國 立 交 通 大 學

應 用 數 學 系

碩 士 論 文

The Laplacian Spectral Radius of a Graph

圖的拉普拉斯譜半徑

研究生 ：林凡軒

指導教授 ：翁志文 教授

中 華 民 國 一 百 零 三 年 六 月

The Laplacian Spectral Radius of a Graph

圖 的 拉 普 拉 斯 譜 半 徑

Student: Fan-Hsuan Lin　　Advisor: Chih-Wen Weng

研究生: 林凡軒　　指導教授：翁志文 教授

國　立　交　通　大　學

應 用 數 學 系

碩 士 論 文

A Thesis
Submitted to Department of Applied Mathematics
College of Science
National Chiao Tung University
in Partial Fulfillment of Requirements
for the Degree of Master
in Applied Mathematics

June 2014
Hsinchu, Taiwan, Republic of China

中 華 民 國 一 百 零 三 年 六 月

圖 的 拉 普 拉 斯 譜 半 徑

研究生：林凡軒　　　　指導教授：翁志文 教授

國立交通大學

應用數學系

摘 要

令 $G = (V, E)$ 是一個點集 V 和邊集 E 的簡單連通圖。我們有一個新的拉普拉斯譜半徑的極值上界，而這個上界改進了一些已知的結果。

關鍵詞：圖、拉普拉斯矩陣、拉普拉斯譜半徑。

The Laplacian Spectral Radius of a Graph

Student: Fan-Hsuan Lin   Advisor: Chih-Wen Weng

Department of Applied Mathematics

National Chiao Tung University

Abstract

Let $G = (V, E)$ be a simple connected graph with the vertex set V and the edge set E. We have a new sharp bound for the Laplacian spectral radius of G, which improves some known upper bounds.

Keywords: Graph, Laplacian matrix, Laplacian spectral radius.

誌　謝

首先，我要感謝我的指導教授翁志文教授，當初來到交大只知道自己對數學有興趣，但是一直沒有一個確定的方向，在老師的建議下我找到了我的研究方向，雖然過程中不是這麼順利，但是在老師耐心的指導下我終於做出一些自己想要的結果。

再來我要感謝家安、硯仁、光祥等學長姊們，在我的碩士論文中給我許多非常有用的建議與協助。另外對於我同屆的同學博喻、伯融、凱帆和許多的同學們，總是在我找不到方法時給了我一些適當的意見。育謙和獻之這兩位我人生中的好友，適時的在我被團團壓力包圍時，給我一些放鬆的時刻。當然還有更多在這段時間幫助過我的人，謝謝你們的支持與鼓勵。

最後我要感謝我的父母，如果沒有你們全意的支持我，我沒有辦法順利的完成我的碩士論文，謝謝你們。

Contents

Chapter 1

Introduction

Let $G=(V,E)$ be a simple connected graph with vertex set $V=\{v_1,v_2,\cdots,v_n\}$ and edge set E. Let $A(G)$ be the **adjacency matrix** of G, i.e. the ij-entry of the matrix is 1 or 0 according to whether v_i and v_j is adjacent or not. Denote by $d_i=|G_1(v_i)|$ the degree of vertex $v_i \in V(G)$, where $G_1(v_i)$ is the set of neighbors of v_i, and let $D(G)$=diag$(d_1,d_2,\cdots,d_n)$ be the diagonal matrix with entries $d_1,d_2,\cdots,d_n$. Then the matrix

$$L(G)=D(G)-A(G)$$

is called the **Laplacian matrix** of a graph G. The **Laplacian spectrum** of G is

$$S(G)=(\ell_1(G),\ell_2(G),\cdots,\ell_n(G)),$$

where $\ell_1(G)\geq \ell_2(G)\geq \cdots \geq \ell_n(G)$ are eigenvalues of $L(G)$ arranged in nonincreasing order. Especially, $\ell_1(G)$ is called **Laplacian spectral radius** of G. Now we list some known upper bounds of Laplacian spectral radius, as follows.

