圖的度數對之研究

學生：黃苓芸　　　　　　　　　　　　指導教授：翁志文

國 立 交 通 大 學

應 用 數 學 系

摘要

簡單圖 G 上一點 v 的**平均二度數**定義為與 v 相鄰之點的度數平均。度數列和平均二度數列在最大拉普拉斯特徵值上界的應用，已有許多研究成果。若 G 中所有點的平均二度數皆為 k ，則 G 稱為**擬 k 正則圖**。在此論文中，我們證明若 G 為擬 k 正則圖，則 k 是整數；進而找出所有擬正則樹。我們也考慮了當 G 的最大度數為 $k^2 - k$ 的情形，並給出一些基本的結果。最後，我們對於擬 3 正則圖給出了更多的結果。並且刻畫出所有十個點之內非正則的擬 3 正則圖。

關鍵字：圖，鄰接矩陣，拉普拉斯矩陣，度數，平均二度數，擬 k 正則。

The Degree Pairs of a Graph

Student: Ling-Yun Huang   Advisor: Chih-Wen Weng

Department of Applied Mathematics

National Chiao Tung University

Abstract

Let v be a vertex in a simple graph G. The **average 2-degree** of v is the average of degrees of vertices adjacent to v. The applications of the degree and average 2-degree sequences on the upper bounds for the maximum eigenvalue of Laplacian matrix of a graph is studied by many authors. The graph G is called **pseudo k-regular** if each vertex in G has average 2-degree k. We prove that if G is pseudo k-regular then k is integral. Moreover, all pseudo regular trees are given in this thesis. We also consider the case when the maximum degree of G is $k^2 - k$, and give some basic results. In the end, we give more results of pseudo 3-regular graphs. And characterize all the pseudo 3-regular graph within ten vertices but not regular.

Keywords: Graph, adjacency matrix, Laplacian matrix, degree, average 2-degree, pseudo k-regular.

Table of Contents

List of Figures

Chapter 1

Introduction

Let G be a graph with vertex set $VG = \{1, 2, \ldots, n\}$ and edge set EG. Let d_i be the **degree** of the vertex $i \in VG$, defined as follows:

$$d_i := |G_1(i)|,$$

where $G_1(i)$ means the set $\{j \in VG \mid ji \in EG\}$ of neighbors of i.

The sequence $\{d_i\}_{i \in VG}$ of G is called a **degree sequence** of G. There is a multitude of equivalent conditions for determining when a given sequence of integers is a degree sequence. Havel [11] in 1955 and Hakimi [9] in 1962 independently obtained recursive conditions for a sequence to be a degree sequence of a graph if and only if the subsequence with its largest element deleted is also a sequence of a graph. In 1973, Wang and Kleitman [19] proved the necessary and sufficient conditions for arbitrary deleting. There are seven criteria for a sequence to be a degree sequence of a graph, which are proposed by Ryser [17] in 1957, Berge [1] in 1973, Fulkerson, Hoffman, and McAndrew [6] in 1965, Bollobàs [2] in 1978, Grünbaum [7] in 1969, Hässelbarth [10] in 1984, and Erdös and Gallai [5] in 1960. And in 1991, Sierksma and Hoogerveen [18] proved that the above seven criteria are equivalent.

Let m_i be the **average** 2**-degree** of the vertex $i \in VG$, defined as follows.

$$m_i := \frac{1}{d_i} \sum_{ji \in EG} d_j.$$

And the sequence $\{m_i\}_{i \in VG}$ of G is called a **average** 2**-degree sequence** of G. We shall give a survey of average 2-degree sequence of a graph.

Let G be a simple graph. The **adjacency matrix** of G is the 0-1 matrix A indexed by VG such that $A_{xy} = 1$ if and only if $xy \in EG$. The **degree matrix** of G is the diagonal matrix D indexed by VG such that D_{xx} is the degree d_x of $x \in VG$. The average 2-degree sequence appears often in the study of maximum eigenvalue $\ell_1(G)$ of the **Laplacian matrix** $L = D - A$ associated with G, where D is the degree matrix and A is the adjacency matrix of G. The following results are about the upper bounds of $\ell_1(G)$:

1. In 1998, Merris gave the following bound [15] :

$$\ell_1(G) \leq \max_{i \in VG} \{d_i + m_i\}.$$

2. Also in 1998, Li and Zhang gave the following bound [14]:

$$\ell_1(G) \leq \max_{ij \in EG} \left\{ \frac{d_i(d_i + m_i) + d_j(d_j + m_j)}{d_i + d_j} \right\}.$$

3. In 2001, Li and Pan gave the following bound [13]:

$$\ell_1(G) \leq \max_{i \in VG} \left\{ \sqrt{2d_i(d_i + m_i)} \right\}.$$

4. In 2004, Das gave the following bound [4]:

$$\ell_1(G) \leq \max_{ij \in EG} \left\{ \frac{d_i + d_j + \sqrt{(d_i - d_j)^2 + 4m_i m_j}}{2} \right\}.$$

5. Also in 2004, Zhang gave the following bounds [21]:

(a)

$$\ell_1(G) \leq \max_{ij \in EG} \left\{ 2 + \sqrt{d_i(d_i + m_i - 4) + d_j(d_j + m_j - 4) + 4} \right\}.$$

(b)

$$\ell_1(G) \leq \max_{i \in VG} \left\{ d_i + \sqrt{d_i m_i} \right\}.$$

(c)

$$\ell_1(G) \leq \max_{ij \in EG} \left\{ \sqrt{d_i(d_i + m_i) + d_j(d_j + m_j)} \right\}.$$

As everyone knows, a graph G is **k-regular** if $d_i = k$ for all vertices $i \in VG$. If $m_i = k$ for all vertices $i \in VG$, G is called **pseudo k-regular** in [20]. For convenience, we rearrange the vertices of G by $1, 2, \cdots, n$ such that $m_1 \geq m_2 \geq \cdots \geq m_n$. Let $a_1(G)$ be the maximum eigenvalue of adjacency matrix A associated with G, and we have following.

