u_i$ and $i \mapsto v_i$ respectively with s indexed 1. Suppose that there exists X violating (U2′) for V. Then X takes the form

$$X = \begin{pmatrix} 0 & y^T \\ y & X' \end{pmatrix},$$

where $y \in \mathbb{R}^{|G|}$ and $X' \in \mathbb{R}^{(|G|)}$ satisfying $X'_{ij} = 0$ for $i = j$ or $ij \notin E(G)$. Then

$$O = VX = \begin{pmatrix} \sec\alpha & \cos\alpha \cdot \mathbf{1}^T \\ O & \sin\alpha \cdot U \end{pmatrix} \begin{pmatrix} 0 & y^T \\ y & X' \end{pmatrix} = \begin{pmatrix} \cos\alpha \cdot \mathbf{1}^T y & \sec\alpha \cdot y^T + \cos\alpha \cdot \mathbf{1}^T X' \\ \sin\alpha \cdot Uy & \sin\alpha \cdot UX' \end{pmatrix}$$

Then nondegeneracy of $i \mapsto u_i$ implies that $X' = O$ and therefore $y = 0$. Thus $X = O$. Moreover, for $i \neq s$,

$$|1 \oplus 0 - v_i|^2 = (1 - \cos\alpha)^2 + (|u_i| \sin\alpha)^2 \leq (1 - \cos\alpha)^2 + (M \sin\alpha)^2 < \epsilon^2,$$

and $|1 \oplus 0 - v_s|^2 = (1 - \sec\alpha)^2 < \epsilon^2$, that is $1 \oplus 0$ is the associated central vector. ∎

Lemma 3.4. *For $d > 1$, if G_1 and G_2 have central gram labellings in $\mathbb{R}^d$, then so does their union.*

Proof. Given $\epsilon > 0$, take $0 < \delta < \epsilon^2/2$ and pick two unit vectors $u, v \in \mathbb{R}^d$ such that $u^T v = 1 - \delta$ and thus $|u - v| = \sqrt{2\delta}$. Choose $\epsilon_1, \epsilon_2 > 0$ such that $\epsilon_1 + \epsilon_2 + \epsilon_1\epsilon_2 < \delta$ and $\epsilon_2 < \epsilon - \sqrt{2\delta}$. Let $i \mapsto u_i$ and $k \mapsto v_k$ be nondegenerate gram labellings of G_1 and G_2 in $\mathbb{R}^d$ with associated central vectors u and v respectively such that $|u_i - u| < \epsilon_1$, $|v_k - v| < \epsilon_2$ for each i, k. Then for each i, k we have

$$\begin{aligned} |u_i^T v_k - u^T v| &= |(u_i - u)^T v_k + u^T (v_k - v)| \\ &\leq |u_i - u||v_k| + |u||v_k - v| \\ &< \epsilon_1(1 + \epsilon_2) + \epsilon_2 < \delta, \end{aligned}$$

and consequently $u_i^T v_k < u^T v + \delta = 1$. Moreover, for each k, $|v_k - u| \leq |v_k - v| + |v - u| < \epsilon_2 + \sqrt{2\delta} < \epsilon$. Note that the nondegeneracy only needs to be checked on each component. Thus the union of $i \mapsto u_i$ and $k \mapsto v_k$ is a nondegenerate gram labelling for $G_1 \cup G_2$ with the associated central vector u. ∎

Lemma 3.5. *Let G_1 be a graph such that $\nu(G_1 \cup K_2) = d$. If a nonedgeless graph G_2 has a central gram labelling in $\mathbb{R}^d$, then $\nu(G_1 \cup G_2) = d$.*

Proof. Let $i \mapsto u_i$ be a nondegenerate gram labelling for $G_1 \cup K_2$ in $\mathbb{R}^d$. Let s and s' denote the vertices of K_2. Since $u_s^T u_{s'} = 1$, we may assume $|u_s| \geq 1$. Take $v = u_s/|u_s|$. Let $\delta = \min_{i \in G_1}(1 - u_i^T v)$ and $M = \max_{i \in G_1} |u_i|$. By assumption we have $\delta > 0$, and take $\epsilon = \delta/M$. Let $k \mapsto v_k$ be a nondegenerate gram labelling for G_2 in $\mathbb{R}^d$ with the associated central vector v such that $|v_k - v| < \epsilon$. Consequently for $i \in G_1$, $k \in G_2$, we have

$$|u_i^T v_k - u_i^T v| \leq |u_i||v - v_k| < M\epsilon = \delta,$$

hence $u_i^T v_k < u_i^T v + \delta \leq 1$. Therefore the union of $i(\in G_1) \mapsto u_i$ and $k \mapsto v_k$ is a nondegenerate gram labelling for $G_1 \cup G_2$. ∎

Proof. *(of Theorem 1.11)* We may assume $\nu(G) \geq 1$. By Lemma 3.3, each graph has a central gram labelling in $\mathbb{R}^{\nu(G)+1}$. Thus the inequality holds by Lemma 3.4. Now suppose $\nu(G \cup K_2) = \nu(G) > 1$ and for each G_i there exists a vertex v_i such that $\nu(G_i - v_i) < \nu(G)$. Let $G'_i = (G_i - v_i) \vee K_1$, which has a central gram labelling in

$\mathbb{R}^{\nu(G)}$ by Lemma 3.3. By Lemma 3.4 and 3.5, $\nu(G \cup \bigcup_{i=1}^{r} G_i') = \nu(G)$. Since $G_i \leq G_i'$, the result then follows from the monotonicity of ν under taking subgraphs. ■

The following proposition with $H = G_i$ and $n = \nu(G)$ helps us to find the first equality in Theorem 1.11.

Proposition 3.6. *For a positive integer $n > 1$, if $|H| \leq n+3$ and its complement $\overline{H}$ is not a subgraph of cycle C_{n+3}, then H has a vertex v such that $\nu(H - v) < n$.*

Proof. By monotonicity of ν under taking subgraphs, it suffices to show for H with order $n+3$ and whose complement aside from isolated vertices is a claw or a cycle on less than $n+3$ vertices. Let v be an isolated vertex in $\overline{H}$. Then $H = H' \vee \{v\}$, where $\nu(H') = n-1$ by Theorem 1.7 and characterization of $\mu = 2$. ■

It is shown that for a graph G and a vertex $v \in G$, we have

Theorem 3.7. *([5]) $\mu(G) \leq \mu(G - v) + 1$. Moreover, if v is connected to all other vertices and $G - v$ is not $\overline{K_2}$ or empty, then the equality holds.*

Consequently, $\mu(G) + 2 \geq \mu(G \vee \overline{K}_2) \geq \mu(G) + 1$ and the second equality holds if G is complete; together with Theorem 1.7 and 1.11, we can describe the behavior of μ under graph join.

Corollary 3.8. *Let $G, G_1, \ldots, G_r$ be graphs distinct from $\overline{K_2}$ such that $|G| - \mu(G) \geq |G_i| - \mu(G_i)$ for each i. Then*

$$\mu(G) + \sum_{i=1}^{r} |G_i| \geq \mu(G \vee G_1 \vee \cdots \vee G_r) \geq \mu(G) + \sum_{i=1}^{r} |G_i| - 1.$$

Moreover, if $\mu(G \vee \overline{K_2}) = \mu(G) + 2$ and $|G| - \mu(G) \geq 3$ and for each G_i there exists a vertex v_i such that $|G| - \mu(G) > |G_i - v_i| - \mu(G_i - v_i)$, then the first equality holds. ■

As $|G| \geq \mu(G) + 2$ iff $G \neq \overline{K}_2$ and is not complete, the corollary would imply the former result that for $G_1, G_2 \neq \overline{K_2}$, $\mu(G_1 \vee G_2) \geq \mu(G_1) + \mu(G_2) + 1$. Also we can fully characterize the value of $\mu(G)$ for complete multipartite graphs, which is an immediate corollary of Lemma 3.4 and Example 3.2. For the convenience of descriptions, let $t \in \mathbb{N}$ and $n_1 \geq n_2 \geq \ldots n_t > 1 = n_{t+1} = \ldots = n_r$ be integers and set $n = \sum_{i=1}^{r} n_i$.

