國 立 交 通 大 學

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碩 士 論 文

關於(2, 3)-圖形零和流數之研究

Zero-Sum Flow Numbers of (2, 3)-Graphs

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中 華 民 國 一 零 五 年 六 月

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摘 要

對一無向圖形 G，令 $E(v)$ 記為圖形中頂點 v 的相鄰邊所構成之集合。圖 G 上一零和流為一組對邊的非零實數編號 f 使得對每一頂點 v 來說，

$$\sum_{e \in E(v)} f(e) = 0$$

皆成立。零和 k-流為一零和流且編號全來自集合 $\{\pm 1, \ldots, \pm(\mathrm{k}-1)\}$。零和流數 $F(G)$ 定義為圖 G 具有零和 k-流之最小正整數 k。在此篇論文中，對一 (2,3)-圖形 G 給出了具有零和流數 3 的充分且必要之條件。此外我們研究由路徑和樹擴展而成之 (2,3)-圖形上的零和流數，名曰，聖誕燈、樹燈，並總結它們的零和流數最多為 5。

關鍵字: 零和流，零和 k-流，零和流數，(2,3)-圖形，聖誕燈，樹燈。

Zero-Sum Flow Numbers of $(2,3)$-Graphs

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Abstract

For an undirected graph G, let $E(v)$ denote the set of edges incident on a vertex $v \in V(G)$. A *zero-sum flow* is an assignment f of non-zero real numbers on the edges of G such that

$$\sum_{e \in E(v)} f(e) = 0$$

for all $v \in V(G)$. A *zero-sum k-flow* is a zero-sum flow with integers from the set $\{\pm 1, ..., \pm(\mathrm{k}-1)\}$. Let *zero-sum flow number* $F(G)$ be defined as the least number of k such that G admits a zero-sum k-flow. In this paper, a necessary and sufficient condition for $(2,3)$-graph G with $F(G) = 3$ is given. Furthermore we study zero-sum flow number of $(2,3)$-graphs expanded from path and tree, namely, the Christmas lamps, the tree lamps, respectively, and conclude that their zero-sum flow numbers are at most 5.

Keywords: zero-sum flow, zero-sum k-flow, zero-sum flow number, $(2,3)$-graph, Christmas lamp, tree lamp.

致謝詞

光陰似箭，歲月如梭，時間匆匆流逝，碩士生涯悄悄地接近尾聲，驀然回首，轉眼間又度過了兩載的人生。回想碩士兩年在新竹的求學歷程，既新奇又充實，在學期間，我受到諸位師長、同學、朋友與家人的協助和鼓勵，感激之情，實難以一言以蔽之。

首先誠摯地感謝我的指導教授翁志文教授，翁老師悉心指導初踏入研究路途的我，不時地指點與修正我的研究方法及語句，耐心地解釋修改的細節，不辭辛勞地反覆修改，幫助我度過了寫作的瓶頸，平日裡教導我做人處事的道理，在此感謝老師的教誨與包容。另外我非常感謝張耀祖教授和傅東山教授在百忙之中抽空來新竹為我們口試，並為這篇論文提供寶貴的意見，使我從中受益良多。我也要感謝陳秋媛教授、傅恒霖教授、符麥克教授、康明軒教授、陳冠宇教授和薛名成教授，在兩年的求學期間，不論在學術上或是日常生活中，都給予我極多的幫助與鼓勵，令我銘感五內。

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我即將踏上新的旅途，希望我在交大所學的，未來能對社會做出貢獻。最後，請各位不吝嗇的再聽我說一句，謝謝你們。

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1 Introduction

Throughout the thesis, a graph is always undirected and connected.

An **orientation** of an undirected graph is an assignment of a direction to each edge. Let G be an undirected graph with vertex set $V(G)$ and edge set $E(G)$. Let D be an orientation of $E(G)$. For a vertex $v \in V(G)$, let $E^+(v)$ ($E^-(v)$, respectively) denote the set of directed edges according to the orientation D with their tails (heads, respectively) at the vertex v.

Suppose $k \in \mathbb{N}$. A k-**flow** on G is an ordered pair (D, f) where D is an orientation of $E(G)$ and f is an assignment of integers with absolute value at most $k-1$ to each edge of G such that

$$\sum_{e \in E^+(v)} f(e) - \sum_{e \in E^-(v)} f(e) = 0$$

for all $v \in V(G)$.

A **nowhere-zero** k-**flow** is a k-**flow** with no zeros. If G is an undirected graph, then we say that it **has a nowhere-zero** k-**flow** if the graph G admits a nowhere-zero k-flow.

Example 1.1. Let G be a cycle C_3 of order 3, then G has a nowhere-zero 2-flow as shown in Figure 1.

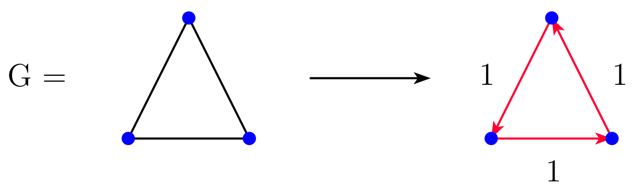

Figure 1: G has a nowhere-zero 2-flow.

Definition 1.2. A **bridge** of a connected graph is an edge whose removal disconnects the graph. A **bridgeless** graph is a graph that contains no bridges.

A famous conjecture of Tutte's says that,

Conjecture 1.3. *(Tutte's 5-flow Conjecture [5]) Every bridgeless graph has a nowhere-zero 5-flow.*

Seymour has proven a result related to this conjecture in 1981.

Theorem A. *(Seymour [4]) Every bridgeless graph has a nowhere-zero 6-flow.*

An interesting problem about nowhere-zero k-flow is the following. Given a graph G, what is the smallest integer k such that G has a nowhere-zero k-flow, i.e., an integer k for which G admits a nowhere-zero k-flow, but it does not admit a $(k-1)$-flow. Let $\Gamma = \Gamma(G)$ denote this minimum k called the **minimum flow number** of G. For $\Gamma(G) = 2$, we have completely known the situation.

Theorem B. *(Tutte [6]) A graph G has a nowhere-zero 2-flow if and only if the degree of each vertex is even.*

S. Akbari, N. Ghareghani, G.B. Khosrovshahi and A. Mahmoody [1] use a linear algebraic approach to look at Tutte's Conjecture in 2009 which provides them with a motivation to adopt a different definition of k-flow in an undirected graph.