Let $B = D^{-1}AD$, where D is the degree matrix and A is the adjacency matrix of G. Then B is a nonnegative irreducible $n \times n$ matrix. By Perron-Frobenius Theorem in [16], we have $a_1(G) \leq m_1$ with equality if and only if G is a pseudo k-regular graph.

In 2011, Chen, Pan and Zhang [3] proved the following.

Theorem 1.1. *Let* $a := \max\{d_i/d_j \mid 1 \leq i, j \leq n\}$. *Then*

$$a_1(G) \leq \frac{m_2 - a + \sqrt{(m_2 + a)^2 + 4a(m_1 - m_2)}}{2}$$

with equality if and only if G is a pseudo k-regular graph.

And in 2014, Huang and Weng [12] proved the following.

Theorem 1.2. *For any* $b \geq \max\{d_i/d_j \mid ij \in EG\}$ *and* $1 \leq l \leq n$,

$$a_1(G) \leq \frac{m_l - b + \sqrt{(m_l + b)^2 + 4b\sum_{i=1}^{l-1}(m_i - m_l)}}{2}$$

with equality if and only if G *is a pseudo* k*-regular graph.*

This thesis studies degree sequence together with average 2-degree sequence of a graph. Thus we define the sequence $\{(d_i, m_i)\}_{i \in VG}$ of pairs as a **degree pairs**.

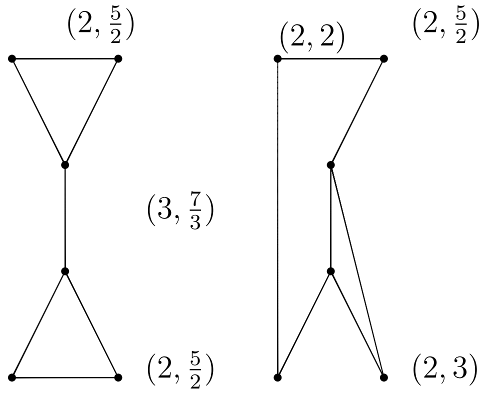


Figure 1.1: Two graphs with different sequences of degree pairs (d_i, m_i).

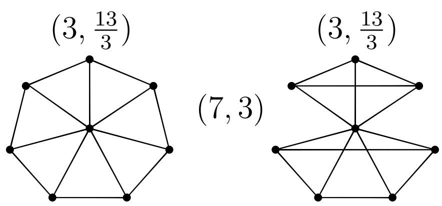


Figure 1.2: Two graphs with the same sequence of degree pairs (d_i, m_i).

This thesis is organized as follows. In Chapter 2, we introduce some basic results about degree pairs. In Chapter 3, we prove that if G is pseudo k-regular then $k \in \mathbb{N}$, and give a family of pseudo k-regular graphs T_k. Furthermore, we prove that T_k is the only pseudo k-regular tree for each k. We also consider the case when the maximum degree of G is $k^2 - k$, and give some basic results. In the end, we give more results of pseudo 3-regular graphs. And characterize all the pseudo 3-regular graph within ten vertices but not regular.

Chapter 2

Degree pairs

Let G be a simple graph with vertex set $VG = \{1, 2, \ldots, n\}$, edge set EG, and sequence $\{(d_i, m_i)\}_{i \in VG}$ degree pairs. The following lemma provides a feasible condition of degree pairs.

Lemma 2.1.

$$\sum_{i \in VG} d_i m_i = \sum_{i \in VG} d_i^2.$$

Proof.

$$\sum_{i \in VG} d_i m_i = \sum_{i \in VG} d_i \frac{\sum_{ji \in EG} d_j}{d_i} = \sum_{i \in VG} \sum_{ij \in EG} d_i = \sum_{i \in VG} d_i^2.$$

□

We give a sequence $A = \{(1, 3), (1, 3), (2, 3), (3, 2), (3, 2)\}$, and a sequence $B = \{(1, 4), (3, 2), (3, 3), (3, 3), (4, 2)\}$. Observe that sequence A matches the condition in Lemma 2.1, and is a sequence of degree pairs of the graph as shown in Figure 2.1. But sequence B does not match the condition in Lemma 2.1, so its not a sequence of degree pairs of any graph.

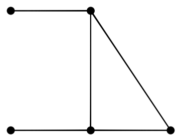


Figure 2.1: A graph with the given sequence A.

Here is another feasible condition for degree pairs.

Lemma 2.2. *There are even number of odd values $d_i m_i$ among $i \in VG$.*

Proof. Since $\sum_{i\in VG} d_i$ is even, there are even number of odd d_i, and so does d_i^2. Hence $\sum_{i\in VG} d_i m_i = \sum_{i\in VG} d_i^2$ is even. □

Corollary 2.3.

$$\sum_{i\in VG} m_i^2 \geq \sum_{i\in VG} d_i^2$$

with equality if and only if $m_i = d_i = k$ for all i.

Proof.

$$(\sum_{i\in VG} d_i^2)(\sum_{i\in VG} m_i^2) \geq (\sum_{i\in VG} d_i m_i)^2 = (\sum_{i\in VG} d_i^2)^2$$

and equality if and only if $m_i = cd_i$ for all $i \in VG$, where $c = 1$ by the Lemma 2.1. This is also equivalent to that all neighbors of a vertex of minimum degree k also have degree k. □

Degree sequence gives hints of graph properties. For example, the well-known fact $|EG| = \frac{1}{2}\sum_{i\in VG} d_i$ expresseds the number of edges of a graph as a sum its degree sequence.