In 1985[1], Anderson and Morley showed the following bound

$$\ell_1(G) \le \max_{v_i \sim v_j} \{d_i + d_j\}, \tag{1.1}$$

where $v_i \sim v_j$ means that v_i and v_j are adjacent. We call $m_i = \frac{1}{d_i} \sum_{v_j \sim v_i} d_j$ **average 2-degree** of vertex v_i. For all $v_i \in V(G)$, we have $d_i + m_i = d_i + \frac{1}{d_i} \sum_{v_j \sim v_i} d_j \le d_i + \max_{v_j \sim v_i}\{d_j\} \le \max_{v_j \sim v_i}\{d_i + d_j\}$. In 1998[7], Merris improved the bound (1.1), as follows

$$\ell_1(G) \le \max_{v_i \in V(G)} \{d_i + m_i\}. \tag{1.2}$$

In 2000[9], Rojo et al. showed the following upper bound

$$\ell_1(G) \le \max_{v_i \sim v_j} \{d_i + d_j - |G_1(v_i) \cap G_1(v_j)|\}. \tag{1.3}$$

In 2001[5], Li and Pan gave a bound, as follows

$$\ell_1(G) \le \max_{v_i \in V(G)} \left\{\sqrt{2d_i(d_i + m_i)}\right\}. \tag{1.4}$$

In 2004[11], Zhang showed the following result, which is always better than the bound (1.4).

$$\ell_1(G) \le \max_{v_i \in V(G)} \left\{d_i + \sqrt{d_i m_i}\right\}. \tag{1.5}$$

In this paper, we obtain a new sharp upper bound of Laplacian spectral radius $\ell_1(G)$, and provide some graphs that satisfy the sharp upper bound. In particular, if G is a **strongly regular graph** with parameters (n, k, λ, μ), then the graph satisfies the sharp upper bound, where strongly regular graph G with parameters (n, k, λ, μ) means that G is a k-regular graph with n vertices and common neighbours of two adjacent(nonadjcent) vertices is a fixed number $\lambda(\mu)$, respectively, and G is denoted by $\mathrm{srg}(n, k, \lambda, \mu)$. See Theorem 3.5 and Corollary 3.13 for those results.

Chapter 2

Preliminaries

Let $G = (V, E)$ be a simple connected graph with vertex set $V = \{v_1, v_2, \cdots, v_n\}$ and edge set E. We define G^c to be the **complement** of G, i.e. G^c has the same vertex set of G and two distinct vertices of G^c are adjacent if and only if they are not adjacent in G. Let an **orientation** σ of a graph G be an assignment of each edge of G a direction to form a digraph G^σ. Let N denote the **directed incidence matrix** of G^σ, i.e. N has rows indexed by the vertices and columns by edges, where the xe-entry of N is $-1, 1$, or 0 when x is the head of e, the tail of e, or not on e, respectively. Hence we have $L(G) = NN^\top$, which implies that $L(G)$ is positive semi-definite. Then we have the following facts[3].

1. $\ell_n(G) = 0$ is an eigenvalue of $L(G)$ corresponding to the eigenvector $\mathbf{1}_n$, where $\mathbf{1}_n$ is the all-ones vector.

2. If $X = (x_1, x_2, \cdots, x_n)^\top$ is an eigenvector of $L(G)$ corresponding to $\ell_i(G)$ $(1 \leq i \leq n-1)$, then $\sum_{i=1}^{n} x_i = 0$.

3. $L(G)+L(G^c) = nI - J$, where I and J are identity matrix and all-ones matrix,

respectively.

4. If X is the eigenvector of $L(G)$ corresponding to $\ell_i(G)$ $(1 \leq i \leq n-1)$, then X is also an eigenvector of $L(G^c)$ corresponding to $n - \ell_i(G)$.

5. $\ell_i(G) \leq n$, for $1 \leq i \leq n$.

Now, we give more definitions.

Definition 2.1. Let $G = (V, E)$ be a simple connected graph with vertex set $V = \{v_1, v_2, \cdots, v_n\}$ and edge set E. The following notations are adopted.

1. $\lambda(G) = \min_{v_i \sim v_j} |G_1(v_i) \cap G_1(v_j)|$.

2. $\mu(G) = \min_{v_i \nsim v_j} |G_1(v_i) \cap G_1(v_j)|$.

3. We call G a **triangulation**, if $\lambda(G) > 0$.

4. A planar graph is called a **maximal planar graph** if every pair of nonadjacent vertices u and v of G, the graph $G + uv$ is nonplanar.

Remark 2.2. Let $G = (V, E)$ be a simple connected graph. Then we have $\mu(G) = \lambda(G^c)$.

Theorem 2.3. *[4] If G is a maximal planar graph and $|V(G)| \geq 4$, then G is a triangulation. Moreover, $\lambda(G) \geq 2$.*

Proof. Because G is a maximal planar graph, every region in G is triangle. When $|V(G)| \geq 4$, we know that every edge in a maximal planar graph belongs to two distinct regions. Therefore, it implies that any two adjacent vertices in G have at least two common neighbors. On the other hand, $\lambda(G) \geq 2$. □

In 2013, Guo et al. improved the bound (1.5) and showed the following result.