A **zero-sum flow** on a graph G is an assignment f of non-zero real numbers on the edges of G such that the total sum of the assignments of all edges incident with any vertex on G is zero. A **zero-sum k-flow** for a graph G is a zero-sum flow with numbers from the set $\{\pm 1, ..., \pm(\mathrm{k}-1)\}$.

Note that G has a nowhere-zero flow is not the same as G has a zero-sum flow. We now only consider the zero-sum flow problem. Let G be a graph, we say **zero-sum rule** holds on a vertex $v \in G$ if the sum of assignments of all edges incident with v is zero.

A similar conjecture of Tutte's 5-flow conjecture is the zero-sum conjecture.

Conjecture 1.4. *(Zero-Sum Conjecture [1]) If G is a graph with a zero-sum flow, then G has a zero-sum 6-flow.*

Example 1.5. Let G be a cycle C_4 of order 4, then G has a zero-sum 2-flow as shown in Figure 2.

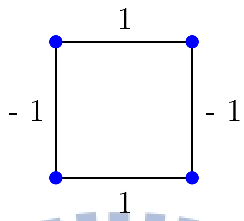

Figure 2: G has a zero-sum 2-flow.

Akbari et al. have proved a necessary and sufficient condition for the existence of zero-sum flow for non-bipartite graphs.

Theorem C. *(Akbari et al. [1]) Suppose G is not a bipartite graph. Then G has a zero-sum flow if and only if for any edge e of G, $G \setminus \{e\}$ has no bipartite component.*

Definition 1.6. Let G be a connected graph. Then G is **k-edge connected** if it remains connected whenever fewer than k edges are removed.

Akbari et al. also show that the zero-sum conjecture is true for the 2-edge connected bipartite graphs.

Theorem D. *(Akbari et al. [1]) Let G be a 2-edge connected bipartite graph. Then G has a zero-sum 6-flow.*

In addition, Akbari et al. have proved that the zero-sum conjecture is true for 3-regular graphs.

Theorem E. *(Akbari et al. [1]) Every* 3*-regular graph has a zero-sum* 5*-flow.*

Remark 1.7. *The following graph shown in Figure 3 [1] shows that in the above theorem, zero-sum 5-flow can not be replaced with zero-sum 4-flow.*

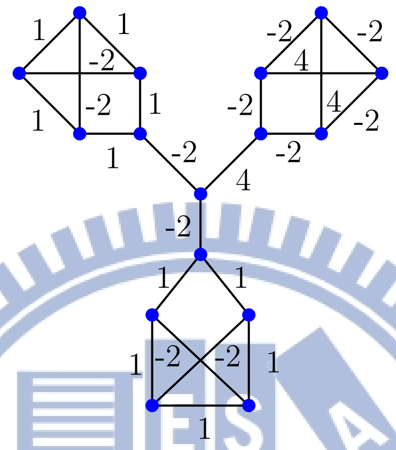

Figure 3: G has a zero-sum 5-flow.

Definition 1.8. Let G be a graph. We call G a $(2, 3)$-graph if the degree of each vertex is 2 or 3.

Moreover, Akbari et al. provide a relation between the $(2, 3)$-graph and zero-sum conjecture.

Theorem F. *(Akbari et al. [1]) If Zero-Sum Conjecture is true for any* $(2, 3)$*-graph, then it is true for any graph.*

For the details on the above theorems and other results, see [1, 3]. T.M. Wang and S.-W. Hu extend the concept minimum flow number in 2011 to the following definition.

Definition 1.9. Let G be a graph. The **zero-sum flow number** $F(G)$ is defined as the least number of k for which G may admit a zero-sum k-flow. $F(G) = \infty$ if no such k exists.

Example 1.10. Let G be a non-bipartite graph as shown in Figure 4. Since there is a bipartite component of G after deleting an edge of $E(G)$, by the theorem C, $F(G) = \infty$.

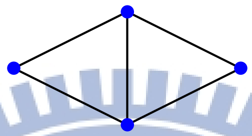

Figure 4: $F(G) = \infty$.

Note that $F(G) = k$ is not the same as $\Gamma(G) = k$. Akbari et al. provide a relation between the $F(G)$ and $\Gamma(G)$.

Definition 1.11. Let G be a graph, then $S(G)$ is a graph obtain from G by augmenting exactly one new vertex on each edge of G.

Lemma G. *(Akbari et al. [1])* $\Gamma(G) = k$ *if and only if* $F(S(G)) = k$.

T.M. Wang et al. [7] show some general properties of small zero-sum flow numbers, so that the estimate of zero-sum flow numbers gets easier. A result on the zero-sum flow numbers is the following.

Theorem H. *(T.M. Wang et al. [7]) A graph G has zero-sum flow number $F(G) = 2$ if and only if G is Eulerian with even size (even number of edges) in each component.*

2 Zero-sum flow number 3 on (2, 3)-graph

Throughout the thesis, a graph is always finite, simple, connected and undirected.

Lemma 2.1. *Any* $(2,3)$*-graph has even number of vertices with degree 3.*

Proof. Let G be a $(2,3)$-graph. Then G has an even number of vertices with odd valency since that $\sum_{x \in V(G)} \deg(x) = 2|E(G)|$. That is, G has even number of vertices with degree 3. □

Definition 2.2. A **loop** is an edge that connects a vertex to itself. A **multigraph** is a graph that can have more than one edge between a pair of vertices and allow loops, which add two to the degree.

Definition 2.3. The **edge subdivision** of an edge e with endpoints $\{u, v\}$ yields a graph containing one new vertex w, and with an edge set replacing e by two new edges, $\{u, w\}$ and $\{w, v\}$. A **subdivision** of a graph G is a graph resulting from the edge subdivision of edges in G.

Lemma 2.4. *Any* $(2,3)$*-graph with at least two vertices of degree 3 is obtained by consecutive edge subdivisions from a* 3*-regular multigraph.*

Proof. If the number of vertices with degree 2 is zero, the proof is done. Suppose the number of vertices with degree 2 is bigger than zero. Then if we merge the two edges of a vertex with degree 2 to an edge and delete that vertex, each remainder vertices has degree 3. That is any $(2,3)$-graph with at least two vertices of degree 3 is obtained by consecutive edge subdivisions from a 3-regular multigraph. □

Suppose G is a (2,3)-graph, by the above theorem H, we obtain $F(G) = 2$ if and only if G is an even cycle. That is the study of $F(G)$ equal to two is completed. So we want to discuss that G is not an even cycle, which means that the number of vertices with degree 3 in G is greater than zero.

