國立交通大學

應用數學系

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摘要

本篇論文主要討論一類特別的圖：笛卡爾積圖。首先，對於樹狀圖與圈狀圖的笛卡爾積，我們討論它的漢彌爾頓性及邊漢彌爾頓性。其次，對於樹狀圖與路徑圖的笛卡爾積，討論其漢彌爾頓性及偶泛圈性。在第二類圖中，我們將樹狀圖分為可完美配對或存在路徑因子兩情況討論，並且用系統性的方法建構出此二圖類的漢彌爾頓圈。論文內亦在已知定理的基礎上補充進一步的結果並且給予新證明方法，尤其證明了在所有討論的圖類中，圖為漢彌爾頓圖與圖為1堅韌兩條件為等價。

關鍵字: 漢彌爾頓性, 邊漢彌爾頓性, 偶泛圈性, 笛卡爾積, 路徑因子, 圖韌性

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abstract

The Cartesian product of two graphs forms a special class of graphs. First, for a given tree through its Cartesian products with cycles, we discuss its Hamiltonicity and edge-Hamiltonicity. Second, for a given tree through its Cartesian products with paths, we discuss its Hamiltonicity and even-pancyclicity. We find several Hamiltonian graphs in the case that the tree has a perfect matching or a path factor. Some well-known results which have been proved are also given in this thesis with modified results or new approach of proofs. In particular, we prove that the two conditions Hamiltonian and 1-tough are equivalent in those graphs we discussed.

Keywords: Hamiltonicity, edge-Hamiltonicty, even-pancyclicity, Cartesian product, path factor, graph toughness

致謝辭

白駒過隙，日月穿梭，風城六載，竟也宛若一夢。初至交大之時，年方十七，懵懂無知；離別之日，只願微薄之力，得為後繼之研究添磚加瓦。

在碩士班兩年的前間，雖然忙碌，但充實。首先最感謝的是我的指導教授—翁志文教授。沒有翁老師的提點和不厭其煩地反覆修改，就不會成就這篇論文。另外我非常感激口試委員傅東山教授與張耀祖教授不辭辛勞地替這篇論文提供寶貴意見，並在百忙之中抽空來新竹為我們口試，使這篇論文更為完整。當然也要感謝陳秋媛教授，多年來在學術上甚或是日常生活中都給我們很大的幫忙照顧。而就讀碩士班這兩年，我也受到傅恒霖教授和符麥克教授各方面極多的幫助，使我受益良多。

學術研究或許總需孤獨，不過與同儕的合作讓我學到更多。感謝我的學術和助教夥伴馨勻及同窗鎮魁，課業和研究上的討論切磋讓我們以飛快的速度相互成長。應數所這個大家庭的每位同學，無論組別，無論專業領域或課餘活動，大家的團結、日常的提攜與幫助亦是我能潛心研究的強力後盾。感謝多年來的室友凱凱、峻嘉、小紀，還有方綺、阿草、士軒、喜得以及我所有的好朋友，你們的支持與陪伴讓我能有歡樂的六年交大時光，相信日後必定值得回憶。特別要感謝的是芷菱及明揚學長，你們的建議和經驗傳承給我很大的信心，讓我確定未來的方向不致迷途。除了教授和同學，我也要感謝應數系的職員、系上系外所有教過我給過我幫助的老師，有你們默默的付出，我們的學習便無後顧之憂。感謝所有學長姐為後輩們開的路，你們走過的路，我們好走許多。感謝所有學弟妹，雖然學長不比你們聰明認真，你們也願意與我互相切磋砥礪。最後感謝我的家人，雖然你們可能聽不懂我的研究，但還是全力支持，讓我備感安心。

在交大六年，我空手而來，卻滿載而歸，希望交大應數也因為我，有了小小的改變。離開交大後，我必不負所託，讓交大應數以我為榮。感謝大家，感謝上帝。

Contents

1 Introduction

Verifying the Hamiltonicity of graphs is a classic issue in the field of graph theory. It is not easy to find the sufficient conditions to ensure the existences of Hamiltonian cycle in graphs. There were several earlier results such as Dirac's theorem and Ore theorem, but the assumptions of these results are still too strong. V. Chavatal[4] introduced a new invariant for graphs, which measures in a simple way how tightly various pieces of a graph hold together; called toughness. He proven that every Hamiltonian graph is 1-tough. But there also exists many 1-tough non-Hamiltonian graphs which told us the two conditions 1-tough and Hamiltonian are not equivalent in general graphs.

In our research we discuss a special class of graphs called Cartesian product graphs. Using the Cartesian product to combine two graphs with established properties makes it possible to construct a new topology with the properties of both worlds, which is of practical interest for network design[5]. Moreover, there are some reasons that we restrict our discussion on Cartesian products $P_n \square T$ and $C_n \square T$ of a tree T and a path P_n of order n or a cycle C_n of order n. First, the spanning tree structure can describe the macro structure of a graph. Namely, simplify the graph without losing the backbone structure. Second, C_n and P_n are able to connect those trees layer by layer. There are some results of Hamiltonicity of the graph class $C_n \square T$ given in [2][6].

We will prove that in the two classes $P_n \square T$ and $C_n \square T$, the two conditions 1-tough and Hamiltonian are equivalent. To the class of $C_n \square T$, we introduce the idea of edge-Hamiltonian and edge-1-tough and their equivalence. We divide the case of $P_n \square T$ into two subcases based on the structure of T (Trees with a perfect matching and trees with a path factor) and construct their own Hamiltonian cycles.

In the view of computational complexity, the problem **HC**, which checks the Hamiltonicity of graph, has been proven to be NP-complete. Hence we want to find another way to check the Hamiltonicity of certain graphs. In other words, finding the equivalent conditions to "Hamiltonian" is important. However, to recognize a given graph being 1-tough or not is also an NP-hard problem[3]. To overcome this disadvantage, in all of our results, we gives one more equivalent statement which is easier to verify, such as the order of n or the structure of trees.

The remaining part of this paper is organized as follows. Section 2 gives some basic notations and preliminaries. Section 3 gives the relations between toughness and Hamiltonicity, includes an improved result to a theorem in [4]. Section 4 presents our research results in the class of Cartesian product of cycles and trees. Section 5 gives two Propositions that will be used in next two sections. Section 6 talks about the Hamiltonian characterization of trees with perfect matching through its Cartesian products with paths and Section 7 presents the Hamiltonian characterization of trees with path factor through its Cartesian products with paths. Section 8 gives a stronger equivalent condition to Hamiltonicity of $P_n \Box T$ called even-pancyclicity. Finally, Section 9 concludes this paper.

2 Notations and preliminaries

Let G be a simple graph with the vertex set $V(G)$ and the edge set $E(G)$. Let $c(G)$ denote the number of connected components of G. We say that G is *Hamiltonian* if G has a Hamiltonian cycle, i.e. a cycle of length $|V(G)|$. A Hamiltonian graph G is said to be *edge-Hamiltonian* if every edge $e \in E(G)$ lies in a Hamiltonian cycle [5]. We will use the sequence of passing vertices to represent a cycle, denoted as $(v_1, v_2, \cdots, v_n, v_1)$.

A vertex is said to be a *leaf* if and only if it has degree 1. The neighbor set of a vertex $v \in V(G)$ is denoted by $N(v)$. The *contraction* of G on an edge e is a graph generated by removing e from G and merging the two end vertices of e simultaneously (also remove the multiple edge if there exists).

In this thesis, we discuss a type of graphs called Cartesian product which is defined as follows:

Definition 2.1. The *Cartesian product* of graphs G and H is a graph, denoted by $G\Box H$, whose vertex set is

$$V(G\Box H) = \{a_b : a \in V(G), b \in V(H)\},$$

and edge set is

$$E(G\Box H) = \{a_b a_c : bc \in E(H)\} \cup \{a_b c_b : ac \in E(G)\}.$$

A *spanning subgraph* of G is a subgraph of G that uses all the vertices of G. A *1-factor* of G is a 1-regular spanning subgraph of G, which is also known as a *perfect mathching*. On the other hand, if M is a set of graphs, an *M-factor* is a spanning subgraph of G where each component of the subgraph is isomorphic to an element in M. Following the definition in [1], a *path factor* is an *M-factor* where M is a set of paths with order at least two. In particular, $M = \{P_2, P_3\}$ has been used frequently in our research, where P_n is a path with n vertices. Moreover, we say that a component X is adjacent to another component Y in a factor of G if there exist vertices $x \in X, y \in Y$, such that x is adjacent to y in G.