Theorem 2.4. *[4, Theorem 3.1] Let $G = (V, E)$ be a simple connected graph with vertex set $V = \{v_1, v_2, \cdots, v_n\}$ and edge set E. We define*

$$M(G) = \max_{v_i \in V(G)} \left\{ \frac{2d_i - \lambda + \sqrt{4d_i m_i - 4\lambda d_i + \lambda^2}}{2} \right\},$$

where $\lambda = \lambda(G)$. Then

$$\ell_1(G) \leq M(G), \tag{2.1}$$

□

In 2013, Guo et al. showed a corollary about maximal planar graph, as follows.

Corollary 2.5. *[4, Theorem 3.3] If G is a maximal planar graph and $|V(G)| \geq 4$, then*

$$\ell_1(G) \leqslant \max_{v_i \in V(G)} \left\{ d_i - 1 + \sqrt{d_i m_i - 2d_i + 1} \right\}.$$

□

Chapter 3

Main Results

3.1 Some Corollary about Theorem 2.4

We will show two easy corollaries of Theorem 2.4.

Corollary 3.1. *If G is a k-regular graph, then*

$$\ell_1(G) \leqslant 2k - \lambda,$$

where $\lambda = \lambda(G)$.

Proof. Because G is a k-regular graph, we have $d_i = m_i = k$, for all v_i. Therefore,

$$\begin{aligned}\ell_1(G) \leq & \max_{v_i \in V(G)} \left\{ \frac{2d_i - \lambda + \sqrt{4d_i m_i - 4\lambda d_i + \lambda^2}}{2} \right\} \\ = & \frac{2k - \lambda + \sqrt{4k \cdot k - 4\lambda k + \lambda^2}}{2} \\ = & 2k - \lambda.\end{aligned}$$

□

Because $\ell_1(G) \leq n$, we get the following corollary.

Corollary 3.2. *If G is a simple connected graph with n vertices, then* $\ell_1(G) \leq \min\{M(G), n\}$.

$\square$

3.2 Main Results

Now we will show our main result. First, we give a proposition, as follows.

Proposition 3.3. *[8] Let $G = (V, E)$ be a simple connected graph with vertex set $V = \{v_1, v_2, \cdots, v_n\}$ and edge set E.*

1. *If $T = A(G)^2$ and $T = (t_{ij})$, we have $t_{ij} = |G_1(v_i) \cap G_1(v_j)|$ and $\sum_{j=1}^{n} t_{ij} = \sum_{j \sim i} d_j = m_i d_i$.*
2. *If $X = (x_1, x_2, \cdots, x_n)^\top$ is a vector, $X^\top L(G) X = \sum_{\substack{j<k \\ v_j \sim v_k}} (x_j - x_k)^2$.*

$\square$

Now, we prove the main theorem, which improves Theorem 2.4.

Theorem 3.4. *Let $G = (V, E)$ be a simple connected graph with vertex set $V = \{v_1, v_2, \cdots, v_n\}$ and edge set E. Let $S(G) = (\ell_1(G), \ell_2(G), \cdots, \ell_n(G))$ be the Laplacian spectrum of G. We define*

$$M'(G) = \max_{v_i \in V(G)} \left\{ \frac{2d_i - \lambda + \mu + \sqrt{B_i}}{2} : B_i \geq 0 \right\}$$

and

$$N'(G) = \min_{v_i \in V(G)} \left\{ \frac{2d_i - \lambda + \mu - \sqrt{B_i}}{2} : B_i \geq 0 \right\},$$

where $B_i = 4d_im_i - 4(\lambda-\mu)d_i + (\lambda-\mu)^2 - 4\mu n,\ \lambda = \lambda(G),$ *and* $\mu = \mu(G)$. *Then*

$$N'(G) \leq \ell(G) \leq M'(G), \tag{3.1}$$

where $\ell(G) \in \{\ell_1(G), \ell_2(G), \cdots, \ell_{n-1}(G)\}$.

Proof. Let $X = (x_1, x_2, \cdots, x_n)^\top$ be the eigenvector of $L(G)$ corresponding to $\ell(G)$. We have