Definition 2.5. A path in G is called a **323-path** if its internal vertices have degree 2 and its two endpoints have degrees 3.

Note that a **323-path** without internal vertices is an edge in G.

Definition 2.6. A family of vertex-disjoint 323-paths in a (2,3)-graph G is called **complete** if each vertex of degree 3 is in exactly one path of the family.

Lemma 2.7. *Let G be a (2,3)-graph with $F(G) \leq 4$. Then there exists a complete family of vertex-disjoint 323-paths in G.*

Proof. Since $F(G) \leq 4$, there exists an assignment f on $E(G)$ such that $f(e) \in$ {-3, -2, -1, 1, 2, 3}. And we know that the case as shown in Figure 5 with numbers {even, even, odd} on edges incident on a vertex of degree 3 is illegal of the zero-sum rule for even $\in \{\pm 2\}$ and odd $\in \{\pm 1, \pm 3\}$.

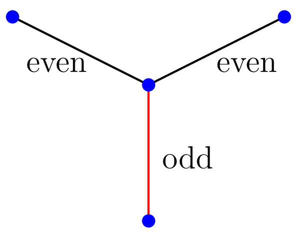

Figure 5: An illegal labeling in $F(G) \leq 4$.

Hence there is no vertex of degree 3 whose two edges have numbers with absolute value 2. Moreover, we know there must exist an edge with even number incident on a vertex v with degree 3 and the case with numbers {even, even, even} on edges incident on v is illegal of the zero-sum rule for even $\in \{\pm 2\}$. Let Ω be the graph induced on the edge set and $\Omega = \{e \mid e \in E(G) \text{ and } |f(e)| = 2\}$. Then Ω is a complete family of vertex-disjoint 323-paths in G since G is a $(2,3)$-graph. □

Now, by using the Lemma 2.7, we find some general properties for the zero-sum flow number $F(G) = 3$.

Definition 2.8. A ***k*-factor** of a graph is a spanning k-regular subgraph. A 1-factor is a perfect matching in G.

T.M. Wang and S.-W. Hu [7] show the following (ii)-(iii) of theorem 2.9 are equivalent.

Theorem 2.9. *Let G be a 3-regular graph. Then the following (i)-(iii) are equivalent.*

(i) $F(G) \leq 4$;

(ii) G has a 1-factor;

(iii) $F(G) = 3$.

In particular, there is no 3-regular graph with $F(G) = 4$.

Proof. (i) $\Rightarrow$ (ii): By the lemma 2.7, there exists a complete family of vertex-disjoint 323-paths in G, we denote it by Ω. Since G is a 3-regular graph, a 323-path is an edge in G. Hence Ω is a perfect matching in G.

(ii) $\Rightarrow$ (iii): Since G is not an even cycle, $F(G) > 2$. Suppose G has a perfect matching Ω in G. For an edge $e \in E(G)$, we give the edge values $f(e) = 2$ if $e \in \Omega$

and $f(e) = \text{-}1$ if $e \notin \Omega$. Since a matching in G is a set of edges without common vertices, the zero-sum rule holds for any vertex $v \in V(G)$. That is $F(G) = 3$.

(iii) $\Rightarrow$ (i): $F(G) = 3$ which is less than 4. □

From the theorem 2.9, it is easy to see $F(G) = 5$ for the graph in Figure 3.

Definition 2.10. Let G be a graph with an assignment of two colors to the edges. Then a path P of G is **alternating** if no two adjacent edges of P have the same color.

Definition 2.11. A path P is **tangent** to Ω at x if x is a vertex of P and Ω but no edges in $E(P) \cap E(\Omega)$ incident on x.

Theorem 2.12. *Let G be a (2,3)-graph other than an even cycle. Then $F(G) = 3$ if and only if the following conditions hold:*

(i) There exists a complete family Ω of vertex-disjoint 323-paths.

(ii) There exists an assignment of two colors to the edges of G such that any path P of G not tangent to Ω is alternating.

Proof. Suppose $F(G) = 3$. Then there exists an assignment f on $E(G)$ such that the edge values $f(e) \in \{\text{-}2, \text{-}1, 1, 2\}$. Let Ω be the graph induced on the edge set and $\Omega = \{e \mid e \in E(G) \text{ and } |f(e)| = 2\}$. By the lemma 2.7, Ω is a complete family of vertex-disjoint 323-paths in G. Then (i) holds. Now we define a coloring on $E(G)$ such that e is blue if $f(e)$ is positive or e is red if $f(e)$ is negative. To prove (ii), suppose on the contrary, P is not alternating. Then there are two edges e_1 and e_2 incident on a vertex $x \in V(P)$ such that $sgn(f(e_1)) = sgn(f(e_2))$. By the zero-sum rule, $f(e_1) = f(e_2) \in \{\pm 1\}$ and the degree of x equals 3 as shown in Figure 6. Since $e \in \Omega$ if $e \in E(G)$ and $|f(e)| = 2$, P is tangent to Ω at $x \in V(\Omega)$. Then (ii) holds.

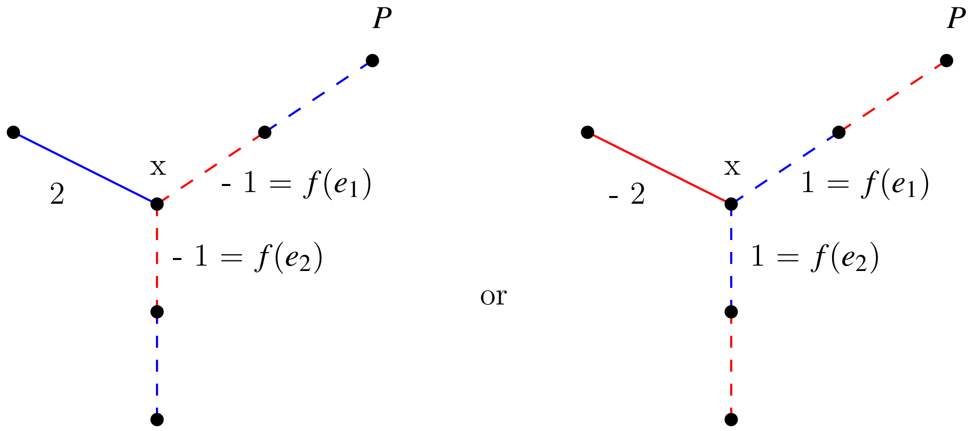

Figure 6: The degree of x is 3 and $f(e_1) = f(e_2) \in \{\pm 1\}$

Conversely, we know $F(G) > 2$ since G is not an even cycle. Suppose (i) and (ii) hold. Define an assignment f on $E(G)$ as follow.