We say that a graph G is *t-tough* if t is a real number such that $|S| \geq t \cdot c(G - S)$ for any vertex subset S whose deletion makes G disconnected ($c(G - S) > 1$). If G is not complete, the largest t such that G is t-tough is called the *toughness* of G, denoted by $t(G)$. We set $t(K_n) = +\infty$ for all n.

3 Toughness and Hamiltonicity

The following result in [4] gives a very important necessary condition for a graph (especially 2-connected or more) being Hamiltonian.

Proposition 3.1 ([4]). *Every Hamiltonian graph is 1-tough.*

Next, we give one necessary condition and one sufficient condition to characterize the edge-Hamiltonian graphs.

First, the sufficient condition is modified from Ore's theorem. A similar statement is also mentioned in **Problem 4F.** of [8]

Theorem 3.2. *Let G be a simple graph of order n such that* $\deg(u) + \deg(v) > n$ *for any nonadjacent vertices u, v. Then G is edge-Hamiltonian.*

Proof. Suppose the theorem failed and G is the counterexample with maximum number of edges. Then for any given edge e which is not contained in a Hamiltonian cycle, there exists a Hamiltonian path $(u_1, u_2, \cdots, u_n)$ passes through e with nonadjacent end-vertices u_1, u_n (otherwise adding an edge can not yield a Hamiltonian cycle that contains e, and G could not be a counterexample with maximum number of edges).

Now consider the sets

$$A = \{i : u_1 \text{ is adjacent to } u_{i+1}\}$$

and

$$B = \{i : u_n \text{ is adjacent to } u_i\}.$$

Since $|A| + |B| > n$ and $A \cup B \subseteq \{1, 2, 3, \cdots, n-1\}$, we know that $|A \cap B| \geq 2$. Hence we can find at least one $i \in A \cap B$ such that the edge $u_i u_{i+1} \neq e$. However, the cycle

$$(u_1, u_{i+1}, u_{i+2}, \cdots, u_n, u_i, u_{i-1}, \cdots, u_1)$$

is a Hamiltonian cycle that contains e, a contradiction. ∎

Notice that the degree bound is sharp, the graph G_2 in Figure 1 is a non-edge-Hamiltonian example whose only pair of nonadjacent vertices satisfies $\deg(u)+\deg(v) = n$.

To give the necessary condition, here we define a new property called *edge*-1-*tough*.

Definition 3.3. We call a graph G *edge*-1-*tough* if the contraction of G on any edge is 1-tough.

The property edge-k-tough can be defined similarly, but we won't use it in this paper. Note that the definition of edge-k-tough is different to k-edge-tough (k-line-tough) which is defined in [4].

Similar to Proposition 3.1, next proposition characterize the relationship between edge-1-tough and edge-Hamiltonicity.

Proposition 3.4. *Every edge-Hamiltonian graph is edge-*1*-tough.*

Proof. For an edge-Hamiltonian graph G and an arbitrary edge $e \in E(G)$, there is a Hamiltonian cycle contain e. Hence the contraction of G on e still contains a Hamiltonian cycle. By Proposition 3.1, the contraction of G on e is 1-tough. Therefore, G is edge-1-tough. ■

An edge-Hamiltonian graph is apparently Hamiltonian. However, surprisingly an edge-1-tough graph is not necessarily 1-tough and vice versa. Here we give an example to verify this argument.

Example 3.5. In Figure 1, the contractions of graph G_1 on any edge $e \in E(G)$ are all isomorphic to G_2 and the contraction of G_2 on $\hat{e}$ is G_3. Since G_2 is $1-$tough, the graph G_1 is edge-1-tough. However G_1 itself is not 1-tough. On the other hand, although G_2 is 1-tough, but the graph G_3, its contraction on $\hat{e}$, is not 1-tough. Hence G_2 is not edge-1-tough.

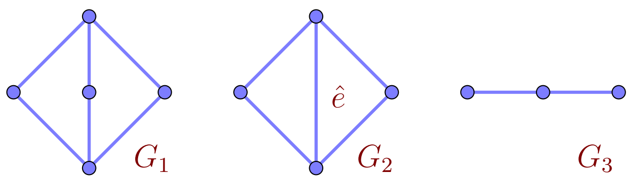

Figure 1

In this thesis we focus on a family of Cartesian product of graphs. This is motivated by Chvátal [4], so let us first generalize one of his result.

Theorem 3.6. *For any given graphs G, H, the toughness $t(G\Box H)$ is less than or equal to $\frac{1}{2}(|V(G)|+|V(H)|)-1$ and the equality holds if and only if both of G and H are complete.*

Proof. Let $|V(G)| = m, |V(H)| = n$ and the Cartesian product of G and H is the graph $G\Box H$ with

$$V(G\Box H) = \{a_x | a \in V(G), x \in V(H)\},$$

$$E(G\Box H) = \{a_x a_y | a \in V(G), xy \in E(H)\} \cup \{a_x b_x | ab \in E(G), x \in V(H)\}.$$

For an arbitrary vertex a_x in $G\Box H$, its neighbor set will be $N(a_x) = \{a_y | xy \in E(H)\} \cup \{c_x | ac \in E(G)\}$. Since $c((G\Box H) - N(a_x)) \geq 2$,

$$t(G\Box H) \leq \frac{|N(a_x)|}{c((G\Box H) - N(a_x))} \leq \frac{\deg_G(a) + \deg_H(x)}{2} \leq \frac{(m-1)+(n-1)}{2}.$$

and the last equality holds if and only if $\deg_G(a) = m-1, \deg_H(x) = n-1$ for all $a \in V(G), x \in V(H)$. That is, both of G and H are complete. ■

4 Cartesian product of cycles and trees

It has been proved in [6] that $C_n\Box T$ is Hamiltonian if and only if $n \geq \Delta(T)$. Our approach is different from the former proofs and can also be applied to our main result, the case of Cartesian product of paths and trees.

For convenience, let $V(C_n) = \{1, 2, \cdots, n\}$ and $E(C_n) = \{12, 23, \cdots, n1\}$.

Lemma 4.1. *For $n \geq 3$, if the graph $C_n\Box T$ is Hamiltonian, then $n \geq \Delta(T)$.*

Proof. If $n < \Delta(T)$, then there exists a vertex $v \in V(T)$ such that $\deg(v) = \Delta(T)$. Hence deleting n vertices $1_v, 2_v, \cdots, n_v$ from $C_n\Box T$ yields $\Delta(T)$ components, which means

$$t(C_n\Box T) \leq n/\Delta(T) < 1.$$

By Theorem 3.1, the graph $C_n \Box T$ is not Hamiltonian. ■

In the proof of Theorem 4.2 and the remaining of this thesis, we prefer to write down the labels continuously even some of them become less than 1 or exceed n. For example, the labels $i, i+1, i+2, i+3, i+4, i+5$ is actually $n-3, n-2, n-1, n, 1, 2$ if the value $i+3$ is equal to n and $i, i-1, i-2, i-3, i-4, i-5$ represent $3, 2, 1, n, n-1, n-2$ when $i = 3$.

Theorem 4.2. *If $n \geq \max(\Delta(T), 3)$, then there exists a Hamiltonian cycle of $C_n \Box T$ which contains exactly $n-1$ of the n edges: $\{1_\ell 2_\ell, 2_\ell 3_\ell, \cdots, n-1_\ell n_\ell, n_\ell 1_\ell\}$, for any leaf ℓ of T.*

Proof. Apply induction on the number of vertices of T. The induction bases are $T = K_1$ and star graphs $T = K_{1,\Delta(T)}$. The graph $C_n \Box K_1$ is actually C_n which is Hamiltonian. Moreover, K_1 has no leaf, so the statement is correct in this case. Let v be the core vertex and $N(v) = \{u^1, u^2, \cdots, u^{\Delta(T)}\}$ are all leaves. We can construct $\Delta(T)+1$ cycles, together contain all vertices of $C_n \Box K_{1,\Delta(T)}$:

$$(1_v, 2_v, \cdots, n_v, 1_v), (1_{u^1}, 2_{u^1}, \cdots, n_{u^1}, 1_{u^1}), \cdots, (1_{u^{\Delta(T)}}, 2_{u^{\Delta(T)}}, \cdots, n_{u^{\Delta(T)}}, 1_{u^{\Delta(T)}}).$$

Merge them by replacing edge pairs

$$\{1_v 2_v, 1_{u^1} 2_{u^1}\}, \{2_v 3_v, 2_{u^2} 3_{u^2}\}, \cdots, \{\Delta(T)_v 1_v, \Delta(T)_{u^{\Delta(T)}} 1_{u^{\Delta(T)}}\}$$

into

$$\{1_v 1_{u^1}, 2_v 2_{u^1}\}, \{2_v 2_{u^2}, 3_v 3_{u^2}\}, \cdots, \{\Delta(T)_v \Delta(T)_{u^{\Delta(T)}}, 1_v 1_{u^{\Delta(T)}}\},$$

respectively, we can connect those $\Delta(T)+1$ cycles into a Hamiltonian cycle of $C_n \Box T$. Since we remove exactly one edge in $\{1_\ell 2_\ell, 2_\ell 3_\ell, \cdots, n-1_\ell n_\ell, n_\ell 1_\ell\}$ for any leaf ℓ, the Hamiltonian cycle contains exactly $n-1$ of the n edges: $\{1_\ell 2_\ell, 2_\ell 3_\ell, \cdots, n-1_\ell n_\ell, n_\ell 1_\ell\}$ for any leaf ℓ.