$$\begin{aligned}
\sum_{i=1}^{n}[d_i - \ell(G)]^2x_i^2 =& \|(D(G) - \ell_\ell(G)I)X\|^2 \\
=& \|(D(G) - L(G))X\|^2 \\
=& \|A(G)X\|^2 \\
=& X^\top TX \\
=& \sum_{i=1}^{n} t_{ii}x_i^2 + 2\sum_{j<k} t_{jk}x_jx_k \\
=& \sum_{i=1}^{n} t_{ii}x_i^2 + \sum_{j<k} t_{jk}(x_j^2 + x_k^2 - (x_j - x_k)^2) \\
=& \sum_{i=1}^{n}((t_{ii} + \sum_{\substack{j=1\\ j\neq i}}^{n} t_{ij})x_i^2) - \sum_{\substack{j<k\\ v_j\sim v_k}} t_{jk}(x_j - x_k)^2 - \sum_{\substack{j<k\\ v_j\nsim v_k}} t_{jk}(x_j - x_k)^2 \\
\leq& \sum_{i=1}^{n} d_im_ix_i^2 - \lambda\sum_{\substack{j<k\\ v_j\sim v_k}}(x_j - x_k)^2 - \mu\sum_{\substack{j<k\\ v_j\nsim v_k}}(x_j - x_k)^2 \\
=& \sum_{i=1}^{n} d_im_ix_i^2 - \lambda X^\top L(G)X - \mu X^\top L(G^c)X \\
=& \sum_{i=1}^{n} d_im_ix_i^2 - \lambda\ell(G)\|X\|^2 - \mu(n - \ell(G))\|X\|^2 \\
=& \sum_{i=1}^{n} d_im_ix_i^2 - \lambda\ell(G)\sum_{i=1}^{n} x_i^2 - \mu(n - \ell(G))\sum_{i=1}^{n} x_i^2.
\end{aligned}$$

Thus, we have

$$\sum_{i=1}^{n}[(d_i - \ell(G))^2 - d_im_i + \lambda\ell(G) + \mu(n - \ell(G))]x_i^2 \leq 0. \tag{3.2}$$

Then there must exist a vertex v_i such that

$$(d_i - \ell(G))^2 - d_i m_i + \lambda\ell(G) + \mu(n - \ell(G))$$

$$=\ell(G)^2 - (2d_i - \lambda + \mu)\ell(G) + (d_i^2 - d_i m_i + \mu n) \leq 0,$$

which implies that

$$\frac{2d_i - \lambda + \mu - \sqrt{B_i}}{2} \leq \ell(G) \leq \frac{2d_i - \lambda + \mu + \sqrt{B_i}}{2}.$$

Therefore,

$$N'(G) \leq \ell(G) \leq M'(G).$$

□

According to Theorem 3.4, we focus on $\ell_1(G)$ and $\ell_{n-1}(G)$ and have the following theorem.

Theorem 3.5. *Let G be a simple connected graph. Then*

$$\ell_1(G) \leq M'(G) \tag{3.3}$$

and

$$\ell_{n-1}(G) \geq N'(G) \tag{3.4}$$

□

We have similar corollary as Corollary 3.1 about $\ell_1(G)$ and $\ell_{n-1}(G)$ on a regular graph.

Corollary 3.6. *If G is k-regular graph, then*

$$\ell_1(G) \leq \frac{2k - \lambda + \mu + \sqrt{4k^2 - 4(\lambda - \mu)k + (\lambda - \mu)^2 - 4\mu n}}{2}$$

and

$$\ell_{n-1}(G) \geq \frac{2k - \lambda + \mu - \sqrt{4k^2 - 4(\lambda - \mu)k + (\lambda - \mu)^2 - 4\mu n}}{2}.$$

□

It is similar to Corollary 3.2, and we have the following corollary.

Corollary 3.7. *If G is a simple connected graph with n vertices, then* $\ell_1(G) \leq \min\{M'(G), n\}$.

□

We give an example and compare results of Theorem 2.4 and 3.5

Example 3.8. In this example, G is the Petersen graph which is $\mathrm{srg}(10, 3, 0, 1)$, as follows. Hence, we have $\lambda = 0$, $\mu = 1$, and $d_i = 3$, for any vertex v_i, and we compute

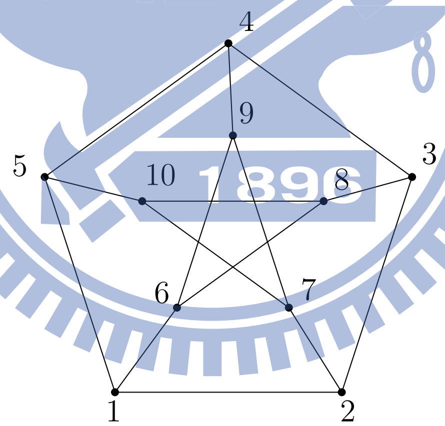


Figure 3.1: Petersen graph

$\ell_1(G) = 5$.