Corollary 3.9. *If $n_1 \geq 4$, then $\nu(\bigcup_{i=1}^{r} K_{n_i}) = n_1 - 2$.* ■

Since ν-minimal graphs for $\nu = 2$ are $3K_2$ and P_4, we have that for complete multipartite graphs with $n_1 \leq 3$, $\nu(\bigcup_{i=1}^{r} K_{n_i})$ equals 1 if $t \leq 2$ and equals 2 if $t > 2$. As a direct consequence of Theorem 1.7, we have

Corollary 3.10.

$$\mu(K_{n_1, n_2, \ldots, n_r}) = \begin{cases} n - n_1 + 1, & \textit{if } n_1 \geq 4, \textit{ or if } n_1 = 3 \textit{ and } t \leq 2; \\ n - 3, & \textit{if } n_1 = 2 \textit{ and } t > 2; \\ n - n_1, & \textit{otherwise.} \end{cases}$$

■

Let $K_n - K_{n_1}$ denote the graph of order n obtained from K_n by deleting edges induced on one of its subgraph K_{n_1}, where $1 < n_1 < n$. Consequently, for any graph G with $K_{n_1,n_2,\dots,n_r} \le G \le K_n - K_{n_1}$,

$$\mu(G) \in \begin{cases} \{n - n_1 + 1\}, & \text{if } n_1 \ge 4, \text{ or if } n_1 = 3 \text{ and } t \le 2; \\ \{n-2, n-3\}, & \text{otherwise (can be explicitly known via } \nu(\overline{G})). \end{cases}$$

Theorem 1.11 and Proposition 3.6 also give some basic characterizations for disconnected ν-minimal graphs. Note that $C_{k+2} \vee \overline{K_2}$ is a maximal planar graph on $k+4$ vertices for all $k \ge 2$, or equivalently, it is a μ-maximal graph for $\mu = 3$ by characterizations for μ. Thus $\overline{C_{k+2}} \cup K_2$ is a ν-minimal graph for $\nu = k$ by Theorem 1.7.

Corollary 3.11. *If a ν-minimal graph for $\nu = k$ is disconnected and distinct from $\overline{C_{k+2}} \cup K_2$ then it takes one of the two forms: $\bigcup_{i=1}^{r+1} G_i$ or $K_2 \cup \bigcup_{i=1}^{r} G_i$, where each G_i is connected, $\nu(G_i) = k-1$ and for every vertex $v \in G_i$, $\nu(G_i - v) = k-1$. In particular, $|G_i| \ge k+3$.* ■

We do not know at most how many components does a ν-minimal graph have in general. It would be informative to find another family of disconnected ν-minimal graphs aside from $\overline{C_{k+2}} \cup K_2$ and $C_{k+3} \cup K_2$, $k \ge 2$, or to prove some necessary conditions. Indeed, we would like to ask whether K_2 is the only detector, that is

Conjecture 3.12. *For $i = 1, 2$, if $\nu(G_i) = \nu(G_i \cup K_2) = k$, is $\nu(G_1 \cup G_2) = k$?*

In classical extremal graph theory, it is asked whether for each graph G, $e(G) := |E(G)| \le (\eta(G) - 1)|G| - \binom{\eta(G)}{2}$, which is known to be true only for $\eta(G) \le 6$ and fail for $\eta(G) > 6$. For more on Hadwiger numbers, we refer one to the survey [7] written by Seymour. McCarty in [8] proved that if one replaces $\eta(G)$ with $\mu(G) + 1$, then the inequality holds for $\mu(G) \le 7$ and she used this to show that the inequality also holds for $\mu(G) \ge n - 6$. Note that $e(\overline{G}) = \binom{|G|}{2} - e(G)$ and

$$\binom{|G|}{2} - \mu(G)|G| + \binom{\mu(G)+1}{2} = \frac{(|G| - \mu(G) - 1)(|G| - \mu(G))}{2}.$$

Consequently by Theorem 1.7, the problem can be reformulated as

Conjecture 3.13. $e(G) \ge \binom{\nu(G)+1}{2}$ *for all graphs G.*

As just mentioned, the conjecture is proved for $\nu(G) \le 5$ and $|G| - 8 \le \nu(G)$. Here we provide a much simpler proof for $\nu(G) \le 5$. Note that if G is a minimal counter example to Conjecture 3.13, then $|G| \ge \nu(G) + 9$. Also, by Corollary 3.11, G must either be connected or $G = G' \cup K_2$ for some connected graph G'. Thus $e(G) \ge \nu(G) + 8$ if G is connected and $e(G') \ge \nu(G) + 6$ if $G = G' \cup K_2$. This shows that the conjecture holds for $\nu \le 4$. Now assume $\nu(G) = 5$. By assumption of G, $e(G) \le 14$. Then in both cases, G is planar and thus by Theorem 1.8, $\nu(G) \le 4$, a contradiction.

We close this section by a minor result on the realizability question of μ-optimal matrices for complete multipartite graphs.

Corollary 3.14. *Each complete multipartite graph has a μ-optimal matrix with coefficients in $\mathbb{Q}$, hence in $\mathbb{Z}$.*

Proof. We remark that the proof of Theorem 1.4 for $\mu(G) \geq 2$ is constructive. To be more precise, given a matrix A satisfying (A1)-(A3) for G with corank $\mu(G)+1$, Kotlov, Lovász and Vempala shows that $A - J$ is a μ-optimal matrix for G. By Corollary 3.10, complete multipartite graphs have $\mu \geq 2$ except for K_2 and $K_{1,2}$. Clearly, K_2 and $K_{1,2}$ have μ-optimal matrices over $\mathbb{Z}$ (for $K_{1,2}$, take $M = \begin{pmatrix} 0 & -\mathbf{1}^T \\ -\mathbf{1} & O \end{pmatrix}$). Thus we restrict ourselves to complete multipartite graphs with $\mu(G) \geq 2$. Since J is a matrix over $\mathbb{Z}$, it suffices to construct a gram labelling in $\mathbb{Q}^{\nu(\overline{G})}$ for $\overline{G}$. The case $\nu(\overline{G}) = 1$ is easy. For $n_1 \leq 3$ and $\nu(\overline{G}) = 2$, one can obtain a desired gram labelling by either using Lemma 3.3 or doing vertex deletions from larger graphs. We show the construction for those with $n_1 \geq 4$. Recall the central gram labellings constructed for complete graphs in Example 3.2. For each clique component C_i in $\overline{G}$, if $|C_i| = 1$, we label it with the zero vector; otherwise, we associate each with a distinct unit vector being the associated central vector and an extended orthonormal basis in $\mathbb{Q}^{\nu(\overline{G})}$. Then we choose $0 < \alpha_i \in \mathbb{Q}$ sufficiently small such that for any two components, their union of labellings satisfies (U1). Then the union of these labellings is the desired one. ∎

4 Decomposing μ-maximal graphs with separating cliques

For a graph $G = (V, E)$, a proper subset $S \subset V$ *separates* G if $G - S$ is disconnected. Moreover, if S induces a clique in G, then it is called a *separating clique* of G. To put it another way, we say $G = (V, E)$ is a *(pure) clique sum* of $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ along the clique S if $V = V_1 \cup V_2$ with $V_1 \cap V_2 = S$ inducing cliques in both G_1 and G_2, and $E = E_1 \cup E_2$. That is, G is obtained from patching G_1 and G_2 together by identifying the set S (a bijection in between S in V_1 and V_2). The clique S is always assumed to be a proper subset of both V_1 and V_2.

Definition 4.1. For a graph G, we say G is *decomposable* if G has a separating clique; otherwise, G is *indecomposable*.

The goal of this section is to show that each decomposable μ-maximal graph can be successively built up by the indecomposable ones via clique sums along cliques of order not larger than μ with certain criterion. We need a crucial theorem that fully describes the behavior of μ under clique sum. This criterion was discovered by van der Holst, Lovász and Schrijver in [4] by the observation that $K_{t+3} - K_3$ is a clique sum of 3 pieces of K_{t+1} along K_t and that $\mu(K_{t+3} - K_3) = t + 1$. This can be obtained by Theorem 3.7 together with $\mu(\overline{K_3}) = 1$ and $K_{t+3} - K_3 = \overline{K_3} \vee K_t$.