The sequence of degree pairs give more hints of graph structure. In general, $d_i m_i \geq |G_1(i)| + |G_2(i)|$, and there are at least $(d_i m_i - n)/2$ triangles based on the vertex i.

Proposition 2.4. *If* $\max_{i\in VG} d_i m_i \geq n$ *then the graph has girth at most* 4.

Proof. If the graph has girth at least 5 then

$$n-1 = |VG|-1 \geq |G_1(i)+G_2(i)| = d_i m_i.$$

for any $i \in VG$. □

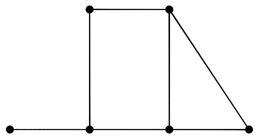


Figure 2.2: A graph has girth at most 4.

In Figure 2.2, we observe that $\max_{i\in VG} d_i m_i = 8 \geq 6 = |VG|$

The distance $d(x,y)$ between two vertices x and y of a graph is the minimum length of the paths connecting them. Let G^2 be **the square of** G, denote the graph with $VG^2 = VG$ and $EG^2 = \{xy \mid d(x,y) \leq 2\}$. The **independence number** of G is $\alpha(G) = \max\{|S| \mid S \subseteq VG$, S is the independent set of $G\}$.

Proposition 2.5.

$$\alpha(G^2) \geq \sum_{i\in VG} \frac{1}{1+d_i m_i},$$

where $\alpha(G^2)$ *is the independence number of the square of* G.

Proof. If a vertex is picked equally in random then the probability of a vertex i appears before those vertices in $G_1(i) \cap G_2(i)$ is $(1+|G_i(i)|+|G_2(i)|)^{-1}$. Hence the expected size of a set consisting of these i is $\sum_{i\in VG}(1+|G_i(i)|+|G_2(i)|)^{-1}$, which is at least $\sum_{i\in VG}\frac{1}{1+d_i m_i}$. □

The following lemma will be used later.

Lemma 2.6. *$d_i \leq m_i(m_j - 1) + 1$ for any j with $ji \in EG$ and $d_j \leq m_i$. Moreover the above equality holds if and only if $d_j = m_i$ and all neighbors of j excluding i have degree* 1.

Proof. Pick j such that $ji \in EG$ and $d_j \leq m_i$. Then $d_j m_j \geq d_i + (d_j - 1) \cdot 1$. Hence

$$m_i(m_j - 1) + 1 \geq d_j(m_j - 1) + 1 \geq d_i.$$

□

Chapter 3

Pseudo k-regular graphs

We now turn to the study of pseudo k-regular graphs, i.e. $m_i = k$ for all i. We try to give some theories for pseudo k-regular graphs.

From the definition of pseudo k-regular graphs, $k \in \mathbb{Q}$, but indeed we have the following.

Proposition 3.1. *If G is pseudo k-regular then $k \in \mathbb{N}$.*

Proof. Let A be the adjacency matrix of G, and note that

$$(d_1, d_2, \ldots, d_n)A = k(d_1, d_2, \ldots, d_n).$$

Being a zero of the characteristic polynomial of A, k is an algebraic integer. Since k is also a positive rational number, k is indeed a positive integer. □

Obviously, any k-regular graph is a pseudo k-regular graph. However, a pseudo k-regular graph may not be a regular graph. An interesting problem is to characterize all the non-regular pseudo k-regular graphs. There are some examples in [12] of pseudo k-regular graphs that are not regular in the following Example 3.2.

Example 3.2. The graphs in Figure 3.1, 3.2, and 3.3 are pseudo k-regular but not regular.

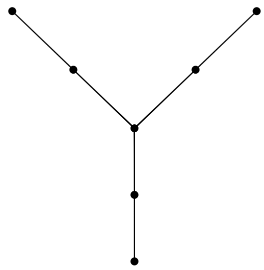


Figure 3.1: A graph with $m_i = 2$.

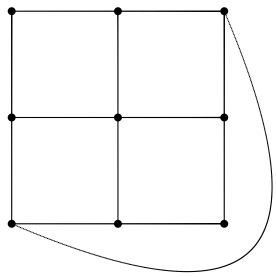


Figure 3.2: A graph with $m_i = 3$.

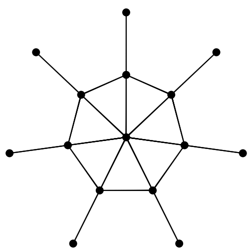


Figure 3.3: A graph with $m_i = 4$.

It is natural to ask when a pseudo k-regular graph attains the maximum number of edges when the order n of a graph is given.

Theorem 3.3. *A pseudo k-regular graph has at most $nk/2$ edges, and the maximum is obtained if and only if the graph is regular.*

Proof. From

$$2k|EG| = \sum_{i \in VG} d_i m_i = \sum_{i \in VG} d_i^2 \geq (\sum_{i \in VG} d_i)^2/n = 4|EG|^2/n,$$

we have $|EG| \leq nk/2$ and equality is obtained if and only if d_i is a constant. □

We shall study the connected pseudo k-regular graphs of order n which attain the minimum number of edges, i.e. pseudo k-regular trees. We also want to study connected pseudo k-regular graphs of order n with maximal degree among such graphs.

Definition 3.4. Let T_k be the tree of order $k^3 - k^2 + k + 1$ whose root has degree $k^2 - k + 1$ and each neighbor of the root has $k - 1$ children as leafs.

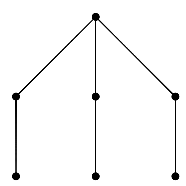


Figure 3.4: The tree T_2.

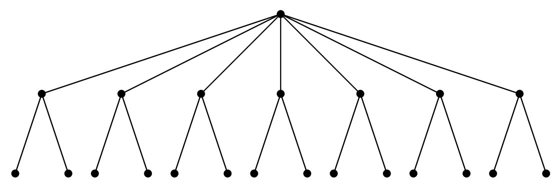


Figure 3.5: The tree T_3.