According to Theorem 2.4

$$
\begin{aligned}
M(G) &= \max_{v_i \in V(G)} \left\{ \frac{2d_i - \lambda + \sqrt{4d_i m_i - 4\lambda d_i + \lambda^2}}{2} \right\} \\
&= \frac{2 \times 3 - 0 + \sqrt{4 \times 3^2 - 0 + 0}}{2} \\
&= 6.
\end{aligned}
$$

According to Theorem 3.5

$$
\begin{aligned}
M'(G) &= \max_{v_i \in V(G)} \left\{ \frac{2d_i - \lambda + \mu + \sqrt{4d_i m_i - 4(\lambda - \mu)d_i + (\lambda - \mu)^2 - 4\mu n}}{2} : B_i \geq 0 \right\} \\
&= \frac{2 \times 3 - 0 + 1 + \sqrt{4 \times 3^2 - 4(0-1)3 + (0-1)^2 - 4 \times 1 \times 10}}{2} \\
&= 5.
\end{aligned}
$$

Therefore, we have $\ell_1(G) = 5 = M'(G) \leq M(G) = 6$. According to this example, we will prove two things, as follows

1. $M'(G) \leq M(G)$, if we consider the condition $\ell_1(G) \leq n$.

2. If G is a strongly regular graph, then $\ell_1(G) = M'(G)$.

In Theorem 3.9 and Corollary 3.13, we will prove two results about those observations.

In Theorem 3.9, we will show that our result of Corollary 3.7 is better than Corollary 3.2.

Theorem 3.9. *Let $G = (V, E)$ be a simple connected graph with vertex set $V = \{v_1, v_2, \cdots, v_n\}$ and edge set E. Then*

$$\min\{M'(G), n\} \leq \min\{M(G), n\}.$$

Proof. We have two cases in this proof.

Case 1: When $M(G) \geq n$, we have $\min\{M(G), n\} = n \geq \min\{M'(G), n\}$

Case 2: When $M(G) < n$. Let ζ_i be the largest root of $f_i(x) = (d_i - x)^2 - d_i m_i + \lambda x = 0$ and ξ_i be the largest root of $g_i(x) = (d_i - x)^2 - d_i m_i + \lambda x + \mu(n - x) = 0$, for $1 \leq i \leq n$, as $B_i \geq 0$, where $\lambda = \lambda(G)$ and $\mu = \mu(G)$. Then we have

$$\zeta_i = \frac{2d_i - \lambda + \sqrt{4d_i m_i - 4\lambda d_i + \lambda^2}}{2}$$

and

$$\xi_i = \frac{2d_i - \lambda + \mu + \sqrt{4d_i m_i - 4(\lambda - \mu)d_i + (\lambda - \mu)^2 - 4\mu n}}{2}.$$

Hence, we can remark that $M(G) = \max_i\{\zeta_i\}$ and $M'(G) = \max_i\{\xi_i\}$. Let $v_i \in V(G)$.. If $B_i \geq 0$, then we have

- $g_i(\zeta_i) = (d_i - \zeta_i)^2 - d_i m_i + \lambda\zeta_i + \mu(n - \zeta_i) = 0 + \mu(n - \zeta_i) > 0$, because $M(G) < n$ implies $\zeta_i < n$.
- $g_i'(\zeta_i) = 2(\zeta_i - d_i) + \lambda - \mu > 2(\frac{2d_i - \lambda}{2} - d_i) + \lambda - \mu = d_i - \mu \geq 0$, because $\mu = \mu(G) = \min_{v_i \nsim v_j} |G_1(v_i) \cap G_1(v_j)| \leq \min_{v_i \in V(G)} |G_1(v_i)| \leq d_i$, for $1 \leq i \leq n$.

Therefore, $\xi_i < \zeta_i$, for $1 \leq i \leq n$, as $B_i \geq 0$.

Finally, we get $M'(G) = \max_i\{\xi_i\} < M(G) = \max_i\{\zeta_i\} < n$.

According to those cases, we complete the proof. □

3.3 Applications of Theorem 3.5

In the section, let $\lambda = \lambda(G)$ and $\mu = \mu(G)$. First, we give a trivial example on $n = 5$ such that the equality in (3.3) holds.