$$f(e) = \begin{cases} 1 & \text{e is blue} \quad \text{and} \quad \text{e is not in a path of } \Omega. \\ -1 & \text{e is red} \quad \text{and} \quad \text{e is not in a path of } \Omega. \\ 2 & \text{e is blue} \quad \text{and} \quad \text{e is in a path of } \Omega. \\ -2 & \text{e is red} \quad \text{and} \quad \text{e is in a path of } \Omega. \end{cases}$$

Since Ω is a complete family of vertex-disjoint 323-paths, there are three situations of a vertex $v \in V(G)$. First, $v \in V(\Omega)$ and v is an internal vertex in a path of Ω, the numbers on the edges incident with v are 2, -2. Second, $v \in V(\Omega)$ and v is an endpoint in a path of Ω, the numbers on the edges incident with v are 2, -1, -1 or -2, 1, 1. Third, $v \notin V(\Omega)$, the numbers on the edges incident with v are 1, -1. Hence, $\sum_{e \in E(v)} f(e) = 0$ for all $v \in V(G)$, where $E(v)$ denote the set of edges incident with a vertex $v \in V(G)$. That is mean $F(G) = 3$. □

Remark 2.13. *Let G be a $(2,3)$-graph with $F(G) = 4$ as shown in Figure 7. The graph G satisfies the condition (i) but does not satisfy the condition (ii) of Theorem 2.12.*

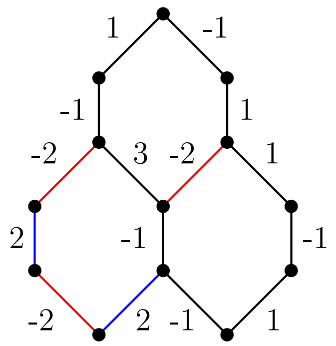

Figure 7: $F(G) = 4$.

3 Zero-sum flow number for Christmas lamp

Definition 3.1. An **edge minor** of a 323-path is obtained by contracting some edges of the 323-path while preserving the parity of the number of edges. A **minor** of a graph G is a graph resulting from the edge minors of 323-paths in G.

Note that we allow the 323-path which is contracted has same endpoints.

Example 3.2. Let $\tilde{G}$ be obtained from G by contracting a 323-paths(yellow) while preserving the parity of the number of edges as shown in Figure 8. Then $\tilde{G}$ is a minor of G.

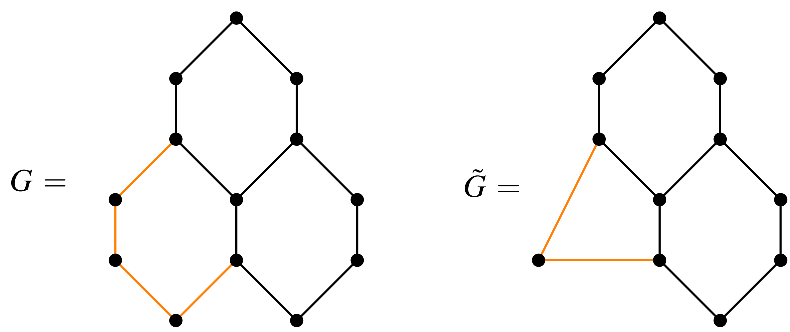

Figure 8: $\tilde{G}$ is a minor of G.

Let G and $\tilde{G}$ be $(2,3)$-graphs. If $\tilde{G}$ is a minor of G, then $F(G) = F(\tilde{G})$. That is, if we remove even number of edges from any 323-path of G, then the zero-sum flow number $F(G)$ does not change.

Theorem 3.3. *Let G be a bridgeless $(2,3)$-graph which the number of edges of any 323-path in G is even. Then $F(G) \leq 6$.*

Proof. Suppose G is a bridgeless $(2,3)$-graph which the number of edges of any 323-path in G is even. Use the property of minor, there is a bridgeless $(2,3)$-graph G_1 such that $F(G_1)$ equals $F(G)$ and there is a bridgeless 3-regular graph G_2 such that $G_1 = S(G_2)$. By the theorem A, we obtain $\Gamma(G_2) \leq 6$. Moreover, by the lemma G, we have $F(S(G_2)) = F(G_1) \leq 6$ which implies $F(G) \leq 6$. □

With the theorem 3.3, we know the zero-sum conjecture is true for any bridgeless $(2,3)$-graph which the number of edges of any 323-path in G is even. Now, we study a special 1-edge connected (2, 3)-graph and find the upper bound of zero-sum flow number of this graph.

Definition 3.4. Let H be a graph and $v \in V(H)$ with neighbors u_1, u_2, $\cdots$, u_s. Let C be a cycle with vertex set $V(C) = \{v_1, v_2, \cdots, v_t\}$, where $t \geq s$. A graph G is said to be obtained from H by **replacing** v by C if $V(G) = V(H) \cup V(C) - \{v\}$ and there exist $1 \leq i_1 < i_2 < \cdots < i_s \leq t$ such that $E(G)$ contains $\{u_k v_{i_k} | 1 \leq k \leq s\} \cup E(H) \cup E(C) - \{vu_i | 1 \leq i \leq s\}$.

Definition 3.5. A graph G is a **Christmas lamp** if G is obtained from a path of order at least 2 by replacing its two endpoints by two odd cycles and some internal vertices by cycles. Moreover, we call C is a lamp of G if C is a subgraph of G and C is a cycle.

Example 3.6. Let P_5 be a path of order 5. Then G is a Christmas lamp obtained from P_5 as shown in Figure 9.