Note that T_1 is exactly the complete graph K_2. For each $k \geq 2$, T_k exists and provides an example for a non-regular pseudo k-regular graph.

Let $\Delta(G) = \max\{d_i \mid i \in VG\}$ be the maximal degree of G. We have the following result.

Theorem 3.5. *Let G be a connected graph with $m_i \leq k$ for all $i \in VG$ and some $k \in \mathbb{N}$. Then $\Delta(G) \leq k^2 - k + 1$. Moreover the following (i)-(ii) are equivalent.*

(i) $\Delta(G) = k^2 - k + 1$.

(ii) *G is the tree T_k.*

Proof. Choose i such that $d_i = \Delta(G)$. Then by Proposition 2.6, $\Delta(G) = d_i \leq m_i(m_j - 1) + 1 = k^2 - k + 1$ for any j with $ji \in EG$ and $d_j \leq m_i$. Moreover $\Delta(G) = k^2 - k + 1$ if and only if $d_j = m_j = m_i = k$ and $d_z = 1$ for all neighbors $z \neq i$ of j. Hence (i) and (ii) are equivalent. □

We have seen that the degree of a neighbor of maximum degree vertex is k in T_k. We are interested in what other vertices have this property.

Lemma 3.6. *Let G be a pseudo k-regular graph. Then the following (i)-(ii) hold.*

(i) *If z is a vertex of degree 1 then k is the degree of the neighbor of z.*

(ii) *If ij is an edge with $2 \leq d_j < k$ then $2 \leq d_i \leq k^2 - 3k + 4$, with the second equality if and only if all neighbors of j except i have degree 2.*

Proof. (i) is clear. To prove (ii), note that $d_i \neq 1$, otherwise $d_j = k$, a contradiction. Indeed $d_z \neq 1$ for any neighbors z of j. Hence

$$d_i + 2(d_j - 1) \leq d_j m_j = d_j k.$$

Hence

$$d_i \leq d_j(k-2) + 2 \leq k^2 - 3k + 4.$$

□

Corollary 3.7. *Let G be a pseudo k-regular graph of order n with a vertex of degree $d_i \geq k^2 - 3k + 5$. Then*

(i) Any neighbor j of i has degree $d_j = k$;

(ii) The order of G is at least $f(k) := \lceil (5k^4 - 31k^3 + 94k^2 - 140k + 100)/k^2 \rceil$.

Proof. (i) From Lemma 3.6 (i) $d_j \neq 1$, and from Lemma 3.6 (ii) $d_j \geq k$. This is true for all neighbors j of i. Hence $d_j = k$.

(ii) From Lemma 2.1 $\sum_{w \in VG} d_w^2 = \sum_{w \in VG} d_w m_w$,

$$d_i^2 + d_i k^2 + \sum_{w \notin \{i\} \cup G_1(i)} d_w^2 = k d_i + k^2 d_i + \sum_{w \notin \{i\} \cup G_1(i)} k d_w.$$

Hence

$$\begin{aligned} k^4 - 7k^3 + 22k^2 - 35k + 25 &\leq \sum_{w \notin \{i\} \cup G_1(i)} d_w(k - d_w) \\ &\leq \left(\frac{k}{2}\right)^2 (n - 1 - (k^2 - 3k + 5)). \end{aligned}$$

□

Note that for $k = 3$, $k^2 - 3k + 5 = 5$ and $f(3) = 11$.

Now we try to characterize the pseudo k-regular graphs. It is easily seen that a graph is pseudo k-regular if and only if each component of it is pseudo k-regular. Hence we just focus on the characterization of connected pseudo k-regular graphs.

The first two cases of pseudo k-regular graphs are easy to settle.

Lemma 3.8. *If G is connected pseudo 1-regular then G is K_2.* □

Lemma 3.9. *If G is connected pseudo 2-regular then G is a cycle or T_2.*

Proof. Note that $\Delta(G) = 2$ or 3, and the first implies that G is a cycle and the latter implies that $G = T_2$. □

Pseudo k-regular graphs is also called harmonic graphs [8], and finite harmonic tree are already given. But for the complete of this thesis we reprove the Theorem as follow.

Theorem 3.10. *[8, Theorem 2.1] If G is a pseudo k-regular tree, then $G = T_k$.*

Proof. By Lemma 3.8 and Lemma 3.9, the assumption holds for each $k \leq 2$. Let $G = (VG, EG)$ be a pseudo k-regular tree with $k \geq 3$. Pick any $v \in VG$ with $d_v \geq 2$ as a root. Since a star is not pseudo k-regular, there exists a leaf x with parent $y \neq v$, such that all children of y are leaves. Then y has degree k by Lemma 3.6 and has $k - 1$ children as leaves. Hence the degree of root $d_v = km_y - (k - 1) = k^2 - k + 1$. This concludes that $G = T_k$ by Definition 3.4. □

We shall study pseudo k-regular graph with the second largest degree $k^2 - k$.

Definition 3.11. Let U_k be the tree of order $k^3 - k^2 + 1$ whose root has degree $k^2 - k$ and each neighbor of the root has $k - 1$ children as leafs.

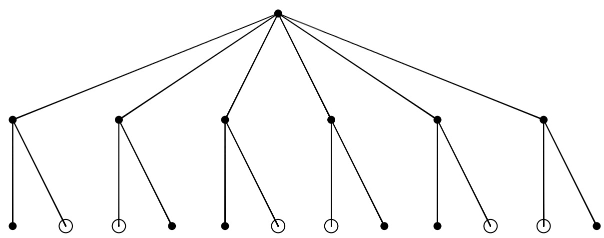


Figure 3.6: The graph U_3 with type A vertices.