Theorem 4.2. *([4]) Let G be a clique sum of G_1 and G_2 along the clique S and $t = \max\{\mu(G_1), \mu(G_2)\}$. If $\mu(G) > t$, then $\mu(G) = t + 1$ and we can contract two or three components of $G - S$ so that the contracted vertices together with S form $K_{t+3} - K_3$ (removing a triangle in K_{t+3}).*

To be more precise, the case $\mu(G) = t + 1$ occurs if and only if $s := |S| \geq t = \mu(G_1) = \mu(G_2)$ and one of the following occurs:

(i) $s = t$ and $G - S$ has at least three components C such that $N_G(C) = S$. In this case, G_1 and G_2 both contain at least one and at most two such components;

(ii) $s = t+1$ and $G - S$ has (exactly) two components C, C' in such that $N_G(C) = N_G(C')$ is of cardinality t. In this case, G_1 and G_2 both contain exactly one of such components;

We first prove a basic fact for μ-maximal graphs.

Proposition 4.3. *A μ-maximal graph is connected.*

Proof. Suppose G is a disconnected μ-maximal graph. Clearly, G is non-edgeless by characterizations for $\mu = 1$. Let G_1 be a non-edgeless component of G and $G_2 = G - G_1$. For $i = 1, 2$, we add a pendant vertex w_i to a vertex v_i of G_i (if G_i is a union of path, then v_i needs to be chosen of degree less than 2). Then let G' be the clique sum of these two graphs by identifying w_1 and w_2 and contracting w_1 to v_1. By minor-monotonicity of μ, Theorem 4.2 and characterizations of $\mu = 1$, $\mu(G') \leq \mu(G)$ and G' properly contains G, a contradiction. ∎

We recall that Theorem 3.7 and minor-monotonicity of μ together say that adding an edge or joining a vertex to a graph increase μ by at most 1. In the following discussions, by adding an edge e to G, we are assuming $e \in E(\overline{G})$.

Proposition 4.4. *Suppose G is a clique sum of G_1 and G_2 along the clique S, and assume that G is μ-maximal. Then $\mu(G) = \max\{\mu(G_1), \mu(G_2)\}$ and the following holds:*

(i) If $|S| < \mu(G)$, then G_1 and G_2 are themselves μ-maximal. In particular, if $\mu(G_i) < \mu(G)$, then G_i is a complete graph.

(ii) If $|S| \geq \mu(G)$, then there exist subgraphs G_1', G_2' of G and a clique $S' \subset S$ of order not larger than $\mu(G)$, such that G is a clique sum of G_1' and G_2' along the clique S'.

Proof. Let $s = |S|$ and $t = \max\{\mu(G_1), \mu(G_2)\}$. Suppose on the contrary that $\mu(G) > t$. Then $\mu(G_1) = \mu(G_2)$ and $s \in \{t, t+1\}$. If $s = t+1$, then both G_1 and G_2 are not complete; if $s = t$, then one of G_1, G_2 is not complete. Since if both are complete, then $G = K_{t+2} - K_2$, contradicting the assumption that $\mu(G) = t+1$. Say G_1 is not complete. Add any edge e to G_1. Then $\mu(G_1 + e) \leq t+1$ and by μ-maximality of G, we have $\mu(G+e) = t+2$, but this would imply $\mu(G_1 + e) = \mu(G_2) = t+1$, a contradiction. Thus $\mu(G) = t$.
Case $s < t$: Suppose there exists an edge e such that $\mu(G_1 + e) \leq t$. By μ-maximality of G, $\mu(G+e) = t+1$ implies $s \geq t$, a contradiction. The proof is the same for G_2.
Case $s = t$: If G_1 and G_2 are both μ-maximal, then we are done. Suppose say G_1 is not μ-maximal. Let e be any edge such that $\mu(G_1 + e) = \mu(G_1)$. By μ-maximality of G, we have $\mu(G+e) = t+1$. By Theorem 4.2, $\mu(G_2) = t = \mu(G_1+e) = \mu(G_1)$ and there must exist a component C in $G_1 - S$ such that $|N_{G_1}(C)| = t-1$ (so that after adding e, there are three components, including C, whose neighbors are S). Let G_1' be the graph induced by $C \cup N_{G_1}(C)$ in G_1 and let $G_2' = G - C$. Then G is the clique sum of G_1' and G_2' along $N_{G_1}(C)$. By the previous case, G_1' and G_2' are μ-maximal, done.
Case $s = t+1$: Let C be any component of $G - S$. Then $|N_G(C)| \leq t$. Let H_1 be the graph induced by $C \cup N_G(C)$. Let $H_2 = G - C$. Then G is the clique sum of H_1 and H_2 along $N_G(C)$ and the result follows from previous cases. ∎

The following consequence, directly derived from the proof of Proposition 4.4, will be frequently used in the rest of the section.

Corollary 4.5. *Suppose G is a clique sum of G_1 and G_2 along the clique S, and assume that G is μ-maximal. For a component C of $G - S$, the graph $C \cup N_G(C)$ is μ-maximal.* ■

Also, by Proposition 4.4, we see that to construct μ-maximal graphs via clique sum, we can only patch against cliques of order at most μ. Now we prove the converse to Proposition 4.4, that is, the criterion in judging whether the clique sum of two μ-maximal graphs is still μ-maximal. We need an easy lemma.

Lemma 4.6. *Let G be a μ-maximal graph and S be a clique in G of order not larger than $\mu(G) - 1$. For $v \in G - S$, let C be the component of $G - S$ that contains v. If $N_G(C) \subset N_G(v)$, then $S = N_G(C)$.*

Proof. Let $S' = N_G(C)$. Suppose on the contrary that $S - S'$ is nonempty and take $w \in S - S'$. Let H be the clique sum of $C \cup S'$ and $K_{|S'|+2}$ with $V(K_{|S'|+2}) = S' \cup \{v, w\}$ along $S' \cup \{v\}$. Note that by Theorem 4.2, $\mu(H) \in \{\mu(C \cup S'), |S'| + 1, |S'| + 2\}$, which is not larger than $\mu(G)$ as by assumption $|S'| \leq \mu(G) - 2$. We patch H with S along $S' \cup \{w\}$ and patch the obtained graph with $G - C$ along S, forming the graph $G + vw$ with $\mu(G + vw) \leq \mu(G)$, which contradicts the μ-maximality of G. ■

For the convenience of stating our result, we introduce the following definition.

Definition 4.7. A *pair* (G, S) consists of a μ-maximal graph G and a clique S in G of order $\mu(G) - 1$. We say a pair (G, S) has *property (P)* if S is not maximal in G and for any vertex $u \in G$ with $S \subset N_G(u)$, $G - (S \cup \{u\})$ have two components C, C' such that $N_G(C) = N_G(C') = S \cup \{u\}$.

Proposition 4.8. *Let G_1 and G_2 be μ-maximal graphs and G be a clique sum of G_1 and G_2 along the clique S. Let $t = \max\{\mu(G_1), \mu(G_2)\}$. If $s := |S| < t$, then G is μ-maximal if and only if one of the following is satisfied.*

(i) *S is a maximal clique in G_1 or G_2;*

(ii) *$s = t - 1$ and (G_1, S) or (G_2, S) has property (P).*

Moreover, in both cases if $\mu(G_i) < t$, then G_i is complete.