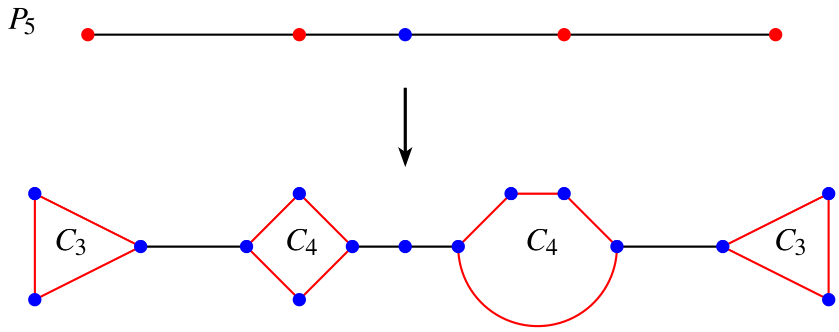

Figure 9: G is a Christmas lamp obtained from a path P_5 of order 5.

Definition 3.7. Suppose G is a Christmas lamp obtained from a path H. A lamp C of G is called **internal** if C is obtained by replacing an internal vertex of H.

Let G be a Christmas lamp, from the theorem C, the zero-sum flow number $F(G) < \infty$. Note that $F(G) > 2$ since G is not an even cycle. The Christmas lamp G in Figure 10 has $F(G) = 5$.

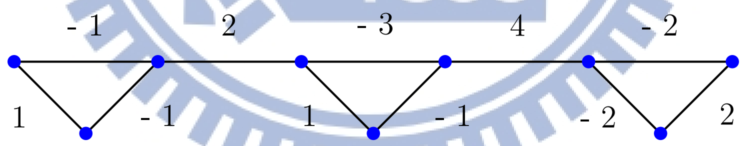

Figure 10: G is a Christmas lamp which $F(G) = 5$.

Theorem 3.8. *Let G be a Christmas lamp based on a path H. Then $F(G) \leq 5$. Moreover, for any labeling for G, with $F(G) \leq 5$, $f(E(H)) \subseteq \{\pm 2, \pm 4\}$.*

Proof. Suppose G is a Christmas lamp. By the property of minor, there are three kinds of lamps in G as shown in Figure 11, namely, the oo lamp, the ee lamp, the o lamp, respectively. An internal lamp has exactly two vertices of degree 3, called end vertices. An internal lamp is an oo lamp if its end vertices of degree 3 divide the lamp into two paths of odd length. An internal lamp is an ee lamp if its end vertices of degree 3 divide the lamp into two paths of even length. A lamp is an o lamp if it has odd order.

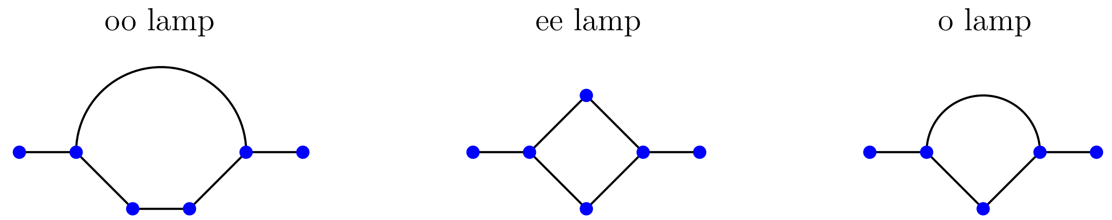

Figure 11: Three kinds of lamps in G

Let G be the christmas lamp obtained from a path $H = u_1u_2\cdots u_n$ by replacing some vertex u_i by cycles C_i. First, we give lables on edges of H consecutively by setting $f(u_1u_2) = 2$ and when $f(u_{i-1}u_i)$ is defined, let

$$f(u_iu_{i+1}) = \begin{cases} -f(u_{i-1}u_i) & \text{if } \mathrm{C_i} \text{ is a single vertex or an ee lamp.} \\ f(u_{i-1}u_i) & \text{if } \mathrm{C_i} \text{ is a oo lamp.} \\ \dfrac{f(u_{i-1}u_i)}{|f(u_{i-1}u_i)|} * 6 \; - \; f(u_{i-1}u_i) & \text{if } \mathrm{C_i} \text{ is a o lamp.} \end{cases}$$

The labels on $E(H)$ are in the set $\{\pm 2, \pm 4\}$. Second, use the above labels on $E(H)$, we give labels on edges of each cycle C_i of G consecutively as follow.

Case 1: For $i = 1$, as u_1 is replaced by an o lamp C_1 in G, where $C_1 = v_1v_2v_3v_1$ and $\{v_1, u_2\} \in E(G)$. Set $f(v_1u_2) = f(u_1u_2)$. If $f(v_1u_2) = a$, we set $f(v_1v_2) = \frac{-a}{2}$, $f(v_2v_3) = \frac{a}{2}$ and $f(v_3v_1) = \frac{-a}{2}$.

Case 2: For $1 < i <$ n, if u_{i-1} is replaced by an cycle C_{i-1}, let u'_{i-1} be the vertex such that $\{u'_{i-1}, u_i\} \in E(G)$. Otherwise, $u'_{i-1} = u_{i-1}$. As u_i is replaced by an o lamp C_i, where C_i is divided into two paths $P_1 = v_1v_3$ and $P_2 = v_1v_2v_3$ such that $\{u'_{i-1}, v_1\}$ and $\{v_3, u_{i+1}\} \in E(G)$. Set $f(u'_{i-1}v_1) = f(u_{i-1}u_i)$ and $f(v_3u_{i+1}) =$

$f(u_iu_{i+1})$. If $f(u'_{i-1}v_1) = a$ and $f(v_3u_{i+1}) = 2a$, we set $f(v_1v_3) = \frac{-3a}{2}$, $f(v_1v_2) = \frac{a}{2}$ and $f(v_2v_3) = \frac{-a}{2}$. If $f(u'_{i-1}v_1) = 2a$ and $f(v_3u_{i+1}) = a$, we set $f(v_1v_3) = \frac{-3a}{2}$, $f(v_1v_2) = \frac{-a}{2}$ and $f(v_2v_3) = \frac{a}{2}$.