We shall select some vertices from a graph and call them **type A** vertices. In general a type A vertex has degree 1 and its unique neighbor j has $d_j = k$ and $m_j = (k^2 - t)/k$, where t is the number of type A neighbors of j (in U_k, $t = 1$).

Let M_k be the graph obtained from U_k by identifying $(k^2 - k)/2$ pairs of type A vertices into $(k^2 - k)/2$ vertices. Then M_k gives a pseudo k-regular graphs with maximum degree $k^2 - k$ for each $k \geq 3$.

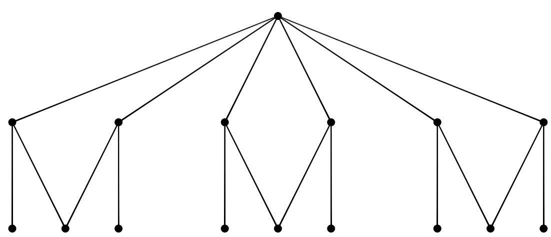


Figure 3.7: The graph M_3.

Proposition 3.12. *If G is a pseudo k-regular graph with a vertex x of degree $k^2 - k$, then the subgraph induced on $\{x\} \cup G_1(x) \cup G_2(x)$ is U_k with possibly even number of vertices in type A being identified in pairs. Moreover a type A vertex not been identified with another one has degree 2 in G.*

Proof. Let y be a neighbor of x. Then y has degree $d_y = k$ by Corollary 3.7(i), and has a neighbor $z \neq x$ of degree $d_z \geq 2$ by Theorem 3.5. Hence $k^2 = d_y m_y \geq d_x + d_z + (d_y - 2) \geq (k^2 - k) + 2 + (k - 2) = k^2$. This implies that $d_z = 2$ and the remaining vertices $w \notin \{x, z\}$ of y have degree $d_w = 1$. Note that z, w have distance two to x. As one neighbor of z has degree k, the other neighbor of z also has degree k. Hence the vertex z might adjacent to some neighbor of x or to some vertex of degree k and at distance 3 to x. $\square$

Let $\mathcal{E}_k$ be a family of graphs constructed as the following. Firstly pick a bipartite $(k-1)$-regular graph of order $2(2k-1)$ with bipartition $X \cup Y$, where $|X| = |Y| = 2k - 1$. Then add a new vertex connecting to all vertices of X. One can check that graphs in $\mathcal{E}_k$ are pseudo k-regular of order $4k - 1$ with maximum degree $2k - 1$.

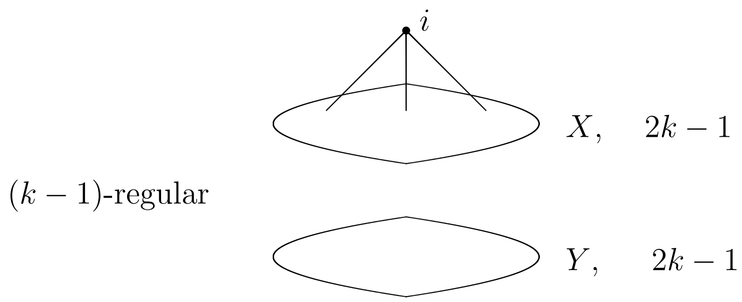


Figure 3.8: The graphs in $\mathcal{E}_k$.

By a **switching** on G, we mean a process to obtain a new graph G' by removing two edges xy and uv such that $d_x = d_u$ and $d_y = d_v$ and adding two new edges xv and yu to form a new graph, where xv and yu are not edges in G. In this case G and G' are called **switching equivalent**.

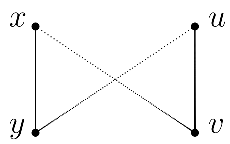


Figure 3.9: Switching.

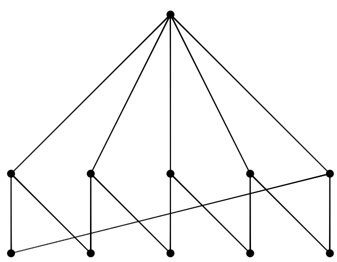


Figure 3.10: The graph $E_3 \in \mathcal{E}_3$.

Every graph in $\mathcal{E}_3$ is switching equivalent to E_3.

From Corollary 3.7 (ii), we know a pseudo 3-regular graph with maximum degree at least 5 has at least $f(3) = 11$ vertices. All the graphs in $\mathcal{E}_k$ are extremal for this property.

Let $\mathcal{F}_k$ be a family of graphs constructed as the following. Firstly pick any $(k-2)$-regular graph H of order $(2k-1)(k-1)$, not necessary connected.

Secondly add $(2k-1)(k-1)$ new vertices of degree 1 by connecting them to vertices of H one by one. Finally partition the vertex set of H into $k-1$ blocks of equal size $2k-1$ and connect all vertices in a block to a new vertex to make it degree $2k-1$. One can check that graphs in $\mathcal{F}_k$ are pseudo k-regular with maximum degree $2k-1$.

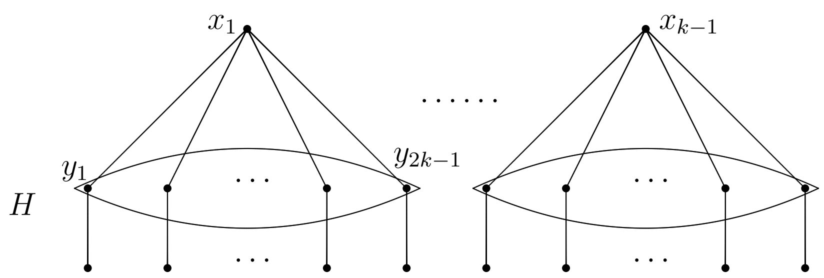


Figure 3.11: The graphs in $\mathcal{F}_3$.