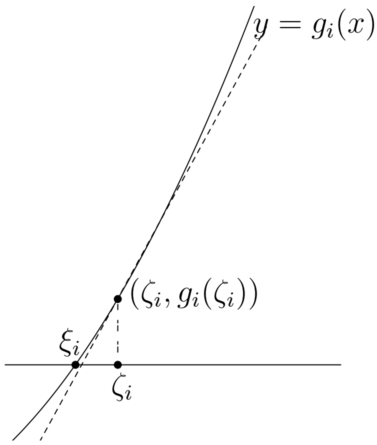


Figure 3.2: compare ζ_i and ξ_i

Example 3.10. When $G = K_5$, we have $\lambda = 3$, $\mu = 0$, and $d_i = 4$, for any vertex

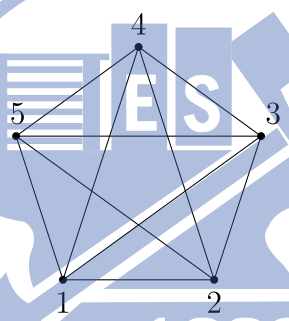


Figure 3.3: K_5

v_i, and we calculate the $\ell_1(G) = 5$. Then, according to Theorem 3.5, we get

$$M'(G) = \frac{2 \times 4 - 3 + 0 + \sqrt{4 \times 4^2 - 4(3-0)4 + (3-0)^2 - 4 \times 0 \times 5}}{2} = 5 = \ell_1(G).$$

Therefore, K_5 is a graph which satisfies the the equality in (3.3) .

We have the following definition about two graphs.

Definition 3.11. [6] Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs with disjoint vertex sets. Then we define the **join** of two graphs G_1 and G_2 is $G_1 \vee G_2 = (V, E)$, where $V = V_1 \cup V_2$ and $E = E_1 \cup E_2 \cup \{xy | x \in V_1 \text{ and } y \in V_2\}$.

Theorem 3.12 is a useful tool to compute $\ell_1(G)$ and eigenvector corresponding to the join of two graphs.

Theorem 3.12. *[6] Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs with disjoint vertex sets and $(|V_1|, |V_2|) = (n, m)$. Let λ_i and ν_j be eigenvalues of $L(G_1)$ and $L(G_2)$ corresponding to the eigenvector v_i and w_j, respectively, where $< \lambda_i >$ and $< \nu_j >$ both are nonincreasing sequences, for all $1 \le i \le n$ and $1 \le j \le m$. Then, $0, \lambda_i + m, \nu_j + n,$ and $n + m$ are eigenvalues of $L(G_1 \vee G_2)$ corresponding to the eigenvector $\mathbf{1}_{n+m}$, $(v_i^\top, \mathbf{0}_m^\top)^\top$, $(\mathbf{0}_n^\top, w_j^\top)^\top$,and $(m\mathbf{1}_n^\top, -n\mathbf{1}_m^\top)^\top$, respectively, for all $2 \le i \le n$ and $2 \le j \le m$.* □

Now, we find some graphs such that the equality in (3.3) holds. First, the equality in (3.2) hold if and only if

$$\sum_{\substack{j<k \\ v_j \sim v_k}} t_{jk}(x_j - x_k)^2 = \lambda \sum_{\substack{j<k \\ v_j \sim v_k}} (x_j - x_k)^2$$

and

$$\sum_{\substack{j<k \\ v_j \nsim v_k}} t_{jk}(x_j - x_k)^2 = \mu \sum_{\substack{j<k \\ v_j \nsim v_k}} (x_j - x_k)^2.$$

It is not easy to find a sufficient and necessary condition which satisfied the above two equations, because we must understand more about the eigenvector. We have an obvious condition to satisfy above two equations, if $|G_1(v_i) \cap G_1(v_j)| = \lambda(G)$ for any edge v_iv_j of G and $|G_1(v_i) \cap G_1(v_j)| = \mu(G)$ for any edge v_iv_j of G^c, then the equality in (3.2) holds. Therefore, when G is a strongly regular graph with some parameter (n,k,λ,μ), G make the equality in (3.2) holds, because λ or μ is a fixed number of common neighbours of two adjacent or nonadjcent vertices, respectively. We will show that it is not a coincidence that the Petersen graph of Example 3.8 satisfies the equality in (3.3). In Corollary 3.13, we will prove all strongly regular graphs satisfy the equality in (3.3).

Corollary 3.13. *If G is a strongly regular graph with parameters (n,k,λ,μ), then*

$$\ell_1(G) = M'(G) \text{ and } \ell_{n-1}(G) = N'(G).$$

Proof. Because $n = 1 + k + \dfrac{k(k-1-\lambda)}{\mu}$, we have

$$\begin{aligned}
&\frac{2k-\lambda+\mu \pm \sqrt{4k^2-4(\lambda-\mu)k+(\lambda-\mu)^2-4\mu n}}{2}\\
=&\frac{2k-\lambda+\mu \pm \sqrt{4k^2-4(\lambda-\mu)k+(\lambda-\mu)^2-4\mu(1+k+\frac{k(k-1-\lambda)}{\mu})}}{2}\\
=&\frac{2k-\lambda+\mu \pm \sqrt{(\lambda-\mu)^2+4(k-\mu)}}{2}.
\end{aligned}$$