Case 3: For $1 < i <$ n, if u_{i-1} is replaced by an cycle C_{i-1}, let u'_{i-1} be the vertex such that $\{u'_{i-1}, u_i\} \in E(G)$. Otherwise, $u'_{i-1} = u_{i-1}$. As u_i is replaced by an oo lamp C_i, where C_i is divided into two paths $P_1 = v_1v_4$ and $P_2 = v_1v_2v_3v_4$ such that $\{u'_{i-1}, v_1\}$ and $\{v_4, u_{i+1}\} \in E(G)$. Set $f(u'_{i-1}v_1) = f(u_{i-1}u_i)$ and $f(v_4u_{i+1}) = f(u_iu_{i+1})$. If $f(u'_{i-1}v_1) = a$ and $f(v_4u_{i+1}) = a$, we set $f(v_1v_4) = \frac{-a}{2}$, $f(v_1v_2) = \frac{-a}{2}$, $f(v_2v_3) = \frac{a}{2}$ and $f(v_3v_4) = \frac{-a}{2}$.

Case 4: For $1 < i <$ n, if u_{i-1} is replaced by an cycle C_{i-1}, let u'_{i-1} be the vertex such that $\{u'_{i-1}, u_i\} \in E(G)$. Otherwise, $u'_{i-1} = u_{i-1}$. As u_i is replaced by an ee lamp C_i, where C_i is divided into two paths $P_1 = v_1v_2v_4$ and $P_2 = v_1v_3v_4$ such that $\{u'_{i-1}, v_1\}$ and $\{v_4, u_{i+1}\} \in E(G)$. Set $f(u'_{i-1}v_1) = f(u_{i-1}u_i)$ and $f(v_4u_{i+1}) = f(u_iu_{i+1})$. If $f(u'_{i-1}v_1) = a$ and $f(v_4u_{i+1}) = -a$, we set $f(v_1v_2) = \frac{-a}{2}$, $f(v_2v_4) = \frac{a}{2}$, $f(v_1v_3) = \frac{-a}{2}$ and $f(v_3v_4) = \frac{a}{2}$.

Case 5: For $i =$ n, if u_{n-1} is replaced by an cycle C_{n-1}, let u'_{n-1} be the vertex such that $\{u'_{n-1}, u_n\} \in E(G)$. Otherwise, $u'_{n-1} = u_{n-1}$. As u_n is replaced by an o lamp C_n, where C_n is a cycle $C = v_1v_2v_3v_1$ such that $\{u'_{n-1}, v_1\} \in E(G)$. Set $f(u'_{n-1}v_1) = f(u_{n-1}u_n)$. If $f(u'_{n-1}v_1) = a$, we set $f(v_1v_2) = \frac{-a}{2}$, $f(v_2v_3) = \frac{a}{2}$ and $f(v_3v_1) = \frac{-a}{2}$.

Since $a \in \{\pm 2\ \pm 4\}$, we find an assignment f on $E(G)$ such that $f(e) \in \{\pm 1, \pm 2, \pm 3, \pm 4\}$ and the zero-sum rule holds on any vertex $v \in V(G)$. By the property of minor, $F(G) \leq 5$. Moreover, for any labeling for G, with $F(G) \leq 5$, assume there is an edge $e \in E(H)$ such that $f(e) \in \{\pm 1, \pm 3\}$. Since there is no vertex with degree 3 whose edges has the labels with three odd numbers or with one odd number and two even numbers, it will lead to $f(u_{n-1}u_n)$ is an odd number. Then, it is impossible make the zero-sum rule hold on all vertices of G since u_n is replaced by an odd cycle in G. Hence, for any labeling for G, with $F(G) \leq 5$, $f(E(H)) \subseteq \{\pm 2, \pm 4\}$. □

Corollary 3.9. *If G is a christmas lamp, then $F(G) = 3$ if and only if G has no internal odd lamp.*

Proof. ($\Leftarrow$) By the theorem 3.8, we can find an assignment $f(e) \in \{\pm 1, \pm 2\}$ on $E(G)$ such that the zero-sum rule holds on any vertex $v \in V(G)$. And we know $F(G) > 2$ since G is not an even cycle. That is $F(G) = 3$.

($\Rightarrow$) Note that the complete family Ω of vertex-disjoint 323-paths in G is uniquely determined. Indeed, it is obtained by deleting all the edges in lamps from G. If G has an internal odd lamp, then the two end vertices divide the odd lamp into two paths of orders in different parity. It is impossible to satisfy the condition (ii) of theorem 2.12, which means $F(G) \neq 3$. Hence, G has no internal odd lamp. $\square$

4 Zero-sum flow number for tree lamp

Definition 4.1. A graph G is a **tree lamp** if G is obtained from a tree of order at least 2 by replacing its leaves by odd cycles and some internal vertices by cycles. Moreover, we call C is a lamp of G if C is a subgraph of G and C is a cycle.

Example 4.2. Let T_4 be a tree of order 4. Then G is a tree lamp obtained from T_4 as shown in Figure 12.

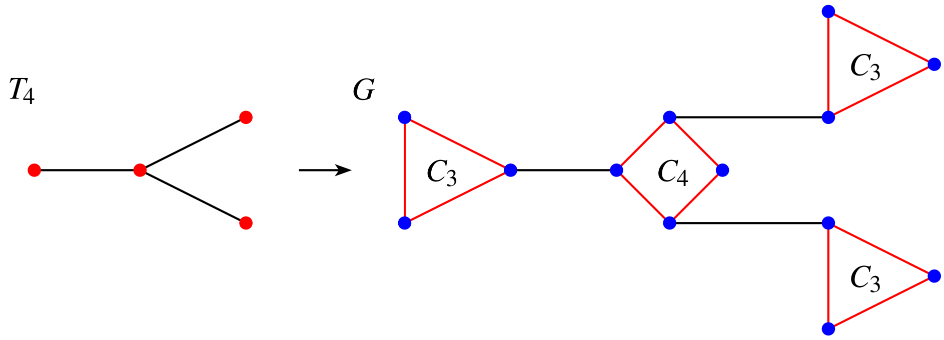

Figure 12: G is a tree lamp obtained from a tree T_4 of order 4.

Let G be a tree lamp, from the theorem C, the zero-sum flow number $F(G) < \infty$. The Christmas lamp is a special case of tree lamp. Similarly, we provide an upper bound of zero-sum flow number of tree lamp.