Assume that for all T' with $|V(T')| < |V(T)|$, $C_n \square T'$ has a Hamiltonian cycle which contains exactly $n-1$ of the n edges: $\{1_\ell 2_\ell, 2_\ell 3_\ell, \cdots, n-1_\ell n_\ell, n_\ell 1_\ell\}$ for any leaf ℓ.

To discuss the Hamiltonicity of $C_n \square T$ where $T \neq K_{1,\Delta(T)}$, we can choose a vertex v with exactly one non-leaf neighbor and

$$N(v) = \{u, u^1, u^2, \cdots, u^d : u \text{ is not a leaf and all the other are leaves.}\}.$$

Note that $d < \Delta(T)$ and we can find a Hamiltonian cycle of $C_n \square (T - (N(v) - u))$ which contains $n-1$ edges

$$1_v 2_v, 2_v 3_v, \cdots, d_v d+1_v, \cdots, n-1_v n_v,$$

up to permutation of labels of C_n. Now, replace $i_v i+1_v$ $(1 \leq i \leq d)$ of the above edges by a path $(i_v, i_{u^i}, i-1_{u^i}, i-2_{u^i}, \cdots, i+1_{u^i}, i+1_v)$ of length $n+1$, respectively, where the internal vertices are along a direction in the cycle of $C_n \square T$ based on u^i. After replacing all the edges, we can extend the cycle into a Hamiltonian cycle of $C_n \square T$. ■

Proposition 3.1 told us that every Hamiltonian graph is 1-tough, but on the other hand, there also exists many 1-tough non-Hamiltonian graph. We want to show that in the class $C_n \square T$, the two conditions are equivalent.

Theorem 4.3. *For $n \geq 3$, the graph $C_n \square T$ is Hamiltonian if and only if it is 1-tough.*

Proof. If $C_n \square T$ is Hamiltonian, then by Theorem 3.1 it is 1-tough.

If $C_n \square T$ is not Hamiltonian, then by Theorem 4.2, n must be less than $\Delta(T)$. There exists a vertex $v \in V(T)$ such that $\deg(v) = \Delta(T)$, hence delete n vertices $1_v, 2_v, \cdots, n_v$ from $C_n \square T$ yields $\Delta(T)$ components, so

$$t(C_n \square T) \leq n/\Delta(T) < 1.$$

In the other words, $C_n \square T$ is not 1-tough. ■

Combining Lemma 4.1, Theorem 4.2 and Theorem 4.3, our first main theorem can be obtained.

Theorem 4.4. *The following three statements:*

(i) $n \geq \Delta(T)$.

(ii) $C_n \Box T$ *is Hamiltonian.*

(iii) $C_n \Box T$ *is* 1*-tough.*

are equivalent for all n *not less than* 3.

Since "1-tough" and "Hamiltonian" are not always equivalent, we think about a problem.

Problem 4.5. To what extent will a 1-tough graph become Hamiltonian?

We will provide two more classes of such graphs in Section 6 and Section 7.

Increasing the value of n by one, we have an improved result of Theorem 4.2. For convenience, we just write a label i directly even if i is larger than n (and in fact, it means $i - n$).

Theorem 4.6. *If* $n > \max(\Delta(T), 2)$ *and* $e \in E(C_n \Box T)$*, then* $C_n \Box T$ *has a Hamiltonian cycle contains* e *and exactly* $n-1$ *of the* n *edges:* $\{1_\ell 2_\ell, 2_\ell 3_\ell, \cdots, n-1_\ell n_\ell, n_\ell 1_\ell\}$ *for any leaf* ℓ *of* T.

Proof. Apply induction on the number of vertices of T. The induction bases are $T = K_1$ and star graphs $T = K_{1,\Delta(T)}$. The graph $C_n \Box K_1$ is actually C_n which forms a Hamiltonian cycle that contains all the edges. Moreover, K_1 has no leaf, so the statement is correct in this case. For $C_n \Box K_{1,\Delta(T)}$,let v be the core vertex and $N(v) = \{u^1, u^2, \cdots, u^{\Delta(T)}\}$ are all leaves, then we can find $\Delta(T)+1$ cycles, together contain all vertices:

$$(1_v, 2_v, \cdots, n_v, 1_v), (1_{u^1}, 2_{u^1}, \cdots, n_{u^1}, 1_{u^1}), \cdots, (1_{u^{\Delta(T)}}, 2_{u^{\Delta(T)}}, \cdots, n_{u^{\Delta(T)}}, 1_{u^{\Delta(T)}}).$$

To show that there is a Hamiltonian cycle containing a chosen edge, we first classify the chosen edge into two different types:

(1) Edges labelled as $i_v i+1_v$ or $i_{u^j} i+1_{u^j}$.

(2) Edges labelled as $i_v i_{u^j}$.

In case (1), replace edge pairs

$$\{i+1_v i+2_v, i+1_{u^1} i+2_{u^1}\}$$

$$\{i+2_v i+3_v, i+2_{u^2} i+3_{u^2}\}$$

$$\vdots$$

$$\{i+\Delta(T)_v i+\Delta(T)+1_v, i+\Delta(T)_{u^{\Delta(T)}} i+\Delta(T)+1_{u^{\Delta(T)}}\}$$

into

$$\{i+1_v i+1_{u^1}, i+2_v i+2_{u^1}\}$$

$$\{i+2_v i+2_{u^2}, i+3_v i+3_{u^2}\}$$

$$\vdots$$

$$\{i+\Delta(T)_v i+\Delta(T)_{u^{\Delta(T)}}, i+\Delta(T)+1_v i+\Delta(T)+1_{u^{\Delta(T)}}\}$$

respectively. Then we get a Hamiltonian cycle of $C_n \square T$. The real label of the last replaced edge is

$$(i+\Delta(T), i+\Delta(T)+1) \text{ or } (i+\Delta(T)-n, i+\Delta(T)+1-n).$$

Since $n \geq \Delta(T)+1$, edges $i_v i+1_v$ and $i_{u^j} i+1_{u^j}$ are in the small cycles initially and haven't been replaced. Hence all type (1) edges are lying in some Hamiltonian cycles.

In case (2), replace edge pairs

$$\{i_v i+1_v, i_{u^j} i+1_{u^j}\}$$

$$\{i+1_v i+2_v, i+1_{u^{j+1}} i+2_{u^{j+1}}\}$$

$$\vdots$$

$$\{i+\Delta(T)-1_v i+\Delta(T)_v, i+\Delta(T)-1_{u^{j+\Delta(T)-1}} i+\Delta(T)_{u^{j+\Delta(T)-1}}\}$$

into

$$\{i_v i_{u^j}, i+1_v i+1_{u^j}\}$$

$$\{i+1_v i+1_{u^j+1}, i+2_v i+2_{u^j+1}\}$$

$$\vdots$$

$$\{i+\Delta(T)-1_v i+\Delta(T)-1_{u^{j+\Delta(T)-1}}, i+\Delta(T)_v i+\Delta(T)_{u^{j+\Delta(T)-1}}\}$$

respectively, then we get a Hamiltonian cycle of $C_n \Box T$. The edge $i_v i_{u^j}$ are not in the small cycles but has been replaced into the Hamiltonian cycle. In this way, all type (2) edges are lying in some Hamiltonian cycles. Combine the two cases, $T = K_{1,\Delta(T)}$ is edge-Hamiltonian. On the other hand, it is easy to check that every Hamiltonian cycles we constructed must satisfy the condition: for any leaf ℓ of T, the Hamiltonian cycle contains exactly $n-1$ edges of $\{1_\ell 2_\ell, 2_\ell 3_\ell, \cdots, n-1_\ell n_\ell, n_\ell 1_\ell\}$.