Here

$$\ell_1(G) = \frac{2k-\lambda+\mu+\sqrt{(\lambda-\mu)^2+4(k-\mu)}}{2}$$

and

$$\ell_{n-1}(G) = \frac{2k-\lambda+\mu-\sqrt{(\lambda-\mu)^2+4(k-\mu)}}{2}$$

are a known result about the graph srg(n,k,λ,μ). Therefore, we complete the proof. □

It is difficult to find a graph, which satisfies the equality in (3.3), though some graphs satisfy the equality in (3.2). The following are some examples, which satisfy the equality in (3.2), but the equality in (3.3) uncertainly holds. In Example 3.14 and Example 3.15, We will show fan graphs such that the equality in

(3.2) holds, but the equality in (3.3) does not hold. After the section, we let $\xi_i = \dfrac{2d_i - \lambda + \mu + \sqrt{4d_im_i - 4(\lambda-\mu)d_i + (\lambda-\mu)^2 - 4\mu n}}{2}$, then we know $M'(G) = \max\limits_i\{\xi_i\}$.

Example 3.14. We usually call $F_\ell = K_1 \vee \ell K_2$ a fan graph. When $G = F_2$ the

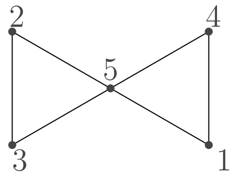


Figure 3.4: F_2

adjacency matrix $A(G)$ and Laplacian matrix are as follows.

$$A(G) = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix},$$

and

$$L(G) = \begin{pmatrix} 2 & 0 & 0 & -1 & -1 \\ 0 & 2 & -1 & 0 & -1 \\ 0 & -1 & 2 & 0 & -1 \\ -1 & 0 & 0 & 2 & -1 \\ -1 & -1 & -1 & -1 & 4 \end{pmatrix}.$$

Hence,

$$A(G)^2 = \begin{pmatrix} 2 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 4 \end{pmatrix},$$

$\lambda = 1$ and $\mu = 1$. According to Theorem 3.12, we get $X = \begin{pmatrix} 1 & 1 & 1 & 1 & -4 \end{pmatrix}^\top$ is a eigenvector corresponding to the eigenvalue $\ell_1(G) = 5$.

We calculate $M'(G)$ and the equality in (3.2) as shown in the following table. $\ell_1(G) = 5 < \frac{8+\sqrt{12}}{2}$, so the inequality (3.3) does not hold. But the equality in

i	d_i	m_i	ξ_i	$(d_i - \ell_1(G))^2 - d_i m_i + \lambda \ell_1(G) + \mu(n - \ell_1(G))$
$1 \sim 4$	2	3	3	$(2-5)^2 - 2 \cdot 3 + 1 \cdot 5 + 1 \cdot (5-5) = 8$
5	4	2	$\frac{8+\sqrt{12}}{2} \approx 5.73$	$(4-5)^2 - 4 \cdot 2 + 1 \cdot 5 + 1 \cdot (5-5) = -2$

Table 3.1: calculate $M'(G)$ on F_2

(3.2) holds, because $\sum_{i=1}^{5} [(d_i - \ell_1(G))^2 - d_i m_i + \lambda \ell_1(G) + \mu(n - \ell_1(G))] x_i^2 = 0$.

Example 3.15. When $G = F_\ell = K_1 \vee \ell K_2$, according Theorem 3.12, we have $X = (\mathbf{1}_{2\ell}^\top, -2\ell)^\top$ is an eigenvector corresponding to the eigenvalue $\ell_1(G) = 2\ell + 1$, we have following result on Table 3.2. Then we have $\ell_1(G) = 2\ell + 1 \leq \frac{4\ell - 1 + \sqrt{8\ell+1}}{2}$. Therefore, all of F_ℓ do not satisfy the equality in (3.3), but the equality in (3.2) holds.

In Examples 3.16, we will show more graphs with $n = 5$ such that the equality in (3.2) holds.