Proof. We first prove the necessity. The last assertion follows from (i) of Proposition 4.4. We show the proof only for the case $s < t - 1$ as the proof for the case $s = t - 1$ is exactly the same. Now suppose $s < t - 1$ and S is not maximal in either G_1 or G_2. For $i = 1, 2$, let v_i be a vertex in G_i with $S \subset N_{G_i}(v_i)$, and let G_i' be the clique sum of G_i and K_{s+2} with $V(K_{s+2}) = S \cup \{v_i, w\}$ along $S \cup \{v_i\}$, where w is an added vertex. We have $\mu(G_i') \leq t$ since $|S \cup \{v_i\}| < t$ for $i = 1, 2$. Let G' be the clique sum of G_1' and G_2' along the clique $S \cup \{w\}$ (as the same way G is patched). Let G'' be the graph obtained from G' by contracting w to v_1 via the edge $v_1 w$. Then $G'' = G + v_1 v_2$ and we have, by minor-monotonicty of μ, $t \geq \mu(G') \geq \mu(G'') \geq \mu(G) = t$, contradicting the μ-maximality of G. For sufficiency, it suffices to show that for any vertices $u \in G_1 - S$ and $v \in G_2 - S$, we have $\mu(G + uv) > t$. Suppose condition (i) is satisfied. Say S is maximal in G_1. Thus G_1 is not complete and hence $\mu(G_1) = t$. Let C be the component of $G_2 - S$ that contains v. By maximality of S, $T := N_{G_1}(u) \cap S \subsetneq S$. If $S' := N_{G_2}(C) = S$, take $w \in S - T = S' - T$. Contract C to w in $G_1 \cup C + uv$ and form the new graph G_1', which properly contains G_1. Then we have $\mu(G + uv) \geq$

$\mu(G_1') > \mu(G_1) = t$. Otherwise, suppose $S' \subsetneq S$. Then G_2 is not complete and thus $\mu(G_2) = t$. Consequently, by Lemma 4.6, $S' \not\subset N_{G_2}(v)$. Note that G_2 can be viewed as the clique-sum of $C \cup S'$ and $G_2 - C$ along S'. By Corollary 4.5, $C \cup S'$ is μ-maximal. As $S' \not\subset N_{G_2}(v)$, $C \cup S'$ is not complete and therefore $\mu(C \cup S') = t$. Let C' be the component of u in $G_1 - S'$. Let $T' := N_{G_2}(v) \cap S' \subsetneq S'$. If $S'' := N_{G_1}(C') = S'$ then similarly we have $\mu(G + uv) > \mu(C \cup S') = t$. Otherwise, suppose $S'' \subsetneq S'$. Continue this process which will ultimately terminate as $|S|$ is finite, and we are done. Suppose condition (ii) is satified. Say (G_1, S) has property (P). Let C be the component of $G_2 - S$ that contains v. Suppose $S' := N_{G_2}(C) = S$. If $S \not\subset N_G(u)$, then as argued above, $\mu(G + uv) > \mu(G_1) = t$ and we are done. If $S \subseteq N_G(u)$, then by assumption of G_1, $G + uv$ can be contracted to $K_{t+3} - K_3$, done. Otherwise, suppose $S' \subsetneq S$. Then the rest of the argument is exactly the same as the case (i). ∎

Proposition 4.9. *Let G_1 and G_2 be μ-maximal graphs and G be a clique sum of G_1 and G_2 along the clique S. Suppose $\mu(G_1) = t = \mu(G_2)$. If $s := |S| = t$, then G is μ-maximal if and only if $G - S$ has at most two components C, C' such that $N_G(C) = N_G(C') = S$.*

Proof. The necessity follows from Theorem 4.2 and proposition 4.4. Conversely, assume $G - S$ has at most two such components. We first consider the case that $N_{G_1}(C) \subsetneq S$ for every component C of $G_1 - S$. Let C be a component of $G_1 - S$ and $S' := N_{G_1}(C)$. Since $|S'| < t$, by Proposition 4.4, $H := C \cup S'$ and $G_1 - C$ are μ-maximal. By assumption, if $S \subset G_1 - C$ is complete, then $G_1 - C = S$. It follows by (i) of proposition 4.8 that $\mu(G_1 - C) = t$ unless $G_1 - C = S$ (if this is the case, then we've finished decomposing G_1). Moreover, since $S' \subsetneq S$, by Proposition 4.8 and assumption of G_1, S' is either maximal in H or $|S'| = t - 1$ and (H, S') has property (P). As C is arbitrary, by successively decomposing and patching up, we have that G is μ-maximal by proposition 4.8. Now suppose $G_1 - S$ has exactly one component C_1 such that $N_{G_1}(C_1) = S$. Theorem 4.2 and assumption of G_1 implies that $H_1 := G_1 - C_1$ is μ-maximal. Indeed, if H_1 is not μ-maximal, then there exists an edge e such that $\mu(H_1 + e) = \mu(H_1) \leq t$. Since $G_1 + e$ is the clique sum of $C_1 \cup S$ and $H_1 + e$ along S. Then by Theorem 4.2, $\mu(G_1 + e) = \max\{\mu(C_1 \cup S), \mu(H_1 + e)\} \leq \mu(G_1)$, contradicting μ-maximality of G_1. Also, by Corollary 4.5, $C_1 \cup S$ is μ-maximal. By Theorem 4.2 and assumption of G_1, $\mu(H_1) = t$ unless $H_1 = S$. We show that $C_1 \cup G_2$ is μ-maximal. The assertion then follows by the previous case. For $u \in C_1$ and $v \in G_2 - S$, let C_2 be the component in $G_2 - S$ containing v. If $N_{G_2}(C_2) \subsetneq S$, then pick $w \in S - N_{G_2}(C_2)$. Contracting C_1 to w in $C_1 \cup G_2 + uv$, we obtain $G_2 + vw$ which properly contains G_2, whence $\mu(C_1 \cup G_2 + uv) > \mu(G_2) = t$. If $N_{G_2}(C_2) = S$, then $C_1 \cup G_2 + uv$ can be contracted into K_{t+2}, hence $\mu(C_1 \cup G_2 + uv) = t + 1$. ∎

Lemma 4.10. *Let G be a μ-maximal graph with a clique S such that $|S| = \mu(G) + 1$. Suppose for each component C of $G - S$, $C \cup N_G(C)$ is complete. Then $|N_G(C)| = \mu(G)$, whence $|C| = 1$ for each component C. Consequently, for each $v \in S$, there is at most one vertex $\hat{v} \in G - S$ such that $N_G(\hat{v}) = S - \{v\}$. Conversely, if a graph satisfies the assertion, it is μ-maximal.*

Proof. Let C be a component of $G - S$, $S' := N_G(C)$ and set $G_C := C \cup S'$, which is by assumption complete. Let $t = |S| = \mu(G) + 1$, $m = |G_C| \leq t$ and $s = |S'|$. We have to show $s = t - 1$. Suppose $s < t - 2$. Take a vertex $u \in S - S'$. If $|C| = 1$, then $m \leq t - 2$ and G is a proper subgraph of the clique sum of $G - C$ and K_{m+1} with $V(K_{m+1}) = G_C \cup \{u\}$ along $S' \cup \{u\}$, whose μ by Theorem 4.2 is not larger than

$\mu(G)$ since $|S' \cup \{u\}| \leq t-2$, contradicting μ-maximality of G. Thus $|C| \geq 2$ and pick $w \in C$. Consider the clique sum G' of $G - C$ and K_m with $V(K_m) = G_C \cup \{u\} - \{w\}$ along $S' \cup \{u\}$. Again by Theorem 4.2, since $|S' \cup \{u\}| \leq t-2$, we have $\mu(G') \leq t-1$. Then consider the clique sum G'' of G' and G_C along $G_C - \{w\}$. Since $|C| > 1$, among vertices in G', only u shares the same neighbor set with $\{w\}$. Therefore by Theorem 4.2, $\mu(G'') \leq t-1$ and as $|C| > 1$, G'' properly contains G, a contradiction. Now suppose $s = t-2$. Let $S - S' = \{u_1, u_2\}$. By μ-maximality of G together with Theorem 4.2, Proposition 4.9 and the assumption of G, we may assume there exists a unique vertex $v_i \in G - S$ such that $N_G(v_i) = S' \cup \{u_i\}$ for $i = 1, 2$. Let $G_i = S' \cup \{u_i, v_i\}$ for $i = 1, 2$. Take $w \in C$ and let $G''' = G_1 \cup G_2 \cup G_C + wv_2$. We show that $\mu(G''') = t-1$. Then by Theorem 4.2, the clique sum G'''' of G''' and $G - C - \{v_1, v_2\}$ along S has $\mu(G'''') = t-1$ since no vertices in $G - S - C - \{v_1, v_2\}$ share the same neighbor set with either v_1 or v_2. Now as G'''' properly contains G, we obtain a contradiction. To show $\mu(G''') = t-1$, observe that G''' is the clique sum of $\{v_1\} \cup S$ and $G_2 \cup G_C + wv_2$ along $S - \{u_1\}$. Note that $\mu(G_2 \cup G_C + wv_2) = \mu(G_2 \cup G_C) = t-1$ (by computing the ν value of their complements) and $\mu(\{v_1\} \cup S) = t-1$. Since $G_2 \cup G_C + wv_2 - (S - \{u_1\}) = C \cup \{v_2\} + wv_2$ is connected, by Theorem 4.2, $\mu(G''') = t-1$. This proves the assertion. The rest of the statement then follows from Theorem 4.2 and Proposition 4.9. ∎