Assume that for all T' with $|V(T')| < |V(T)|$ We can find a Hamiltonian cycle of $C_n \Box T'$ which satisfies the following two conditions simultaneously:

- The Hamiltonian cycle contains an arbitrarily chosen edge e.
- For any leaf ℓ of T, the Hamiltonian cycle contains exactly $n-1$ edges of the set $\{1_\ell 2_\ell, 2_\ell 3_\ell, \cdots, n-1_\ell n_\ell, n_\ell 1_\ell\}$.

To discuss the edge-Hamiltonicity of $C_n \Box T$, since $T \neq K_{1,\Delta(T)}$, we can choose a vertex v with exactly one non-leaf neighbor and

$$N(v) = \{u, u^1, u^2, \cdots, u^d : u \text{ is not a leaf and all the other are leaves.}\}.$$

Note that $d < \Delta(T)$ and we can find d cycles:

$$(1_{u^1}, 2_{u^1}, \cdots, n_{u^1}, 1_{u^1}), \cdots, (1_{u^d}, 2_{u^d}, \cdots, n_{u^d}, 1_{u^d}).$$

and a Hamiltonian cycle of $C_n \square (T - (N(v) - u))$.

To construct a Hamiltonian cycle containing a chosen edge, we need to classify the chosen edge into three different types:

(1) Edges labelled as $i_v i+1_v$ or $i_{u^j} i+1_{u^j}$.

(2) Edges labelled as $i_v i_{u^j}$.

(3) All the other edges.

In case (1), by Theorem 4.2, we can find a Hamiltonian cycle of the graph $C_n \square (T - (N(v) - u))$ that contains $n-1$ edges

$$1_v 2_v, 2_v 3_v, \cdots, i-2_v i-1_v, i_v i+1_v \cdots, n-1_v n_v, n_v 1_v.$$

Replace edge pairs

$$\{i+1_v i+2_v, i+1_{u^1} i+2_{u^1}\}$$

$$\{i+2_v i+3_v, i+2_{u^2} i+3_{u^2}\}$$

$$\vdots$$

$$\{i+d_v i+d+1_v, i+d_{u^d} i+d+1_{u^d}\}$$

into

$$\{i+1_v i+1_{u^1}, i+2_v i+2_{u^1}\}$$

$$\{i+2_v i+2_{u^2}, i+3_v i+3_{u^2}\}$$

$$\vdots$$

$$\{i+d_v i+d_{u^d}, i+d+1_v i+d+1_{u^d}\}$$

respectively, then we get a Hamiltonian cycle of $C_n \Box T$. The real label of the last replaced edge is

$$(i+d, i+d+1) \text{ or } (i+d-n, i+d+1-n).$$

Since $n \geq \Delta(T)+1 > d+1$, edges $i_v i+1_v$ and $i_{u^j} i+1_{u^j}$ are in the small cycles initially and haven't been replaced. Hence all type (1) edges are lying in some Hamiltonian cycles.

In case (2), by Theorem 4.2, we can find a Hamiltonian cycle of the graph $C_n \Box (T - (N(v)-u))$ that contains $n-1$ edges

$$1_v 2_v, 2_v 3_v, \cdots, i-2_v i-1_v, i_v i+1_v \cdots, n-1_v n_v, n_v 1_v.$$

Replace edge pairs

$$\{i_v i+1_v, i_{u^j} i+1_{u^j}\}$$

$$\{i+1_v i+2_v, i+1_{u^{j+1}} i+2_{u^{j+1}}\}$$

$$\vdots$$

$$\{i+d-1_v i+d_v, i+d-1_{u^{j+d-1}} i+d_{u^{j+d-1}}\}$$

into

$$\{i_v i_{u^j}, i+1_v i+1_{u^j}\}$$

$$\{i+1_v i+1_{u^j+1}, i+2_v i+2_{u^j+1}\}$$

$$\vdots$$

$$\{i+d-1_v i+d-1_{u^{j+d-1}}, i+d_v i+d_{u^{j+d-1}}\}$$

respectively, then we get a Hamiltonian cycle of $C_n \Box T$. The edge $i_v i_{u^j}$ is not in the small cycles but has been replaced into the Hamiltonian cycle. In this way, all type (2) edges are lying in some Hamiltonian cycles.

In case (3), by induction hypothesis, any chosen edge e of this type must lie in a Hamiltonian cycle of $C_n \square (T-(N(v)-u))$, and the Hamiltonian cycle also contains $n-1$ edges of $\{1_v 2_v, 2_v 3_v, \cdots, n-1_v n_v, n_v 1_v\}$. For convenience, let the $n-1$ edges

$$1_v 2_v, 2_v 3_v, \cdots, i-2_v i-1_v, i_v i+1_v \cdots, n-1_v n_v, n_v 1_v.$$

Then replace edge pairs

$$\{i_v i+1_v, i_{u^j} i+1_{u^j}\}$$

$$\{i+1_v i+2_v, i+1_{u^{j+1}} i+2_{u^{j+1}}\}$$

$$\vdots$$

$$\{i+d-1_v i+d_v, i+d-1_{u^{j+d-1}} i+d_{u^{j+d-1}}\}$$

into

$$\{i_v i_{u^j}, i+1_v i+1_{u^j}\}$$

$$\{i+1_v i+1_{u^j+1}, i+2_v i+2_{u^j+1}\}$$

$$\vdots$$

$$\{i+d-1_v i+d-1_{u^{j+d-1}}, i+d_v i+d_{u^{j+d-1}}\}$$

respectively, forms a Hamiltonian cycle of $C_n \square T$. Since the chosen edge e hasn't been replaced, we are done.

Combining three cases, we conclude by induction that any edge e in $C_n \square T$ is lying in a Hamiltonian cycle which satisfies the condition that for any leaf ℓ of T, the Hamiltonian cycle contains exactly $n-1$ edges of $\{1_\ell 2_\ell, 2_\ell 3_\ell, \cdots, n-1_\ell n_\ell, n_\ell 1_\ell\}$. ■

By the equivalence of edge-1-tough and edge-Hamiltonicity. Our second main result is the following

Theorem 4.7. *The following three statements:*

(i) $n > \Delta(T)$.

(ii) $C_n \square T$ *is edge-Hamiltonian.*

(iii) $C_n \square T$ *is edge-*1*-tough and* $1-$*tough.*

(iv) $C_n \square T$ *is edge-*1*-tough.*

are equivalent for all n *not less than* 3.

Proof. The statement (**i**) ⇒ (**ii**) has been proved in Theorem 4.6. (**ii**) ⇒ (**iii**) can be obtained directly from Proposition 3.4 and Proposition 3.1. (**iii**) ⇒ (**iv**) is straightforward. Hence we only need to prove (**iv**) ⇒ (**i**).

We prove this by contradiction. If $n \leq \Delta(T)$, find a vertex $v \in V(T)$ such that $\deg(v) = \Delta(T)$. Let G be the contraction of $C_n \square T$ on the edge $1_v 2_v$. Delete $n-1$ vertices $1_v(=2_v), 3_v, 4_v, \cdots, n_v$ yields $\Delta(T)$ components. Therefore,

$$t(G) \leq n - 1/\Delta(T) < 1.$$

In the other words, $C_n \square T$ is not edge-1-tough. ■

5 Cartesian product of paths and trees

Different from $C_n \square T$, there are some trees T making $P_n \square T$ non-Hamiltonian for all n.

Proposition 5.1. *Let* T *be a tree. If there is a vertex* $v \in T$ *with more than two leaf-neighbors, then* $P_n \square T$ *is not Hamiltonian for any* n.

Proof. Suppose $P_n \square T$ is Hamiltonian, and there exists a vertex $v \in T$ with 3 leaf-neighbors a, b and c. In $P_n \square T$, all of a, b and c have degree 2, which means that all of edges va, vb, vc are in the Hamiltonian cycle, a contradiction. ■

A bipartite graph is called *balanced* if both of its bipartition have the same size. Actually, a Hamiltonian bipartite graph must be balanced. An even n makes the bipartite graph $P_n \Box T$ balanced no matter T is balanced or not. Besides, for an odd n, T needs to satisfy some requirements to make $P_n \Box T$ Hamiltonian. Here gives a simple proposition.

Proposition 5.2. *If n is odd and T is unbalanced, then $P_n \Box T$ is not Hamiltonian.*

6 Cartesian product of path and trees with perfect matching

In this section, we consider those trees with perfect matching and their Cartesian products with paths.

For convenience, let $V(P_n) = \{1, 2, 3, \cdots, n\}$ and $E(P_n) = \{12, 23, 34, \cdots, (n-1)n\}$ in all remaining parts of this paper.