Theorem 4.3. *If G is a tree lamp, then $F(G) \leq 5$.*

Proof. Suppose G is a tree lamp obtained from a tree T_n of order $n \geq 2$. We shall prove $F(G) \leq 5$ and the corresponding labels in $E(T_n)$ are in the set $\{\pm 2, \pm 4\}$ by induction on n. When $n = 2$, G is a Christmas lamp and this is a special case of theorem 3.8. Since any tree with order bigger than two has one vertex v with neighbors $u_1, u_2, u_3, \cdots, u_s$ such that $s \geq 2$ and degree$(u_i) = 1$ for $2 \leq i \leq s$. Pick the leaf u_2 in T_n, where $n \geq 3$. Let $\tilde{G}$ be the graph obtained from G by deleting the lamp C_{u_2} based on u_2 and the edge $\{v', u_2'\}$ where $u_2' \in V(C_{u_2})$ and $v' \in V(G) \setminus V(C_{u_2})$. Note that $\tilde{G}$ is a tree lamp based on $T_{n-1} := T_n - u_2$. By the induction, $F(\tilde{G}) \leq 5$ and the corresponding labels in $E(T_{n-1})$ are in the set $\{\pm 2, \pm 4\}$. There are two cases of v in G as follows.

Case 1: v is not replaced by a cycle in G. If u_k is replaced by a cycle C_{u_k} in G for $1 \leq k \leq s$, let u_k' be the vertex such that $u_k' \in V(C_{u_k})$ and $\{v, u_k'\} \in E(G)$. Otherwise, $u_k' = u_k$. If $s = 2$, there is a vertex u_1' such that $\{v, u_1'\} \in E(\tilde{G})$ and $f(vu_1') \in \{\pm 2, \pm 4\}$. We give the label on edge $\{v, u_2'\}$ by setting $f(vu_2') = -f(vu_1')$. When u_2 is replaced by an odd cycle C_{u_2} in G, since $|f(vu_2')|$ equal 2 or 4, we can give the labels on $E(C_{u_2})$ such that $f(E(G)) \subseteq \{\pm 1, \pm 2, \pm 3, \pm 4\}$ and the zero-sum rule holds on all vertices of G. If $s \geq 3$, since u_3 is replaced by an odd cycle C_{u_3} in $\tilde{G}$, there is a vertex $u_3' \in V(C_{u_3})$ such that $\{v, u_3'\} \in E(\tilde{G})$ and $f(vu_3') \in \{\pm 2, \pm 4\}$. If $|f(vu_3')| = 4$, we give the label on edge $\{v, u_2'\}$ by setting $f(vu_2') = \frac{f(vu_3')}{2}$ and we give the new label on edge $\{v, u_3'\}$ by setting $f(vu_3') = \frac{f(vu_3')}{2}$. If $|f(vu_3')| = 2$, we give the label on edge $\{v, u_2'\}$ by setting $f(vu_2') = -f(vu_3')$ and we give the new label on edge $\{v, u_3'\}$ by setting $f(vu_3') = 2 \times f(vu_3')$. When u_2 is replaced by an odd cycle C_{u_2} in G, since $|f(vu_2')|$ and $|f(vu_3')|$ equal 2 or 4, we can give the labels on $E(C_{u_2})$ and new labels on $E(C_{u_3})$ such that $f(E(G)) \subseteq \{\pm 1, \pm 2, \pm 3, \pm 4\}$ and the zero-sum rule holds on all vertices of G. Since $f(e) \in \{\pm 2, \pm 4\}$ if $e \in \{vu_k' | 1 \leq k \leq s\}$, we obtain the corresponding labels in $E(T_n)$ are in the set $\{\pm 2, \pm 4\}$ and $F(G) \leq 5$.

Case 2: v is replaced by a cycle C in G, where $V(C) = \{v_1, v_2, \cdots, v_t\}$ and $t \geq s$. There exist $1 \leq i_1 < i_2 < \cdots < i_s \leq t$ such that $E(G)$ contains $\{v_{i_k} u_k' | 1 \leq k \leq s\}$. If u_k is replaced by a cycle C_{u_k} in G for $1 \leq k \leq s$, let u_k' be the vertex such that $u_k' \in V(C_{u_k})$ and $\{v_{i_k}, u_k'\} \in E(G)$. Otherwise, $u_k' = u_k$. First, since the edge $\{v_{i_1} u_1'\} \in E(\tilde{G})$ and $f(v_{i_1} u_1') \in \{\pm 2, \pm 4\}$, we give the new labels on edges $\{v_{i_1}, v_{i_1+1}\}$, $\{v_{i_1+1}, v_{i_1+2}\}$, $\cdots$, $\{v_{t-1}, v_t\}$, $\{v_t, v_1\}$, $\{v_1, v_2\}$, $\cdots$, $\{v_{i_1-1}, v_{i_1}\}$ consecutively by setting $f(v_{i_1} v_{i_1+1}) = \frac{-f(v_{i_1} u_1')}{|f(v_{i_1} u_1')|}$ and when $f(v_{i-1} v_i)$ is changed, let $f(v_i v_{i+1}) = f(v_{i-1} v_i)$ if i $\in i_1, i_2, \cdots, i_s$ or $f(v_i v_{i+1}) = -f(v_{i-1} v_i)$ if i $\notin i_1, i_2, \cdots, i_s$. Note that $v_{t+1} = v_1$ and $v_0 = v_t$. Second, if $f(v_{i_1} u_1') \neq f(v_{i_1} v_{i_1+1}) + f(v_{i_1-1} v_{i_1})$ and $|f(v_{i_1} u_1')| = 2$. We give the new labels on edges $\{v_i, v_{i+1} \mid i_1 \leq i < i_s\}$ consecutively by setting $f(v_{i_1} v_{i_1+1}) = \frac{f(v_{i_1} u_1')}{|f(v_{i_1} u_1')|}$ and when $f(v_{i-1} v_i)$ is changed, let $f(v_i v_{i+1}) = f(v_{i-1} v_i)$ if i $\in i_1, i_2, \cdots, i_s$ or $f(v_i v_{i+1}) = -f(v_{i-1} v_i)$ if i $\notin i_1, i_2, \cdots, i_s$. And, we give the new labels on edges $\{v_{i_1}, v_{i_1-1}\}$, $\{v_{i_1-1}, v_{i_1-2}\}$, $\cdots$, $\{v_{i_s+1}, v_{i_s}\}$ consecutively by setting $f(v_{i_1} v_{i_1-1}) = -f(v_{i_1} u_1') - f(v_{i_1} v_{i_1+1})$ and when $f(v_{i+1} v_i)$ is changed, let $f(v_i v_{i-1}) = -f(v_{i+1} v_i)$ for i $\in \{i_1 - 1, i_1 - 2, \cdots, i_s + 1\}$. If $f(v_{i_1} u_1') \neq f(v_{i_1} v_{i_1+1}) + f(v_{i_1-1} v_{i_1})$ and $|f(v_{i_1} u_1')| = 4$. We give the new labels on edges $\{v_{i_1}, v_{i_1-1}\}$, $\{v_{i_1-1}, v_{i_1-2}\}$, $\cdots$, $\{v_{i_s+1}, v_{i_s}\}$ consecutively by setting $f(v_{i_1} v_{i_1-1}) = -f(v_{i_1} u_1') - f(v_{i_1} v_{i_1+1})$ and when $f(v_{i+1} v_i)$ is changed, let $f(v_i v_{i-1}) = -f(v_{i+1} v_i)$ for i $\in \{i_1 - 1, i_1 - 2, \cdots, i_s + 1\}$. Third, we give the new labels on edges $\{v_{i_k} u_k' | 3 \leq k \leq s\}$ with $f(v_{i_k} u_k') = -f(v_{i_k-1} v_{i_k}) - f(v_{i_k} v_{i_k+1})$ and we give the label on edge $\{v_{i_2} u_2'\}$ with $f(v_{i_2} u_2') = -f(v_{i_2-1} v_{i_2}) - f(v_{i_2} v_{i_2+1})$. Since $f(e) \in \{\pm 1, \pm 3\}$ if $e \in E(C)$, $f(v_{i_k} u_k') \in \{\pm 2, \pm 4\}$ for $2 \leq k \leq s$. When u_2 is replaced by an odd cycle C_{u_2} in G, we can give the labels on $E(C_{u_2})$ and new labels on $E(C_{u_k})$ for $2 \leq k \leq s$ such that $f(E(G)) \subseteq \{\pm 1, \pm 2, \pm 3, \pm 4\}$ and the zero-sum rule holds on all vertices of G. Finally, we obtain the corresponding labels in $E(T_n)$ are in the set $\{\pm 2, \pm 4\}$ and $F(G) \leq 5$.