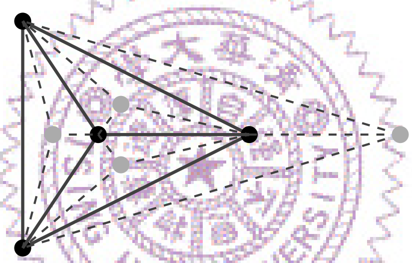

Figure. An example of graphs in Lemma 4.10, where S is the black K_4 and for each $v \in S$, there exists a unique grey vertex $\hat{v}$ with $N_G(\hat{v}) = S - \{v\}$. Removing any number of grey vertices also leads to a graph in Lemma 4.10.

Let G_1 and G_2 be μ-maximal graphs and G be a clique sum of G_1 and G_2 along the clique S. Now the only undealt case is $|S| = t = \mu(G_1)$ and $\mu(G_2) = t-1$. Suppose G is μ-maximal. Let C be a component of $G_2 - S$. Then $N_{G_2}(C) \subsetneq S$. Therefore, by (i) of Proposition 4.4, we have $C \cup N_{G_2}(C)$ is complete. Thus by Lemma 4.10, for each $v \in S$, there exists at most one vertex $\hat{v} \in G_2 - S$ such that $N_{G_2}(\hat{v}) = S - \{v\}$. Then by proposition 4.8, G is maximal if and only if for each $v \in S$ such that $\hat{v}$ exists, $(G_1, S - \{v\})$ has property (P).

The following proposition provides an insight into pairs (G, S) with property (P).

Proposition 4.11. *Let (G, S) be a pair with property (P). Then there exists an indecomposable μ-maximal subgraph H of G such that (H, S) form a pair with a unique vertex $v \in H$ such that $S \subset N_H(v)$.*

Proof. For $u \in G$ such that $S \subset N_G(u)$ and a component C of $G - (S \cup \{u\})$ with $N_G(C) = S \cup \{u\}$, define $m_{(u,C)} := |\{v \in C | S \subset N_G(v)\}|$. Let $m = \min\{m_{(u,C)}\}$, where the minimum is taken over all such (u, C). Choose u_0, C_0 such that $m_{(u_0,C_0)} =$

m. Suppose $m > 0$. Let C_0' be the other component of $G - (S \cup \{u_0\})$ with $N_G(C_0') = S \cup \{u_0\}$. Since $m > 0$, there exists $u_1 \in C_0'$ such that $S \subset N_G(u_1)$. Let C_1 be the component of $G - (S \cup \{u_1\})$ containing u_0. Since $u_1 \notin C_0$ is connected and $N_G(C_0) = N_G(C_0') = S \cup \{u_0\}$, we have $C_0 \subset C_1$ and $N_G(C_1) = S \cup \{u_1\}$. Suppose for each $1 \leq i \leq k$, we've obtained a vertex u_i with $S \subset N_G(u_i)$ in G, all distinct, and a connected component C_i of $G - (S \cup \{u_i\})$ containing u_{i-1} with $N_G(C_i) = S \cup \{u_i\}$. Moreover, $C_{i-1} \subset C_i$ for each $1 \leq i \leq k$. Let C_k' be the other component of $G - (S \cup \{u_k\})$ with $N_G(C_k') = S \cup \{u_k\}$. Since $m > 0$, C_k' has a vertex u_{k+1} such that $S \subset N_G(u_{k+1})$. Let C_{k+1} be the component of $G - (S \cup \{u_{k+1}\})$ containing u_k. Since $u_{k+1} \notin C_k$ is connected and $N_G(C_k) = N_G(C_k') = S \cup \{u_k\}$ we have $C_k \subset C_{k+1}$ and $N_G(C_{k+1}) = S \cup \{u_{k+1}\}$. Ultimately, we would obtain an infinite sequence of distinct vertices in G, contradicting the finiteness of G. Thus $m = 0$, that is, there is no vertex other than u_0 which is the common neighbor of S in $H := C_0 \cup S \cup \{u_0\}$. Note that by Corollary 4.5, H is μ-maximal with $\mu(H) = \mu(G)$. If there exists a separating clique S' of H, then clearly by the choice of H, $S' \not\subset S \cup \{u_0\}$ and $S \cup \{u_0\} \not\subset S'$. Let C' be the component of $H - S'$ containing $S \cup \{u_0\} - S'$ and let $H' := C' \cup N_H(C')$. Since C_0 is connected and $N_G(C_0) = S \cup \{u_0\}$, $C_0' := H' - S \cup \{u_0\}$ is connected and $N_H(C_0') = S \cup \{u_0\}$. Again, by Corollary 4.5, H' is μ-maximal with $\mu(H') = \mu(G)$. Continue the process until there is no separating cliques and we would obtain the desired graph. ■

Corollary 4.12. *The existence of pairs (G, S) with property (P) is equivalent to the existence of pairs (H, S), where H is indecomposable with a unique vertex $v \in H$ satisfying $S \subset N_H(v)$.*

Proof. By Proposition 4.11, it remains to show the converse. Suppose such pair (H, S) exists. Let G be a clique sum of two pieces of H along $S \cup \{v\}$. By Proposition 4.9, G is μ-maximal with $\mu(G) = \mu(H)$. Then since H is indecomposable, $C := H - (S \cup \{v\})$ is connected with neighbor set $S \cup \{v\}$. Thus (G, S) is a pair with property (P). ■

By characterizations of μ, for $\mu = 2$, each vertex in a μ-maximal graph is of degree at least 2. For $\mu = 3$, every edge in a μ-maximal graph is contained in two distinct K_3(faces). Consequently, by Corollary 4.12, there exist no pairs (G, S) with property (P) for $\mu \leq 3$. For $\mu \geq 4$, the existence of such a pair (G, S) is unknown. We suggest that such pairs do not exist. Moreover, we would like to ask the following question. If it is true, it would serve as a generalization of topological properties of graphs.

Question 4.13. Let S be a clique in an indecomposable μ-maximal graph G. Suppose S is not maximal. Is it true that if $|S| < \mu(G)$, then there are at least two vertices u, v such that S is contained in both $N_G(u)$ and $N_G(v)$?

Now we prove a basic property of indecomposable μ-maximal graphs, which is an easy consequence of the behavior of μ under ΔY transformations. The ΔY *transformation* works as follows: for a given graph, select a triangle of it, and add a new vertex adjacent to all vertices of the triangle, and delete the edges of triangles. The $Y\Delta$ *transformation* is the inverse action. That is, for a given graph, select a vertex v of degree 3, and make its neighbors pairwise adjacent, and delete v.

Theorem 4.14. *([5]) Let G be a graph and if G' arise from G by applying a ΔY transformation to a triangle. Then $\mu(G) \leq \mu(G')$ and the equality holds if $\mu(G) \geq 4$.*

To put it another way, Theorem 4.14 says that μ is decreasing under YΔ transformations on vertices of degree 3. Note that by the characterizations of μ, the only indecomposable μ-maximal graphs for $\mu \leq 2$ are complete graphs.