Lemma 6.1. *For $n \geq 3$ and a tree T with a perfect matching, if the graph $P_n \Box T$ is Hamiltonian then $n \geq \Delta(T)$.*

Proof. If $n < \Delta(T)$, then there exists a vertex $v \in V(T)$ such that $\deg(v) = \Delta(T)$. Hence deleting n vertices $1_v, 2_v, \cdots, n_v$ from $P_n \Box T$ yields $\Delta(T)$ components, which means

$$t(P_n \Box T) \leq n/\Delta(T) < 1.$$

By Theorem 3.1, the graph $P_n \Box T$ is not Hamiltonian. ■

Theorem 6.2. *If $n \geq \max(\Delta(T), 3)$, then there exists a Hamiltonian cycle of $P_n \Box T$ which contains exactly $n - \deg(v)$ edges of $\{i_v i+1_v : i = 1, 2, \cdots, n-1\}$ for any vertex v.*

Proof. Apply induction on the number of vertices of T. The induction base is a single edge. Let T be the single edge uu_1, then $P_n \Box T$ has a Hamiltonian cycle

$$(1_u, 2_u, \cdots, n_u, n_{u_1}, n-1_{u_1}, \cdots, 1_{u_1})$$

which contains all the $n-1$ edges $\{i_u i+1_u : i=1,2,\cdots,n-1\}$ and the $n-1$ edges $\{i_{u_1} i+1_{u_1} : i=1,2,\cdots,n-1\}$.

Assume that for all T' with perfect matching and $|V(T')| < |V(T)|$, the graph $P_n \Box T'$ has a Hamiltonian cycle which contains exactly $n-\deg(v)$ of the edges: $\{i_v i+1_v : i = 1,2,\cdots,n-1\}$ for any vertex v.

To discuss the Hamiltonicity of $P_n \Box T$, since T has a perfect matching, we can find a leaf u and its neighbor u_1 with $N(u_1)=\{u,u_2\}$. The subtree T_1 induced by $V(T)-\{u,u_1\}$ is a tree with perfect matching and the tree's order is less than $|V(T)|$. By induction hypothesis, there is a Hamiltonian cycle of $P_n \Box T_1$ which contains exactly $n-(\deg(u_2)-1)$ of the edges $\{i_{u_2} i+1_{u_2} : i=1,2,\cdots,n-1\}$ (Here $\deg(u_2)$ denote the degree of u_2 in T). Since $n \geq \Delta(T)$, $n-(\deg(u_2)-1) \geq \Delta(T)-(\deg(u_2)-1) > 0$. This told us the Hamiltonian cycle must contain an edge $i'_{u_2} i'+1_{u_2}$ for some i'.

Together with the cycle

$$(1_u, 2_u, \cdots, i'_u, i'+1_u, \cdots, n_u, n_{u_1}, n-1_{u_1}, \cdots, 1_{u_1})$$

and replace edge pair

$$\{i'_u i'+1_u, i'_{u_1} i'+1_{u_1}\}$$

into

$$\{i'_u i'_{u_1}, i'+1_u i'+1_{u_1}\}.$$

We can connect those 2 cycles into a Hamiltonian cycle of $P_n \Box T$.

To check the Hamiltonian cycle containing exactly $n-\deg(v)$ edges of $\{i_v i+1_v : i = 1,2,\cdots,n-1\}$ for any vertex v, we can check in three cases:

- vertex u_1.
- vertex u_2.
- all the other vertices.

Before connecting the two cycles, the small cycle contains all $n-1$ of the edges $\{i_{u_1}i+1_{u_1} : i = 1, 2, \cdots, n-1\}$. We replace one of the edges to connect two cycles, so the resulting Hamiltonian cycle contains $n-2$ of the edges $\{i_{u_1}i+1_{u_1} : i = 1, 2, \cdots, n-1\}$ where $n-\deg(u_1)$ is exactly $n-2$. Similarly, in the beginning there are $n-(\deg(u_2)-1)$ of the edges $\{i_{u_2}i+1_{u_2} : i = 1, 2, \cdots, n-1\}$ in the cycle. After connecting two cycles, one of them are replaced, so the Hamiltonian cycle contains exactly $n-\deg(u_2)$ of the edges $\{i_{u_2}i+1_{u_2} : i = 1, 2, \cdots, n-1\}$. Besides this two vertices, the degree of any other vertex has not change and edges correspond to them have not been replaced. Hence, by induction hypothesis, the case of the other vertices can be easily checked. ∎

By Lemma 6.1 and Theorem 6.2, we know that the two statements

- $n \geq \Delta(T)$.
- $P_n \Box T$ is Hamiltonian.

are equivalent if T has a perfect matching. Moreover, they are also equivalent to another statement. The statement is shown in next theorem.

Theorem 6.3. *For $n \geq 3$ and a tree T with a perfect matching, the graph $P_n \Box T$ is Hamiltonian if and only if it is 1-tough.*

Proof. If $P_n \Box T$ is Hamiltonian, then by Theorem 3.1 it is 1-tough.

If $P_n \Box T$ is not Hamiltonian, then by Theorem 6.2, n must be less than $\Delta(T)$. There exists a vertex $v \in V(T)$ such that $\deg(v) = \Delta(T)$, hence delete n vertices $1_v, 2_v, \cdots, n_v$ from $P_n \Box T$ yields $\Delta(T)$ components, so

$$t(P_n \Box T) \leq n/\Delta(T) < 1.$$

In the other words, $P_n \Box T$ is not 1-tough. ∎

7 Cartesian product of path and trees with path factor

In the previous section, we discuss the case that T has a perfect matching. What about T has no perfect matching? We will discuss all the remaining cases that $P_n \square T$ is possible to be Hamiltonian. Here we gives several propositions, constructions and theorems below to prove the main result.

Proposition 7.1. *A graph has a path factor if and only if it has a* $\{P_2, P_3\}$*-factor.*

Proof. The sufficient condition is obvious.

To prove the necessary condition, suppose a tree has a path factor. In other words, the tree has a $\{P_2, P_3, \cdots\}$-factor. By an easy induction, we can see that P_n itself has a $\{P_2, P_3\}$-factor. For example, we can reduce a P_4 to two P_2's and a P_5 to a P_2 with a P_3. So in fact, this proposition is trivial. ■

Furthermore, the following theorem given in [1] also characterizes the path factor property.

Theorem 7.2 ([1]). *A graph* G *has a* $\{P_2, P_3\}$*-factor if and only if*

$$i(G - S) \leq 2|S|$$

for every $S \subseteq V(G)$*, where* $i(G - S)$ *denotes the number of isolated vertices in the graph* $G - S$*.*

After adding one more property that we are going to verify later, we will conclude that :

Theorem 7.3. *Let* G *be a connected graph. Then the following are equivalent:*

- G *has a path factor.*

- *G has a $\{P_2, P_3\}$-factor.*
- *$i(G - S) \leq 2|S|$, for every $S \subseteq V(G)$.*
- *$P_n \square G$ is Hamiltonian for some n.*

for any given G.

To construct a Hamiltonian cycle of $P_n \square T$ where T is a tree with a path factor, we first construct Hamiltonian cycles of $P_n \square P_2$ and $P_n \square P_3$, respectively.

Construction 7.4. The Hamiltonian cycles we choose are as follows.

- The graph $P_n \square P_2$ has two copies of vertices of $V(P_n)$, labelled as $\{1_u, 2_u, \cdots, n_u\}$ and $\{1_v, 2_v, \cdots, n_v\}$. We can find a Hamiltonian cycle:

$$(1_u, 2_u, \cdots, n_u, n_v, (n-1)_v, \cdots, 1_v, 1_u).$$

- Since $P_n \square P_3$ is unbalanced for all odd n, we only consider the case that n is even. Our construction of the Hamiltonian cycle of $P_n \square P_3$ is as follows. Let the three vertices of P_3 be a, b, and c from left to right, then the union of edge sets $\{i_a(i+1)_a : i \equiv 0, 1, 3 \pmod 4 \text{ and } 0 < i < n\}$, $\{i_b(i+1)_b : i \equiv 0, 2 \pmod 4 \text{ and } 0 < i < n\}$, $\{i_c(i+1)_c : i \equiv 1, 2, 3 \pmod 4 \text{ and } 0 < i < n\}$, $\{i_a i_b : i \equiv 2, 3 \pmod 4 \text{ and } 0 < i < n\}$, $\{i_b i_c : i \equiv 0, 1 \pmod 4 \text{ and } 0 < i < n\}$, and $\{1_a 1_b, 1_b 1_c, n_a n_b, n_b n_c\}$. forms a Hamiltonian cycle of $P_n \square P_3$.