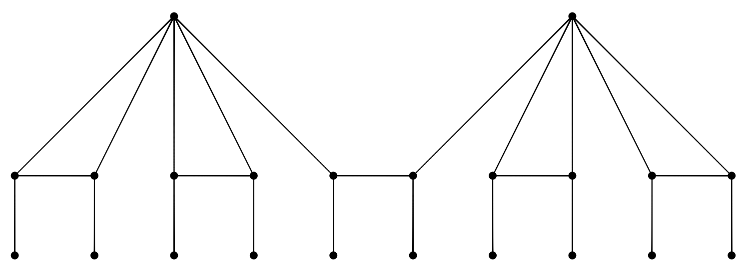


Figure 3.12: The graph $F_3 \in \mathcal{F}_3$.

Every graph in $\mathcal{F}_3$ is switching equivalent to F_3.

Now we restrict our attention to pseudo 3-regular graph G.

Note that the maximum degree $3 \leq \Delta(G) \leq k^2 - k + 1 = 7$ and the case $\Delta(G) = 7$ is solved by Theorem 3.5 and Theorem 3.10.

The local structure of a maximum degree $\Delta(G) = 6$ is obtained in Proposition 3.12 for $k = 3$.

The following lemma is immediate from Corollary 3.7.

Lemma 3.13. *Let G be a pseudo 3-regular graph with a vertex i of degree $d_i = 5$. Then all neighbors j of i have degree $d_j = 3$, and the neighbors of j have degree sequence $(5, 2, 2)$ or $(5, 3, 1)$.* □

Proposition 3.14. *If G is a pseudo 3-regular graph with a vertex i of degree 5, then the subgraph induced on $G_1(i)$ is union of disjoint edges or isolated vertices, and each endpoint of an edge is adjacent to a vertex of degree 1 in $G_2(i)$ and each isolated vertex is adjacent to two vertices in $G_2(i)$ with degrees $(3, 1)$ or $(2, 2)$.* □

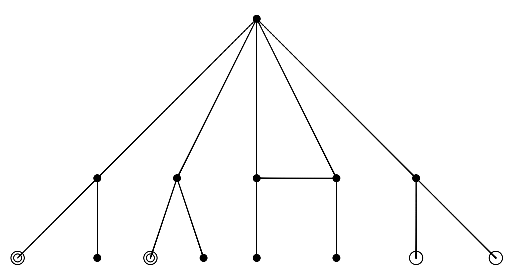


Figure 3.13: Graphs with $\Delta(G) = 5$.

Now we study the local structure of a vertex of degree 4 in a pseudo k-regular graph.

Lemma 3.15. *Let G be a pseudo 3-regular graph. Then the neighbor degree sequence of a vertex of degree 4 is* $(3, 3, 3, 3)$, $(4, 3, 3, 2)$, *or* $(4, 4, 2, 2)$.

Proof. Let (a, b, c, d) be a degree sequence of the neighbors of a vertex i of degree $d_i = 4$, where $a \geq b \geq c \geq d$. Note that $a \leq 4$ otherwise $d_i = 3$ by Corollary 3.7 (i). Then $a+b+c+d = d_i \cdot 3 = 12$. By checking all possible such sequences (a, b, c, d), we find these are as listed in the lemma or $(4, 4, 3, 1)$, which is impossible since the neighbor of a leaf must have degree 3. □

Proposition 3.16. *If G is a pseudo 3-regular graph with a vertex i of degree 4 and the neighbor degree sequence of i is* $(3, 3, 3, 3)$, *then the subgraph induced on $G_1(i)$ is union of disjoint edges or isolated vertices, and each endpoint of an edge is adjacent to a vertex of degree 2 in $G_2(i)$ (possibly identified in pairs) and each isolated vertex is adjacent to two vertices in $G_2(i)$ with degrees* $2, 3$ *or degrees* $1, 4$. □

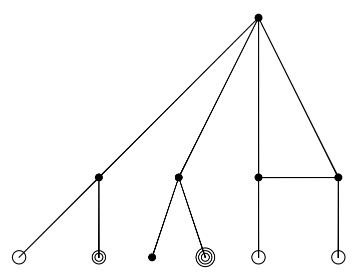


Figure 3.14: Graphs with $\Delta(G) = 4$ and the neighbor degree sequence of a vertex of degree 4 is $(3, 3, 3, 3)$.

In Figure 3.14 we have $1 + |G_1(i)| + |G_2(i)| \geq 7$.

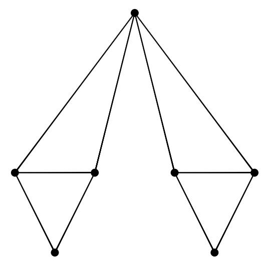


Figure 3.15: The graph has $\Delta(G) = 4$ with degree sequence $(3, 3, 3, 3)$.

Proposition 3.17. *If G is a pseudo 3-regular graph with a vertex i of degree 4 and the neighbor degree sequence of i is $(4, 3, 3, 2)$, then the neighbor of i with degree 2 in G is isolated in $G_1(i)$, and the neighbor of i with degree 3 in G has at most one neighbor in $G_1(i)$.* □

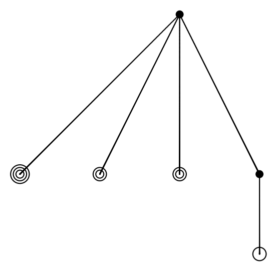


Figure 3.16: Graphs with $\Delta(G) = 4$ and the neighbor degree sequence of a vertex of degree 4 is $(4, 3, 3, 2)$.

In Figure 3.16 we have $1 + |G_1(i)| + |G_2(i)| \geq 8$.