Proposition 4.15. *Let G be an indecomposable μ-maximal graph, not complete. If S is a minimal separating set of G and $|S| < \mu(G)$, then $G - S$ has at most $|S| - 2$ component.*

Proof. Let $C_1, \ldots, C_r$ denote the connected components of $G - S$. Note that since S is minimal, $N_G(C_i) = S$ for all i. We show that if $|S| \leq 2$ or $r > |S| - 2$, then S must induce a clique in G contradicting the assumption of indecomposability of G. Suppose $|S| \geq 3$ and $r > |S| - 2$. For each component C_i, we will construct a graph C_i' that contains $C_i \cup S$ as a subgraph, in which S induces a clique, and $\mu(C_i') \leq \mu(G)$. Patch up these C_i' along S as $C_i \cup S$ are patched and form the new graph G'. Then we have, by Theorem 4.2, $\mu(G) \geq \mu(G')$ and the μ-maximality of G implies $G' = G$, which proves the assertion. Now we show the construction for $i = 1$. For $3 \leq m \leq |S| - 1$, contract components C_m to any $|S| - 3$ distinct vertices in S. By deleting edges if necessary, we may assume the other three vertices x, y, z in S induces a coclique. Then contract C_2 into a vertex w and, deleting edges if necessary, we may assume w is adjacent to x, y, z only. Do a YΔ operation to the claw, and delete other components C_m for $m > |S| - 1$. We form the desired graph C_1'. The case $|S| = 1$ is trivial and the case $|S| = 2$ is similarly proved, without the need of using $Y\Delta$ transformations. ■

Corollary 4.16. *Let G be an indecomposable μ-maximal graph, not complete. Then G is 4-connected.*

Proof. If G is planar and S is a minimal separating set of cardinality 3, then $G - S$ have at most 2 components, since if not, then G would have a $K_{3,3}$ minor. Similarly argued as in proposition 4.15, we would obtain that S is a clique, a contradiction. Then the assertion follows from proposition 4.15. ■

We would like to ask whether indecomposable μ-maximal graphs, not complete, are $\mu(G)$-connected or even weakly, do such graphs have minimum degree $\delta(G) \geq \mu(G)$. Note that indecomposability is required for connectivity. Consider the graph G whose complement is $C_k \cup K_2$, $k \geq 7$. As mentioned in section 1, since $C_k \cup K_2$ is ν-minimal for $\nu = 3$ and has maximum coclique of order $\lfloor k/2 \rfloor + 1$, G is μ-maximal with $\mu(G) = k - 2$ and $\omega(G) = \lfloor k/2 \rfloor + 1 < k - 2$. It follows by proposition 4.8 that for any clique sum of two pieces of G along any maximum clique is μ-maximal. However, such graphs have vertex-connectivity at most $\lfloor k/2 \rfloor + 1$.

It would be interesting to find methods (in terms of μ or ν) in deriving an indecomposable μ-maximal graph from one another. Note that the most trivial case is that if G is indecomposable μ-maximal, then so is $G \vee K_1$ (In terms of ν and $\overline{G}$, an isolated vertex is added.) Another question is to find how small could a maximal clique be for (indecomposable) μ-maximal graphs with given $\mu = k$.

5 μ and ν under Cartesian products

In [3], Goldberg proved that $\mu(G \square K_m) \geq \mu(G) + \mu(K_m)$ for all connected graphs G and positive integers m by explicit constructions of Colin de Verdière matrices for

$G\Box K_m$ with specific corank. He further asked whether K_m could be replaced with any other connected graphs. By using both graph theoretic approaches and explicit constructions, we prove that the statement holds for complete bipartite graphs and graphs with $\mu \leq 5$.

Proposition 5.1. *Let G, G_1, G_2 be connected graphs. The following holds*

(i) $\mu(G_1\Box(G_2 \vee K_1)) \geq \mu(G_1\Box G_2) + 1$;

(ii) $\mu(G\Box K_m) \geq \mu(G) + m - 1$;

(iii) if G_2 can be obtained from G by doing several ΔY operations on triangles, then $\mu(G_1\Box G_2) \geq \mu(G_1\Box G)$.

Proof. (i) Let v denote the added vertex to G_2. By contracting v's copy of G_1 in $G_1\Box(G_2 \vee K_1)$ into a vertex, we obtain $(G_1\Box G_2) \vee K_1$. Then the result follows by minor-monotonicity of μ and Theorem 3.7. (ii) Since $G\Box K_1 = G$ and $K_m = K_{m-1} \vee K_1$, the result follows by repeatedly applying (i). (iii) By doing the same series of ΔY transformations on triangles in each copy of G in $G_1\Box G$, we obtain a subgraph of $G_1\Box G_2$. The result follows by minor-monotonicity of μ and Theorem 4.14. ■

For two matrices A, B, we denote their tensor product by $A \otimes B$, where the (i, j)th block of $A \otimes B$ is $a_{ij}B$.

Proposition 5.2. $\mu(G\Box K_{3,3}) \geq \mu(G) + 4$ *for a connected graph G.*

Proof. Let A be a μ-optimal matrix for G with the negative eigenvalue $\lambda_1(A) = -3$. By Perron-Frobenius theorem, we may choose a corresponding eigenvector $\xi > 0$. Let B denote the adjacency matrix of $K_{3,3}$. Let $n = |G|$. Then $M = I_6 \otimes A - B \otimes I_n + 3I_6 \otimes I_n \in \mathscr{O}_{G\Box K_{3,3}}$ has corank$(M) = \mu(G) + 4$ and $\lambda_1(M) = -3$. We show that M has SAP and the result follows.

We first partition M into 6×6 blocks: $M = [M_{ij}]$, $1 \leq i, j \leq 6$. Also we view $\{1, 2, 3\}$ and $\{4, 5, 6\}$ as two cocliques of order 3 in $K_{3,3}$. By the definition of M, we have:

$$M_{ij} = \begin{cases} A + 3I_n & \text{if } i = j \\ -I_n & \text{if } i \sim j \\ O_n & \text{if } i \nsim j \end{cases} \tag{1}$$

Suppose X fully annihilates M. We have to show $X = O$. Similarly, we partition X into 6×6 blocks: $X = [X_{ij}]$, $1 \leq i, j \leq 6$. For any i, j, let $[i, j]$ denote the (i, j)th block of MX. Since $MX = O$, (j, l)th column of X can be expressed as $\mathbf{1}_6 \otimes u_{jl} + v_{jl} \otimes \xi$ where $u_{jl} \in \ker(A)$ and $v_{jl} \in \ker(B)$. Then lth column of $\sum_{i=1}^{6} X_{ij}$ is $6u_{jl}$. Hence $A\sum_{i=1}^{6} X_{ij} = O$. Also, by (1) we have

$$O = [i, j] = \sum_{k=1}^{6} M_{ik}X_{kj} = (A + 3I_n)X_{ij} - \sum_{k\sim i} X_{kj} \tag{2}$$

Let p, q, r be three vertices in $K_{3,3}$ that form a coclique. By (2), we have

$$O = [p, j] + [q, j] + [r, j] = (A + 3I) \sum_{i=p,q,r} X_{ij} - 3\sum_{k\sim p} X_{kj} \tag{3}$$

Using $A\sum_{i=1}^{6} X_{ij} = O$ and (3),

$$O = A(A+3I)\sum_{i=p,q,r} X_{ij} - 3A\sum_{k\sim p} X_{kj} = (A^2+6A)\sum_{i=p,q,r} X_{ij} = A\sum_{i=p,q,r} X_{ij} \quad (4)$$

since $A+6I$ is nonsingular. Again by (2), we have

$$O = [p,j] - [q,j] = (A+3I)(X_{pj} - X_{qj}) \quad (5)$$

(4) and (5) together yield:

$$O = A(2[p,j] - [q,j] - [r,j]) = (A+3I)(2AX_{pj} - AX_{qj} - AX_{rj}) = 3(A+3I)AX_{pj}$$