Figure 2 shows a pair of examples in the case that $n = 10$.

Lemma 7.5. *If the graph $P_n \square G$ is Hamiltonian, then G has a path factor.*

Proof. If the graph $P_n \square G$ is Hamiltonian, it has a Hamiltonian cycle. Let H be a Hamiltonian cycle of $P_n \square G$ and H_1 be the subgraph of H induced by vertices $\{1_v : v \in V(G)\}$.

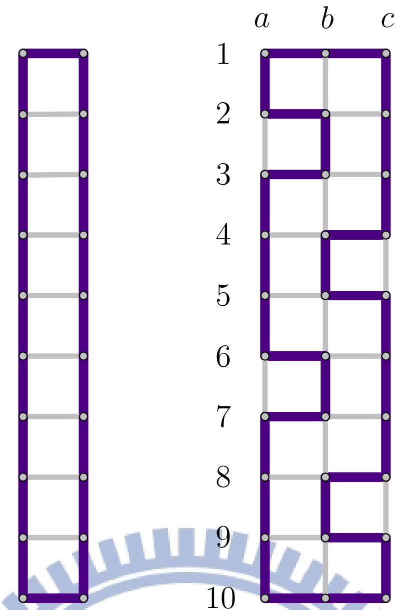

Figure 2: Hamiltonian cycles of $P_{10}\Box P_2$ and $P_{10}\Box P_3$

Since H is a cycle, its proper subgraph is a union of paths. In addition, H_1 has no isolated vertex since $|N(1_v) - \{1_u : uv \in E(G)\}| = 1$ for each vertex $v \in V(G)$. Hence, $H_1 = \{1\}\Box F$ where F is a path factor of G. ■

For a given vertex $v \in V(T)$, we can define

- $\mathcal{S}_v^A = \{i_v(i+1)_v : i = 0, 1, 3 \pmod 4\}$
- $\mathcal{S}_v^B = \{i_v(i+1)_v : i = 0, 2 \pmod 4\}$
- $\mathcal{S}_v^C = \{i_v(i+1)_v : i = 1, 2, 3 \pmod 4\}$

and base on this notation, we can state a main theorem as follow.

Theorem 7.6. *Let T be a tree with a path factor and an even integer $n \geq 4(\Delta(T)-1)+2$. Then there exists a Hamiltonian cycle H of $P_n\Box T$ such that for any vertex $v \in V(T)$, $|E(H) \cap \mathcal{S}_v^A| \geq \lceil n/4 \rceil - \deg(v)$, $|E(H) \cap \mathcal{S}_v^B| \geq \lceil n/4 \rceil - \deg(v)$ and $|E(H) \cap \mathcal{S}_v^C| \geq \lceil n/4 \rceil - \deg(v)$.*

Proof. Apply induction on the number of vertices of T. The induction bases are $T = P_2$ or $T = P_3$. The Hamiltonian cycles are constructed in Construction 7.4. Let $V(P_2) = \{a_1, a_2\}$, then the Hamiltonian cycle of $P_n \Box P_2$ contains edges $i_{a_1}(i+1)_{a_1}$ and $i_{a_2}(i+1)_{a_2}$ for every $i = 1, 2, \cdots, n-1$. Hence, the requirement is satisfied. Let $V(P_3) = \{b_1, b_2, b_3\}$, from Construction 7.4 we know that the vertical edge set correspond vertices b_1, b_2, or b_3 are $\mathcal{S}^A_{b_1}, \mathcal{S}^B_{b_2}, \mathcal{S}^C_{b_3}$, respectively. We also know that any of $\mathcal{S}^A_v, \mathcal{S}^B_v, \mathcal{S}^C_v$ intersect to each other by at least 1 edge in every 4 edges. Hence the number of vertical edges we want to count is at least $\lfloor n/4 \rfloor$ which is larger than $\lceil n/4 \rceil - \deg(v)$ for any vertex v, so the requirement is satisfied.

Assume that for all T' with path factor and $|V(T')| < |V(T)|$, the graph $P_n \Box T'$ has a Hamiltonian cycle which contains at least $\lceil n/4 \rceil - \deg(v)$ edges in all of $\mathcal{S}^A_v, \mathcal{S}^B_v, \mathcal{S}^C_v$ for any vertex v.

In the graph $P_n \Box T$, since T has a path factor, it has a $\{P_2, P_3\}$-factor. Furthermore, there are two possibilities:

1. One of the leaves of T belongs to a P_2 in the $\{P_2, P_3\}$-factor.

2. All leaves of T are belong to P_3 in the $\{P_2, P_3\}$-factor.

If case **1.** occur, we can find a leaf u and its neighbor u_1 with $N(u_1) = \{u, u_2\}$. The subtree T_1 induced by $V(T) - \{u, u_1\}$ is a tree with path factor and the tree's order is less than $|V(T)|$. By induction hypothesis, there is a Hamiltonian cycle of $P_n \Box T_1$ which contains at least $\lceil n/4 \rceil - (\deg(u_2) - 1)$ of the edges in all of $\mathcal{S}^A_{u_2}, \mathcal{S}^B_{u_2}, \mathcal{S}^C_{u_2}$ (Here $\deg(u_2)$ denote the degree of u_2 in T). Since $n \geq 4(\Delta(T) - 1) + 2$, $\lceil n/4 \rceil - (\deg(u_2) - 1) \geq \Delta(T) - (\deg(u_2) - 1) > 0$. This implies that the Hamiltonian cycle must contain an edge $i'_{u_2} i' + 1_{u_2} \in \mathcal{S}^A_{u_2}$.

Together with the cycle

$$(1_u, 2_u, \cdots, i'_u, i' + 1_u, \cdots, n_u, n_{u_1}, n - 1_{u_1}, \cdots, 1_{u_1})$$

and replace edge pair

$$\{i'_u i' + 1_u, i'_{u_1} i' + 1_{u_1}\}$$

into

$$\{i'_u i'_{u_1}, i' + 1_u i' + 1_{u_1}\}.$$

We can connect those 2 cycles into a Hamiltonian cycle of $P_n \square T$.

To check the Hamiltonian cycle containing at least $\lceil n/4 \rceil - \deg(v)$ of the edges in all of $\mathcal{S}_v^A, \mathcal{S}_v^B, \mathcal{S}_v^C$ for any vertex v, we can check in three cases:

- vertex u_1.
- vertex u_2.
- all the other vertices.

Before connecting the two cycles, the small cycle contains all $n-1$ of the edges $\{i_{u_1} i + 1_{u_1} : i = 1, 2, \cdots, n-1\}$. We only replace one of the edges to connect two cycles, so the requirement can be satisfied. Similarly, in the beginning there are at least $\lceil n/4 \rceil - (\deg(u_2) - 1)$ of the edges in all of $\mathcal{S}_{u_2}^A, \mathcal{S}_{u_2}^B, \mathcal{S}_{u_2}^C$ and also in the cycle. After connecting two cycles, one of the edges are replaced, so the Hamiltonian cycle contains at least $\lceil n/4 \rceil - \deg(u_2)$ of the edges in all of $\mathcal{S}_{u_2}^A, \mathcal{S}_{u_2}^B, \mathcal{S}_{u_2}^C$. Besides this two vertices, the degree of any other vertex has not change and edges correspond to them have not been replaced. Hence, by induction hypothesis, the case of the other vertices can be easily checked. Finally we done the proof of case one.

If case **2.** occur, we follow the label of P_3 in Construction 7.4, there are two sub-cases:

2.i. One of the leaves of T belongs to a P_3 in the $\{P_2, P_3\}$-factor where that P_3 is adjacent to another components in vertex b.

2.ii. All leaves of T are belong to P_3 in the $\{P_2, P_3\}$-factor and those P_3 are not adjacent to another components in vertex b.

If case **2.i.** occur, we can find a pair of leaves u_a, u_b and its neighbor u_1 with $N(u_1) = \{u_a, u_b, u_2\}$. The subtree T_1 induced by $V(T) - \{u_a, u_b, u_1\}$ is a tree with path factor and the tree's order is less than $|V(T)|$. By induction hypothesis, there is a Hamiltonian cycle of $P_n \Box T_1$ which contains at least $\lceil n/4 \rceil - (\deg(u_2) - 1)$ of the edges in all of $\mathcal{S}^A_{u_2}, \mathcal{S}^B_{u_2}, \mathcal{S}^C_{u_2}$ (Here $\deg(u_2)$ denote the degree of u_2 in T). Since $n \geq 4(\Delta(T)-1)+2$, $\lceil n/4 \rceil - (\deg(u_2) - 1) \geq \Delta(T) - (\deg(u_2) - 1) > 0$. This implies that the Hamiltonian cycle must contain an edge $i'_{u_2} i'+1_{u_2} \in \mathcal{S}^B_{u_2}$.