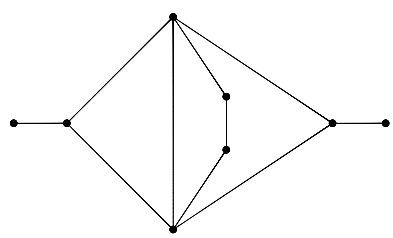


Figure 3.17: The graph has $\Delta(G) = 4$ with degree sequence $(4, 3, 3, 2)$.

Proposition 3.18. *If G is a pseudo 3-regular graph with a vertex i of degree 4 and the neighbor degree sequence of i is $(4, 4, 2, 2)$, then the neighbor of i with degree 2 in G is not connected to a neighbor of i with degree 4 in G.* $\square$

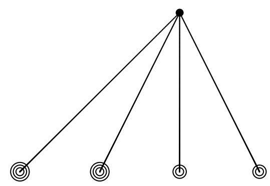


Figure 3.18: Graphs with $\Delta(G) = 4$ and the neighbor degree sequence of a vertex of degree 4 is $(4, 4, 2, 2)$.

In Figure 3.18 we have $1 + |G_1(i)| + |G_2(i)| \geq 9$.

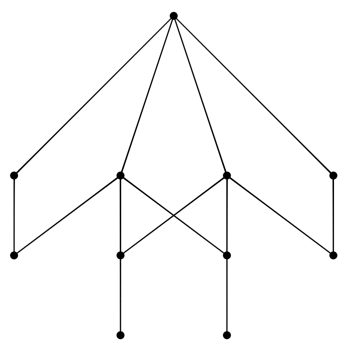


Figure 3.19: The graph has $\Delta(G) = 4$ with degree sequence $(4, 4, 2, 2)$.

We will list all pseudo 3-regular graphs which are not regular of order within 10. From Corollary 3.7(ii), such graphs have maximum degree 4.

Lemma 3.19. *Let G be a connected pseudo 3-regular graph with $\Delta(G) = 4$ and $a_j := |\{i \mid d_i = j\}|$ for $j = 1, 2, 3, 4$. Then*

(i) $a_1 + a_2 = 2a_4$,

(ii) $|VG| = a_3 + 3a_4$,

(iii) $a_1 \leq a_3$,

(iv) a_1, a_2, a_3 *have same parity.*

Proof. (i) and (ii) follow from solving

$$0 = \sum_{i \in VG} (m_i - d_i)d_i = \sum_{i \in VG} (3 - d_i)d_i = a_1 \cdot 2 + a_2 \cdot 2 + a_4(-4).$$

(iii) follows since there exists an injection from the set of degree one vertices into set of degree 3 vertices. Since there are even number of vertices of odd degrees, $a_1 + a_3$ is even. The remaining follows from (i) and (ii). This proves (iv). □

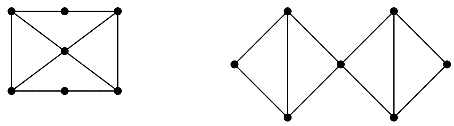
From the above lemma, the following is the possible sequence of (n, a_4, a_3, a_2, a_1) for a connected pseudo 3-regular graph of order n with $\Delta(G) = 4$ and $7 \leq n \leq 10$.

$$
\begin{aligned}
&(n, a_4, a_3, a_2, a_1)\\
=&(10,3,1,5,1),(10,2,4,4,0),(10,2,4,2,2),(10,2,4,0,4),(10,1,7,1,1)\\
=&(9,3,0,6,0),(9,2,3,3,1),(9,2,3,1,3),(9,1,6,2,0),(9,1,6,0,2)\\
=&(8,2,2,4,0),(8,2,2,2,2),(8,1,5,1,1)\\
=&(7,2,1,3,1),(7,1,4,2,0),(7,1,4,0,2).
\end{aligned}
$$

One can check directly that there is no graph whose corresponding sequence (n, a_4, a_3, a_2, a_1) is $(10, 3, 1, 5, 1)$, $(10, 2, 4, 2, 2)$, $(10, 1, 7, 1, 1)$, $(9, 2, 3, 1, 3)$, $(9, 1, 6, 0, 2)$, $(8, 2, 2, 4, 0)$, $(8, 1, 5, 1, 1)$, $(7, 2, 1, 3, 1)$, or $(7, 1, 4, 0, 2)$.

Small pseudo 3-regular but not 3-regular graphs are listed as follows.

$|VG| = 7$**:**



(Switching equivalent)

Figure 3.20: Graphs with sequence $(n, a_4, a_3, a_2, a_1) = (7, 1, 4, 2, 0)$.

$|VG| = 8$**:**

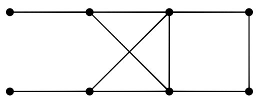
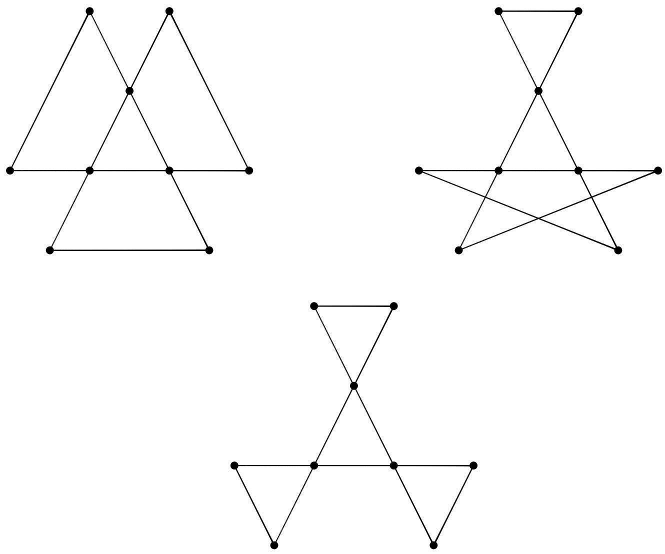


Figure 3.21: The graph with sequence $(n, a_4, a_3, a_2, a_1) = (8, 2, 2, 2, 2)$.