As a consequence, $AX_{pj} = c_{pj}\xi\xi^T$ for some constant c_{pj}. For each i, since $A \circ X_{ii} = O$ and $I_n \circ X_{ii} = O$, we have AX_{ii} is zero in diagonal. Then $c_{ii} = 0$ and thus $X_{ii} = O$ by SAP of A. It follows that u_{jl} is a constant multiple of ξ and hence $u_{jl} = 0$ for each j, l. As $M \circ X = O$, we have $X_{ij} = O$ for $i \sim j$. For $i \nsim j$, by (5), $O = (A+3I)(X_{ij} - X_{jj}) = (A+3I)X_{ij}$, implying that X_{ij} is a constant multiple of $\xi\xi^T$. That is, $X = C \otimes \xi\xi^T$ for some $C \in \mathbb{R}^{(6)}$, with $B \circ C = I_6 \circ C = O$. On the other hand, $O = MX = -BC \otimes \xi\xi^T$ implies that $BC = O$. Since $-B$ is a μ-optimal matrix for $K_{3,3}$, by SAP we have $C = O$ and thus $X = O$. ■

Theorem 5.3. *If $\mu(G_2) \leq 5$ or G_2 is complete bipartite, then $\mu(G_1 \square G_2) \geq \mu(G_1) + \mu(G_2)$ for connected graphs G_1, G_2.*

Proof. Note that if $G \preceq G_2$, then $G_1 \square G \preceq G_1 \square G_2$. Therefore by minor-monotonicity of μ, it suffices to show for the set of forbidden minors. Also, graphs in $\{K_2\} \cup \bigcup_{k=1}^{4} \mathscr{F}_k$, except for $K_{3,3}$ and $K_{3,3,1}$, can be obtained from complete graphs by performing a series of ΔY transformation on triangles. For $G_2 = K_{p,q}$ with $p \geq q \geq 3$, $K_{3,3} \vee K_{q-3} \preceq K_{q,q} \leq K_{p,q}$. The result then follows by Proposition 5.1, 5.2 and the characterizations of μ for $\mu \leq 4$. ■

Pendavingh[9] proved that for a connected graph G, $e(G) \geq \binom{\mu(G)+1}{2}$ unless $G = K_{3,3}$, offering an optimal bound of μ in terms of the number of edges, which provides a necessary condition to determine whether a graph is a forbidden minor for $\mu \leq k$. For $\mu > 5$, by Pendavingh's bound and Theorem 4.14, any graph that can be obtained from complete graphs by performing a series of ΔY operations and $Y\Delta$ operations are forbidden minors. Therefore by (iii) of Proposition 5.1, $\mu(G_1 \square G_2) \geq \mu(G_1) + \mu(G_2)$ holds for a large number of graphs with $\mu \geq 6$. However, there are forbidden minors that can't be obtained by ΔY transformations on triangles, such as the complement of icosahedron [9]. Note that the inequality in Theorem 5.3 could be worse for sparse graphs. To show this, we need the following theorem, which is an easy consequence of minor-monotonicity of μ and Theorem 4.2.

Theorem 5.4. *([5]) Let G be a graph with $\mu(G) \geq 3$. If G' arise from G by subdividing an edge, then $\mu(G) = \mu(G')$.*

For $n \geq m \geq 3$, $\mu(K_{1,n}) = \mu(K_{1,m}) = 2$. Let $G = K_{1,n} \square K_{1,m}$. Note that G can be obtained from $K_{n,m} \vee K_1$ by subdividing each edge in $K_{n,m}$. By Theorem 5.4, we have $\mu(G) = \mu(K_{n,m}) + 1 = m + 2$. As a consequence, we have the following results. Let $\Delta(G)$ denote the maximum degree of G.

Proposition 5.5. *For $i = 1, 2$, let G_i be a graph with $\Delta(G_i) \geq 3$ and let $m_i = \max_{G \preceq G_i} \Delta(G)$. Then $\mu(G_1 \Box G_2) \geq \min\{m_1, m_2\} + 2$.* ■

We say a subset $S \subseteq V(G)$ is a *connected dominating set* if S induces a connected graph in G and all vertices in $G - S$ have a neighbor in S. Using Proposition 5.5, a lower bound of μ under Cartesian products in terms of connected dominating set is provided.

Corollary 5.6. *Let G_1 and G_2 be connected graphs with maximum degree at least 3. Let S_i be a connected dominating set of G_i, $i = 1, 2$. Then $\mu(G_1 \Box G_2) \geq \min\{|G_1| - |S_1|, |G_2| - |S_2|\} + 2$.* ■

Example 5.7. For $d \geq 3$, [10] has provided a connected dominating set for the hypercube $Q_d := \Box^d K_2$ of order $2^{d-2} + 2$. Using Corollary 5.6, $\mu(Q_{2d}) = \mu(Q_d \Box Q_d) \geq 2^d - 2^{d-2} = 3 \cdot 2^{d-2}$. In general, $\mu(Q_d) \geq 3 \cdot 2^{\lfloor d/2 \rfloor - 2}$.

Since $G_1 \Box G_2$ is a subgraph of $G_1 \boxtimes G_2$, $\mu(G_1 \Box G_2) \leq \mu(G_1 \boxtimes G_2)$. However, the bound in Proposition 5.5 is also tight for the strong product of graphs. Let $n \geq m \geq 3$, and $G = K_{1,n} \boxtimes K_{1,m}$. Let G' be the graph obtained from $K_{n,m}$ by subdividing each edge, adding a pendant vertex to the new vertex, doing a $Y\Delta$ transformation to the claw. Since $\mu(K_{n,m}) = m + 1 \geq 4$, by Theorem 4.2, 4.14 and 5.4, $\mu(G') = m + 1$. Since $G = G' \vee K_1$, by Theorem 3.7, we have $\mu(G) = m + 2$.

Proposition 5.8. *$\mu(K_m \times K_n) \geq (m-1)(n-1)$.*

Proof. Assume $n \geq m$. If $m = 1$, the inequality clearly holds. Since $K_2 \times K_2 = 2K_2$, the inequality holds. We may assume $n \geq 3, m \geq 2$. Let $M = -(J_m - I_m) \otimes (J_n - I_n) + I_m \otimes I_n$. Clearly, M is a discrete schrödinger operator for $K_m \times K_n$ with spectrum $\{1 - (m-1)(n-1)^{(1)}, 0^{((m-1)(n-1))}, m^{(n-1)}, n^{(m-1)}\}$. We show that M has SAP and the result follows. We first partition M into $m \times m$ blocks: $M = [M_{ij}], 1 \leq i, j \leq m$. By the definition of M, we have:

$$M_{ij} = \begin{cases} I_n & \text{if } i = j \\ -J_n + I_n & \text{if } i \neq j \end{cases} \tag{6}$$

Suppose X fully annihilates M. We have to show $X = O$. Similarly, we partition X into $m \times m$ blocks: $X = [X_{ij}]$, $1 \leq i, j \leq m$. Since X fully annihilates M, X_{ii} is zero in diagonal for all i, and X_{ij} is diagonal for all $i \neq j$. For any i, j, let $[i, j]$ denote the (i, j)th block of MX. As $MX = O$, we have

$$O = [i, j] = \sum_{k=1}^{m} M_{ik} X_{kj} \tag{7}$$

Using (6), (7) is equivalent to

$$\sum_{k=1}^{m} X_{kj} = J_n \sum_{k \neq i} X_{kj} \tag{8}$$

Summing (8) over i, then we obtain

$$m \sum_{k=1}^{m} X_{kj} = (m-1) J_n \sum_{k=1}^{m} X_{kj}$$

Let $X' = \sum_{k=1}^{m} X_{kj}$. Dividing $m-1$ on both sides, we have

$$J_n X' = \frac{m}{m-1} X'$$

Since $n \geq 3$, $\frac{m}{m-1}$ is not an eigenvalue of J_n. Thus, $X' = O$ which implies that $X_{jj} = O$. Now for $i \neq j$,

$$O = [i,j] - [j,j] = J_n X_{ij} - J_n X_{jj} = J_n X_{ij}$$

Since X_{ij} is diagonal, $X_{ij} = 0$ and therefore $X = O$. ■

Corollary 5.9. $\mu(G_1 \times G_2) \geq (\omega(G_1) - 1)(\omega(G_2) - 1)$

Proof. Since $K_{\omega(G_1)} \times K_{\omega(G_2)}$ is a subgraph of $G_1 \times G_2$, the result follows from minor-monotonicity of μ and Proposition 5.8. ■

Corollary 5.10. *Assume G_1, G_2 non-edgeless. Then $\nu(G_1 \square G_2) \leq |G_1| + |G_2| - 2$*

Proof. By Theorem 1.7 and Proposition 5.8, we have

$$\nu(G_1 \square G_2) \leq \nu(K_{|G_1|} \square K_{|G_2|}) = |G_1||G_2| - \mu(K_{|G_1|} \times K_{|G_2|}) - 1 \leq |G_1| + |G_2| - 2.$$

■

Indeed, we can obtain a nondegenerate gram labelling for $G_1 \square G_2$ naively as constructed below.