Now consider the P_3 with vertex set $\{u_a, u_1, u_b\}$ and edge set $\{u_a u_1, u_1 u_b\}$. We have constructed a Hamiltonian cycle of $P_n \Box P_3$ that contains all the edges of $\mathcal{S}^B_{u_1}$. Hence together with this Hamiltonian cycle of $P_n \Box P_3$, and replace edge pair

$$\{i'_{u_1} i'+1_{u_1}, i'_{u_2} i'+1_{u_2}\}$$

into

$$\{i'_{u_1} i'_{u_2}, i'+1_{u_1} i'+1_{u_2}\}.$$

We can connect those 2 cycles into a Hamiltonian cycle of $P_n \Box T$.

To check the Hamiltonian cycle containing at least $\lceil n/4 \rceil - \deg(v)$ of the edges in all of $\mathcal{S}^A_v, \mathcal{S}^B_v, \mathcal{S}^C_v$ for any vertex v, we can check in three cases:

- vertex u_1.
- vertex u_2.
- all the other vertices.

Before connecting the two cycles, the small cycle contains at least $\lceil n/4 \rceil$ of the edges in all of $\mathcal{S}^A_{u_1}, \mathcal{S}^B_{u_1}, \mathcal{S}^C_{u_1}$. We only replace one of the edges to connect two cycles, and $\lceil n/4 \rceil - 1 \geq \lceil n/4 \rceil - \deg(u_1) = \lceil n/4 \rceil - 3$, so the requirement is satisfied. Similarly, in the beginning there are at least $\lceil n/4 \rceil - (\deg(u_2) - 1)$ of the edges in all of $\mathcal{S}^A_{u_2}, \mathcal{S}^B_{u_2}, \mathcal{S}^C_{u_2}$ and also in the cycle. After connecting two cycles, one of the edges are replaced, so the Hamiltonian cycle contains at least $\lceil n/4 \rceil - \deg(u_2)$ of the edges all of $\mathcal{S}^A_{u_2}, \mathcal{S}^B_{u_2}, \mathcal{S}^C_{u_2}$. Besides this two vertices, the degree of any other vertex has not change and edges correspond to them have not been replaced. Hence, by induction hypothesis, the case of the other vertices can be easily checked. The proof of case **2.i.** is done.

If case **2.ii.** occur, we can find a leaf u and its neighbor u_1 with $N(u_1) = \{u, u_2\}$ and $N(u_2) = \{u_1, u_3\}$. The subtree T_1 induced by $V(T) - \{u, u_1, u_2\}$ is a tree with path factor and the tree's order is less than $|V(T)|$. By induction hypothesis, there is a Hamiltonian cycle of $P_n \square T_1$ which contains at least $\lceil n/4 \rceil - (\deg(u_3) - 1)$ of the edges in all of $\mathcal{S}^A_{u_3}, \mathcal{S}^B_{u_3}, \mathcal{S}^C_{u_3}$ (Here $\deg(u_3)$ denote the degree of u_3 in T). Since $n \geq 4(\Delta(T) - 1) + 2$, $\lceil n/4 \rceil - (\deg(u_3) - 1) \geq \Delta(T) - (\deg(u_3) - 1) > 0$. This implies that the Hamiltonian cycle must contain an edge $i'_{u_3} i' + 1_{u_3} \in \mathcal{S}^A_{u_3}$.

Now consider the P_3 with vertex set $\{u, u_1, u_2\}$ and edge set $\{uu_1, u_1u_2\}$. We have constructed a Hamiltonian cycle of $P_n \square P_3$ that contains all the edges of $\mathcal{S}^A_{u_2}$. Hence together with this Hamiltonian cycle of $P_n \square P_3$, and replace edge pair

$$\{i'_{u_2} i' + 1_{u_2}, i'_{u_3} i' + 1_{u_3}\}$$

into

$$\{i'_{u_2} i'_{u_3}, i' + 1_{u_2} i' + 1_{u_3}\}.$$

We can connect those 2 cycles into a Hamiltonian cycle of $P_n \square T$.

To check the Hamiltonian cycle containing at least $\lceil n/4 \rceil - \deg(v)$ of the edges in all of $\mathcal{S}^A_v, \mathcal{S}^B_v, \mathcal{S}^C_v$ for any vertex v, we can check in three cases:

- vertex u_2.
- vertex u_3.
- all the other vertices.

Before connecting the two cycles, the small cycle contains at least $\lceil n/4 \rceil$ of the edges in all of $\mathcal{S}_{u_2}^A, \mathcal{S}_{u_2}^B, \mathcal{S}_{u_2}^C$. We only replace one of the edges to connect two cycles, and $\lceil n/4 \rceil - 1 \geq \lceil n/4 \rceil - \deg(u_2) = \lceil n/4 \rceil - 2$, so the requirement is satisfied. Similarly, in the beginning there are at least $\lceil n/4 \rceil - (\deg(u_3) - 1)$ of the edges in all of $\mathcal{S}_{u_3}^A, \mathcal{S}_{u_3}^B, \mathcal{S}_{u_3}^C$ and also in the cycle. After connecting two cycles, one of the edges are replaced, so the Hamiltonian cycle contains at least $\lceil n/4 \rceil - \deg(u_3)$ of the edges all of $\mathcal{S}_{u_3}^A, \mathcal{S}_{u_3}^B, \mathcal{S}_{u_3}^C$. Besides this two vertices, the degree of any other vertex has not change and edges corresponding to them have not been replaced. Hence, by induction hypothesis, the case of the other vertices can be easily checked. The proof of case **2.ii.** is done.

Combining the above cases completes the proof. ∎

Now we can fill up the remaining parts of Theorem 7.3.

Proof of Theorem 7.3. We are going to prove that a graph G has a path factor if and only if $P_n \square G$ is Hamiltonian for some n. First, if G has a path factor, then we can find a spanning tree T of G with a path factor. By Theorem 7.6, we can find an integer n to make $P_n \square T$ Hamiltonian. Obviously, the Hamiltonian cycle of $P_n \square T$ is also the Hamiltonian cycle of $P_n \square G$.

On the other hand, by Lemma 7.5, if $P_n \square G$ is Hamiltonian for some n, G must have a path factor and this completes the proof. ∎

Based on Theorem 7.2, we give a related property in the case of trees. Let T be a tree with partite sets T_A and T_B.

Proposition 7.7. *If there exists a vertex subset $S \subseteq V(T)$ such that $i(T - S) > 2|S|$, then there exists a vertex subset $S' \subseteq T_A$ or $S' \subseteq T_B$ such that $i(T - S') > 2|S'|$.*

Proof. Suppose the statement is incorrect, then

$$\forall S' \subseteq T_A, i(T - S') \leq 2|S'|,$$

$$\forall S' \subseteq T_B, i(T - S') \leq 2|S'|.$$

Now, for any given $S \subseteq V(T)$, if S is contained in T_A or T_B, then $i(T-S) \leq 2|S|$.

If not, which means $S \cap T_A \neq \emptyset$ and $S \cap T_B \neq \emptyset$. Let $S \cap T_A = S_A, S \cap T_B = S_B$ then we have

$$i(T - S_A) \leq 2|S_A| \text{ and } i(T - S_B) \leq 2|S_B|.$$

Since isolated vertices of $T - S_A, T - S_B$ must be vertices in B and A, respectively, the value $i(T-S)$ will be the summation of $i(T-S_A), i(T-S_B)$ and the number of isolated vertices of $T-S$ which was not isolated in both of $T-S_A$ and $T-S_B$.

Let x be an isolated vertex of $T-S$ but not an isolated vertex in $T-S_A$ and $T-S_B$. Without loss of generality, let $x \in T_A$. But this also told us $n(x) \subseteq T_B$. Hence, if x is isolated in $T-S$, it must be already isolated in $T-S_B$, a contradiction. We finally know that there is not any such vertex. So,

$$i(T-S) = i(T-S_A) + i(T-S_B) \leq 2|S_A| + 2|S_B| = 2|S|,$$

a contradiction, and we get the proof. ∎

Theorem 7.8. *A tree T has a path factor if there exists an integer n such that $P_n \square T$ is 1-tough.*

Proof. Suppose that T doesn't have a path factor and T has partite sets T_A and T_B. By Theorem 7.2, there exists a vertex subset S such that $i(T-S) > 2|S|$. Moreover, by Proposition 7.7, we can restrict S to be contained in a single partite set, without loss of generality, say $S \subseteq T_A$.