By the induction on n, the corresponding labels in $E(T_n)$ are in the set $\{\pm 2, \pm 4\}$

and $F(G) \leq 5$. That is mean if G is a tree lamp, then $F(G) \leq 5$. □

Definition 4.4. Let G and H be two graphs. Define the sum graph $G+H$ of G and H with vertex set $V(G+H) = V(G) \cup V(H)$ and edge set $E(G+H) = E(G) \cup E(H)$.

The reason why we are interested in the tree lamp is that some $(2,3)$-graphs G with finite $F(G)$ are the sum graph of some edge disjoint tree lamps and even cycles. For instance, the following $(2,3)$-graph G, shown in Figure 13, with 9 vertices and zero-sum flow number 6 is the sum graph $H_1 + H_2$ where H_1 is an even cycle and H_2 is a tree lamp. In addition, a $(2,3)$-graph with infinite $F(G)$ can not be the sum graph of some edge disjoint tree lamps and even cycles, the graph shown in Figure 4 is an example.

The graph G shown in Figure 13 was discovered in [1] through an exhaustive search.

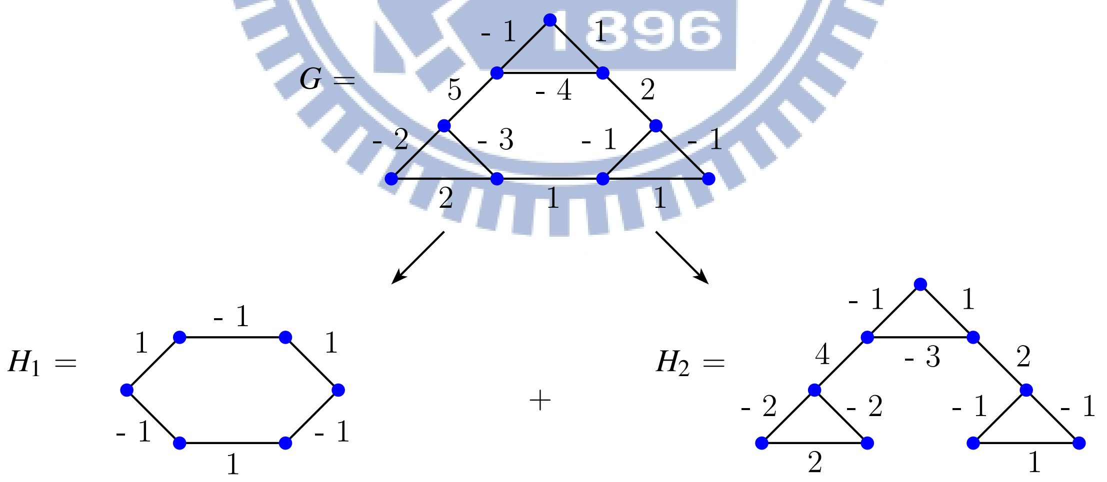

Figure 13: G has zero-sum 6-flow and G is the sum graph $H_1 + H_2$ of H_1 and H_2, where H_1 is an even cycle and H_2 is a tree lamp.

5 Summary

A zero-sum k-flow on a graph G is a zero-sum flow with numbers from the set $\{\pm1, ..., \pm(\mathrm{k}-1)\}$. The zero-sum flow number $F(G)$ of G is the least number k for which G may admit a zero-sum k-flow. In the Section 2, we give a necessary and sufficient condition for a $(2,3)$-graph to have zero-sum flow number 3, so that in some special cases to determine if a $(2,3)$-graph G has $F(G) = 3$ becomes easier. Furthermore, in the Sections 3 and 4, we study the zero-sum flow number of Christmas lamps and tree lamps, which are graphs expanded from paths and trees. At the end of Sections 3 and 4, we conclude that Christmas lamps and tree lamps respectively has zero-sum flow numbers at most 5.

We list some open problems for further study:

1. Give a necessary and sufficient condition for a $(2,3)$-graph to have zero-sum flow number 4.

2. Any $(2,3)$-graph G with finite $F(G)$ is the sum graph of some edge disjoint tree lamps and even cycles.

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