Lemma 5.11. *Let $n = |G|$. Given any real numbers $\{c_i\}_{i=1}^n$ larger than 1, there exists a full-rank (hence nondegenerate) gram labelling $i \mapsto u_i$ in $\mathbb{R}^n$ for G such that $|u_i| = c_i$ for each i.*

Proof. It suffices to construct a gram matrix of rank n for G satisfying (A1) with respect to $\overline{G}$ with $(c_1^2, c_2^2, \ldots, c_n^2)$ in the diagonal. Let A be the adjacency matrix of G and take $m > 1$ such that $A + mI$ is positive definite. Note that the sum of positive definite matrices is still positive definite. Let D be a diagonal matrix with positive diagonal. Set $B = A + D$. Then write $B = V^T V$ for some full-rank square matrix V of order n and let v_i be the ith column of V. For $\alpha \in (0, \pi/2)$, consider the labelling $i \mapsto u_i := \cos\alpha \oplus (\sin\alpha) v_i$. As shown in Lemma 3.3, this is a gram labelling for G_2 in $\mathbb{R}^{n+1}$ with each u_i of norm $\sqrt{1 + (B_{ii} - 1)\sin^2\alpha}$. Choose α and D such that the norm of u_i is c_i. Then its gram matrix is the desired one. ■

Proposition 5.12. *Let G_1, G_2 be graphs. If $i \mapsto u_i$ is a nondegenerate gram labelling of G_1 in $\mathbb{R}^d$ such that $|u_i| = c > 1$ for all i, then $\nu(G_1 \square G_2) \leq d + |G_2|$.*

Proof. Let $n = |G_2|$. By Lemma 5.11, there exists a gram labelling $k \mapsto v_k$ of G_2 in $\mathbb{R}^n$ such that $\{v_k\}$ spans $\mathbb{R}^n$ and $|v_k| = c$ for all k. Consider the labelling $(i,k) \mapsto w_{(i,k)} := (c^2+1)^{-1/2}(u_i \oplus v_k)$. We show that it is a nondegenerate gram labelling for $G_1 \square G_2$. For each $(i,k), (j,l)$,

$$w_{(i,k)}^T w_{(j,l)} = (c^2+1)^{-1}(u_i^T u_j + v_k^T v_l) \begin{cases} = \frac{2c^2}{c^2+1} & \text{if } (i,k) = (j,l) \\ = 1 & \text{if } i = j, k \sim l \text{ or } i \sim j, k = l \\ < 1 & \text{otherwise} \end{cases}$$

Suppose there exist $X \in \mathbb{R}^{(|G_1||G_2|)}$ such that X is zero in the diagonal and $X_{(i,k)(j,l)} = 0$ for $(i,k)(j,l) \notin E(G_1 \Box G_2)$, satisfying that for each (i,k),

$$\sum_{i \sim j} X_{(j,k)(i,k)} w_{(j,k)} + \sum_{k \sim l} X_{(i,l)(i,k)} w_{(i,l)} = 0.$$

As v_k is not in the span of other v_l, we have that $X_{(i,l)(i,k)} = 0$ for each l. Therefore the equation is independent of k and the nondegeneracy of $i \mapsto u_i$ forces $X = 0$ and we are done. ∎

Observe that in the proof, each vector in the labelling for $G_1 \Box G_2$ is of same norm larger than 1. Therefore, we can repeatedly apply the procedure above, hence we have

Corollary 5.13. $\nu(\Box_{i=1}^d G_i) \leq \sum_{i=1}^d |G_i|$. ∎

Now we can derive Proposition 1.9 as an easy consequence.

Proof. *(of Theorem 1.9)* Suppose $\{G_i\}_{i \in \mathbb{N}}$ are non-edgeless graphs with orders less than N. By Example 5.7, $\mu(Q_d) \geq C \cdot 2^{d/2}$ for some positive constant C. Then for d large, $\mu(G_1 \Box \cdots \Box G_d) \geq \mu(Q_d) \geq C \cdot 2^{d/2} \gg dN \geq \nu(G_1 \Box \cdots \Box G_d)$, whence the assertion follows. ∎

Now we can provide a family of examples for nice graphs with μ much larger than ν using the previously obtained inequality. We first recall some definitions. Let $G = (V, E)$ be a connected graph. Two distinct vertices $u, v \in V$ are said to be *twins* if $N_G(u) - \{v\} = N_G(v) - \{u\}$. We say G is *twin-free*, if G has no twins. We say G is *vertex-transitive* (resp. *edge-transitive*) if the automorphism group Aut(G) acts transitively on V (resp. on E). The examples are constructed by the following fact.

Proposition 5.14. *Let G be a connected vertex-transitive graph with $|G| > 1$. For $d \geq 2$, $\Box^d G$ is vertex-transitive. Moreover, $\Box^d G$ is edge-transitive if G is. Also, $\Box^d G$ is twin-free unless $G = K_2$ and $d = 2$.*

Proof. For $u \in \Box^d G$, write $u = (u_1, u_2, \ldots, u_d)$. Given vertices $u, v \in \Box^d G$, there exists $\varphi_i \in$Aut(G) such that $\varphi_i(u_i) = v_i$ for each i. Then since $\varphi_1 \times \varphi_2 \times \cdots \times \varphi_d \in$Aut($\Box^d G$) and maps u to v, $\Box^d G$ is vertex-transitive. Moreover, assume G is edge-transitive. For $uv, u'v' \in E(\Box^d G)$, we have by definition that $u_i \sim v_i$ and $u'_j \sim v'_j$ for some i, j and that $u_k = v_k$, $u'_l = v'_l$ for $k \neq i$, $j \neq l$. By commutativity of Cartesian product and previous discussion, we may assume $i = j$ by permuting coordinates and all other coordinates are equal. By edge-transitivity of G, there exists $\varphi \in$Aut(G) that maps $u_i v_i$ to $u'_i v'_i$. As φ can be viewed as an automorphism acting only on ith coordinate of $\Box^d G$, and thus $\Box^d G$ is edge-transitive. Given vertices $u, v \in \Box^d G$, by permuting, we may assume $1 \leq k \leq d$ is the largest integer such that $u_k \neq v_k$. If $k < d$, then choose $w \in N_G(u_{k+1})$ and we have $(u_1, \ldots, u_k, w, u_{k+2}, \ldots, u_d) \in N_{\Box^d G}(u) - N_{\Box^d G}(v)$; suppose $k = d$ and some vertex in $\{u_i, v_i\}_{i=1}^d$ is of degree at least 2 in G. Again by permuting and symmetry, we may assume $|N_G(u_1)| \geq 2$. Take $w \in N_G(u_1) - \{v_1\}$. Then $(w, u_2, \ldots, u_d) \in N_{\Box^d G}(u) - N_{\Box^d G}(v)$. Suppose $k = d$ and each u_i,v_i is of degree 1 in G. If $G \neq K_2$, then $u_i v_i \notin E(G)$ for each i. Take w to be the neighbor of u_1 in G and we have $(w, u_2, \ldots, u_d) \in N_{\Box^d G}(u) - N_{\Box^d G}(v)$. If $G = K_2$ and $d \geq 3$, then

$(v_1, u_2, u_3, \ldots, u_d) \in N_{\square^d G}(u) - N_{\square^d G}(v)$. This shows that u, v are not twins. ■

Finally, we propose the following question, which is analogue to that of μ.

Question 5.15. Is $\nu(G_1 \square G_2) \leq \nu(G_1) + \nu(G_2) + 2$ for all graphs G_1, G_2?

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