$|VG| = 9$:



(Switching equivalent)

Figure 3.22: Graphs with sequence $(n, a_4, a_3, a_2, a_1) = (9, 3, 0, 6, 0)$.

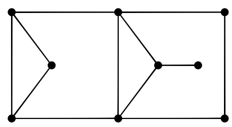
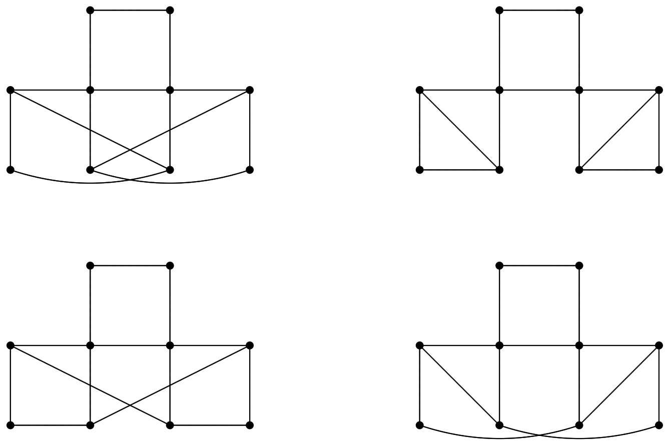


Figure 3.23: The graph with sequence $(n, a_4, a_3, a_2, a_1) = (9, 2, 3, 3, 1)$.

(Switching equivalent)

Figure 3.24: Graphs with sequence $(n, a_4, a_3, a_2, a_1) = (9, 1, 6, 2, 0)$.

$|VG| = 10$:



(Switching equivalent)

Figure 3.25: Graphs with sequence $(n, a_4, a_3, a_2, a_1) = (10, 2, 4, 4, 0)$.

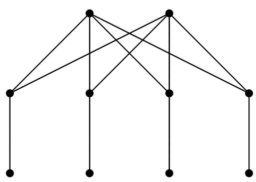


Figure 3.26: The graph with sequence $(n, a_4, a_3, a_2, a_1) = (10, 2, 4, 0, 4)$.

Under what kind of partial information of the pairs (d_i, m_i), one can conclude the diameter of G is at most 6.

In our study of pseudo k-regular graph with a vertex of the maximum degree $k^2 - k + 1$, the obtained graph T_k has diameter 4.

The vertices with large degrees should also play an important role in other graphs.

Bibliography

[1] C. Berge, *Graphs and Hypergraphs*, North-Holland, Amsterdam, and American Elsevier, New York (1973).

[2] B. Bollobàs, *Extremal Graph Theory*, Academic Press, New York (1978).

[3] Y. Chen and R. Pan, and X. Zhang, Two sharp upper bounds for the signless Laplacian spectral radius of graphs, *Discrete Mathematics, Algorithms and Applications* **3** (2011), 185-191.

[4] K.C. Das, A characterization on graphs which achieve the upper bound for the largest Laplacian eigenvalue of graphs, *Linear Algebra and its Applications* **376** (2004), 173-186.

[5] P. Erdös, T. Gallai, Graphs with prescribed degrees of vertices (Hungarian), *Matematikai Lapokt* **11** (1960), 264-274.

[6] D.R. Fulkerson, A.J. Hoffman, and M.H. McAndrew. Some properties of graphs with multiple edges, *Canadian Journal of Mathematics* **17** (1965), 166-177.

[7] B. Grünbaum, Graphs, and complexes, *Report of the university of Washington, Seattle, Mathematic* **572B** (1969).

[8] S. Gruñewald, Harmonic trees, *Applied mathematics letters* **15** (2002), 1001-1004.

[9] S.L. Hakimi, On the realizability of a set of integers as degrees of the vertices of a graph, *Journal of the Society for Industrial and Applied Mathematics* **10** (1962), 496-506.

[10] W. Hässelbarth, Die Verzweigtheit von Graphen, *Match* **16** (1984), 3-17.

[11] V. Havel, A remark on the existence of finite graphs (Hungarian), *Casopis Pest. Mat.* **80** (1955), 477-480.

[12] Y. P. Huang and C. W. Weng, Spectral radius and average 2-degree sequence of a graph, *Discrete Mathematics, Algorithms and Applications* **6** (2014).

[13] J.S. Li and Y.L. Pan, De Caen's inequality and bounds on the largest Laplacian eigenvalue of a graph, *Linear Algebra and its Applications* **328** (2001), 153-160.

[14] J.S. Li and X.D. Zhang, On Laplacian eigenvalues of a graph, *Linear Algebra and its Applications* **285** (1998), 305-307.

[15] R. Merris, A note on Laplacian graph eigenvalues, *Linear Algebra and its Applications* **285** (1998), 33-35.

[16] H. Minc, *Nonnegative Matrices*, John Wiley and Sons, New York, 1988.

[17] H.J. Ryser, Combinatorial properties of matrices of zeros and ones, *Canadian Journal of Mathematics* **9** (1957), 371-377.

[18] G. Sierksma, H. Hoogeveen, Seven criteria for integer sequences being graphic, *Journal of Graph Theory* **15** (1991), 223-231.

[19] D.L. Wang and D.J. Kleitman, On the existence of N-connected graphs with prescribed degrees ($n \geq 2$), *Networks* **3** (1973), 225-239.

[20] A. M. Yu, M. Lu and F. Tian, On the spectral radius of graphs, *Linear Algebra and its Applications* **387** (2004), 41-49.

[21] X.D. Zhang, Two sharp upper bounds for the Laplacian eigenvalues, *Linear Algebra and its Applications* **376** (2004), 207-213.