Now we want to prove that for any n, $G = P_n \square T$ is not 1-tough, so we need to find the suitable vertex cutset X. If n is even, consider a way that choose X to be the union of T_A in odd layers and T_B in even layers (Actually, this choose a part of the bipartition of G). But in this way, the cardinality $|X|$ and the number of components $c(G-X)$ will both equal to $\frac{|V(G)|}{2}$, so we need to do a little modification. Since $S \subseteq T_A$, those isolated vertices are all in T_B. In the last layer, modify X by replacing those isolated vertices in T_B by vertices in S. This makes the size of X change by $-i(T-S)+|S|$, and the number of components change by $-|S|$. Finally, we find a new vertex cutset X' such that $|X'| = \frac{|V(G)|}{2} - i(T-S) + |S|$ and $c(G-X') = \frac{|V(G)|}{2} - |S|$,and

$$t(G) \leq \frac{|X'|}{c(G-X')} = \frac{\frac{|V(G)|}{2} - i(T-S) + |S|}{\frac{|V(G)|}{2} - |S|} < \frac{\frac{|V(G)|}{2} - 2|S| + |S|}{\frac{|V(G)|}{2} - |S|} = 1.$$

If n is odd, the choice of vertex cutset is dependent on T. First, if T is unbalanced, then G is unbalanced. Hence we can choose the smaller partite set to be the vertex cutset. If T is balanced, then similar to the case that n is even, we choose X to be the union of T_A in even layers and T_B in odd layers (and this choose a part of the bipartition of G again, $|X| = c(G-X) = \frac{|V(G)|}{2}$). Modify X in the last layer by replacing those isolated vertices in T_B by vertices in S makes the size of X change by $-i(T-S)+|S|$, and the number of components change by $-|S|$. Finally, we find a new vertex cutset X' such that

$$t(G) \leq \frac{|X'|}{c(G-X')} = \frac{\frac{|V(G)|}{2} - i(T-S) + |S|}{\frac{|V(G)|}{2} - |S|} < \frac{\frac{|V(G)|}{2} - 2|S| + |S|}{\frac{|V(G)|}{2} - |S|} = 1.$$

This complete the proof. ■

Combining Theorem 7.8 together with Proposition 3.1, Lemma 7.5 and Theorem 7.6, we know that for all even integer $n \geq 4\Delta(T) - 2$, the following are equivalent:

- T has a path factor.
- $P_n \Box T$ is Hamiltonian.
- $P_n \Box T$ is 1-tough.

8 Hamiltonicity and even-pancyclicity

We say that a graph G is pancyclic if for each vertex $v \in V(G)$, v is contained in cycles of length 3 to $|V(G)|$. Similarly, we say that a graph G is even-pancyclic if for each vertex $v \in V(G)$, v is contained in cycles of all even length 4 to $|V(G)|$. Details of this definition can be found in [7]. We only discuss the even-pancyclicity of $P_n \square T$ rather than pancyclicity since $P_n \square T$ is a bipartite graph which contains no odd cycles.

If we defined a *grid* to be a C_4 induced by vertices $\{i_v, i+1_v, i+1_u, i_u\}$ for any $i = 1, 2, \cdots, n-1$ and $u \in N(v)$, then $P_n \square T$ can be stack up grid by grid. To show that the even-pancyclicity after we constructed a Hamiltonian cycle, we need to classify those grids.

Definition 8.1. Based on Construction 7.4, we define the word *inside* as follow. First, all grids of $P_n \square P_2$ are on the *inside* of the Hamiltonian cycle of $P_n \square P_2$. Second, grids $\{i_a, i+1_a, i+1_b, i_b : i \equiv 0, 1, 3 \pmod 4\}$ and $\{i_b, i+1_b, i+1_c, i_c : i \equiv 1, 2, 3 \pmod 4\}$ are on the *inside* of the Hamiltonian cycle of $P_n \square P_3$. Finally, a grid $\{i_v, i+1_v, i+1_u, i_u\}$ of $P_n \square T$ is on the *inside* of the Hamiltonian cycle of $P_n \square T$ with respect to the construction given in the proof of Theorem 6.2 and Theorem 7.6 if and only if one of the three conditions is satisfied.

- The edge uv forms a P_2 in the path factor (or the perfect matching) and the grid is on the inside of the Hamiltonian cycle of $P_n \square P_2$.
- The edge uv belongs to a P_3 in the path factor and the grid is on the inside of the Hamiltonian cycle of $P_n \square P_3$.
- The two vertices $u, v \in V(T)$ belong to different components of the path factor (or the perfect matching) of T and edges $i_v i_u$ and $i+1_v i+1_u$ belong to the Hamiltonian cycle.

Furthermore, for the convenience of proof, here gives one more definition.

Definition 8.2. We define a graph $\hat{G} = (\hat{V}, \hat{E})$ corresponding to a construction of Hamiltonian cycle of $P_n \square T$ where $\hat{V} = \{g : g \text{ is a grid on the inside of the Hamiltonian cycle}\}$ and $\hat{E} = \{g_1 g_2 : g_1, g_2 \in \hat{V} \text{ and the two grids share an edge}\}$.

Lemma 8.3. *When applying our Hamiltonian cycle construction of $P_n \square T$ in Theorem 6.2 and Theorem 7.6, the corresponding graph $\hat{G}$ is a tree.*

Proof. Apply induction on $|V(T)|$, the induction bases are $\hat{G}$ corresponding to $P_n \square P_2$ and $P_n \square P_3$ which are clearly trees (Actually, they are paths).

Assume that for any $|V(T')| < |V(T)|$, the graph correspond to $P_n \square T'$ is a tree. In the construction of Hamiltonian cycle, we construct the Hamiltonian cycle by connecting the Hamiltonian cycle of $P_n \square P_k$ and that of $P_n \square (T - P_k)$ where k may be 2 or 3. The corresponding graph to each of them can be considered as a coalescence of path and tree, and that is still a tree. ■

Theorem 8.4. *In Theorem 6.2 and Theorem 7.6, all those Hamiltonian $P_n \square T$ are also even-pancyclic.*

Proof. First, choose a Hamiltonian cycle of the graph. For any given vertex v, we can find a grid g which contains v. Moreover, for any $k = 1, 2, \cdots, |V(\hat{G})|$, we can find a connected sub-tree of $\hat{G}$ of order k which contains g. In fact, the sub-tree of order k represents a cycle of length $2k + 2$ in $P_n \square T$ since adding a vertex into the sub-tree means adding a grid into the cycle (and this increase the length of cycle by 2). Therefore v is contained in cycles of every even lengths and hence $P_n \square T$ is even-pancyclic. ■

9 Conclusion and future works

To conclude our main results and contributions, we list them in Table 1.

Graph class	Equivalent statements
$C_n \square T$	$n \geq \Delta(T)$ [2][6]
	Hamiltonian [2][6]
	1-tough
$C_n \square T$	$n > \Delta(T)$
	edge-Hamiltonian
	edge-1-tough and 1-tough
	edge-1-tough
$P_n \square T$ T has a 1-factor	$n \geq \Delta(T)$
	Hamiltonian
	even-pancyclic
	1-tough
$P_n \square T$ with $n \geq 4\Delta(T) - 2$	T has a path factor
	Hamiltonian
	even-pancyclic
	1-tough

Table 1: The equivalence between Hamiltonicity and other conditions

Except the equivalence between Hamiltonicity and degree conditions, all the other results are found by ourselves. In fact, we characterize all the trees T that makes $P_n \square T$ possible to be Hamiltonian.

This thesis also gives several constructions of Hamiltonian cycle in Cartesian product graphs and further generalize the Hamiltonian properties to edge-Hamiltonian and even-pancyclicity in $C_n \square T$ and $P_n \square T$, respectively. More importantly, those ideas we use, including Hamiltonian cycle, Tree, Cartesian product, path factor, edge-Hamiltonian graph, even-pancyclicity are all widely applied in network theory and also be expected to have more development. Here are some unsolved problems,

- Although we find some n to make $P_n \square T$ Hamiltonian when T has a path factor, such n are still too large. Is there any other construction with smaller n?

- How if we replace the graph C_n into "1-tough graph G", is there any good results in $G \square T$?

- In general, the two terms "Hamiltonian" and "1-tough" are not equivalent but for those graphs we focus on, they are. Is there any other type of graphs making this equivalence holds?

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