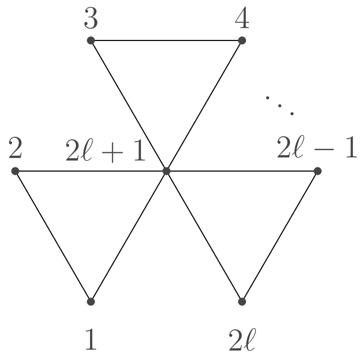


Figure 3.5: F_ℓ

i	d_i	m_i	ξ_i	$(d_i-\ell_1(G))^2-d_im_i+\lambda\ell_1(G)+\mu(n-\ell_1(G))$
$1\sim 2\ell$	2	$\ell+1$	$\frac{3+\sqrt{8\ell+1}}{2}$	$(2-2\ell-1)^2-2(\ell+1)+1(2\ell+1)+1\cdot 0=4\ell^2-4\ell$
$2\ell+1$	2ℓ	2	$\frac{4\ell-1+\sqrt{8\ell+1}}{2}$	$(2\ell-2\ell-1)^2-2\ell\cdot 2+1(2\ell+1)+1\cdot 0=2-2\ell$

Table 3.2: calculate $M'(G)$ on F_ℓ

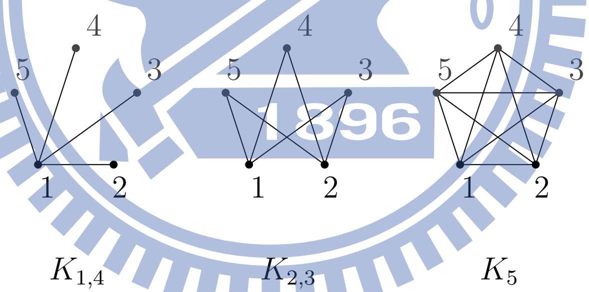


Figure 3.6: $K_{1,4}$, $K_{2,3}$ and K_5

Example 3.16. According to Theorem 3.12, we have all graphs of this example $\ell_1(G)=5$. In the following table, we list some graphs such that the equality in (3.2) holds, and we compare $M'(G)$ and $\ell_1(G)$ on Table 3.3.

Therefore, in this example, all complete bipartite graphs, which satisfy the equality in (3.2), with $n=5$ do not satisfy the equality in (3.3).

G	$L(G)$	$M'(G)$	$\ell_1(G)$
$K_{1,4}$	$\begin{pmatrix} 4 & -1 & -1 & -1 & -1 \\ -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 \end{pmatrix}$	$\frac{8+\sqrt{16}}{2} = 6$	5
$K_{2,3}$	$\begin{pmatrix} 3 & 0 & -1 & -1 & -1 \\ 0 & 3 & -1 & -1 & -1 \\ -1 & -1 & 2 & 0 & 0 \\ -1 & -1 & 0 & 2 & 0 \\ -1 & -1 & 0 & 0 & 2 \end{pmatrix}$	$\frac{8+\sqrt{12}}{2} \approx 5.73$	5
K_5	$\begin{pmatrix} 4 & -1 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 & -1 \\ -1 & -1 & 4 & -1 & -1 \\ -1 & -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & -1 & 4 \end{pmatrix}$	5	5

Table 3.3: compare $K_{1,4}$, $K_{2,3}$ and K_5

In Examples 3.17, we will show more graphs with $n = 6$ such that the equality in (3.2) holds.

Example 3.17. According to Theorem 3.12, all graphs of this example have $\ell_1(G) = 6$. We list the result of the compare of $\ell_1(G)$ and $M'(G)$ on Table 3.4. In this

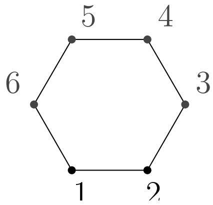
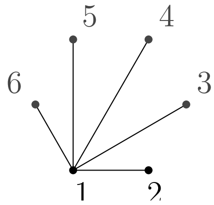
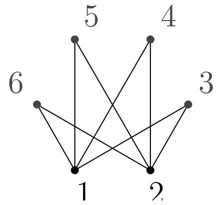
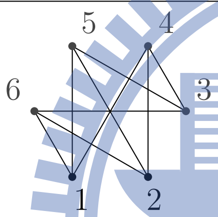
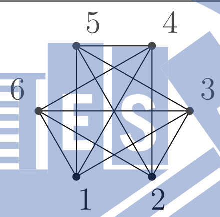
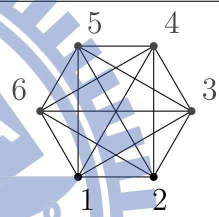
		
C_6	$K_{1,5}$	$K_{2,4}$
$\ell_1(G) = M'(G)$	$\ell_1(G) \neq M'(G)$	$\ell_1(G) \neq M'(G)$
		
$K_{3,3}$	$K_{2,2,2}$	K_6
$\ell_1(G) = M'(G)$	$\ell_1(G) = M'(G)$	$\ell_1(G) = M'(G)$

Table 3.4: compare $\ell_1(G)$ and $M'(G)$ with $n = 6$

example, we obtain the following result. If G is a complete k-partite graph, then G does not satisfy $M'(G) = \ell_1(G)$, unless G is a regular graph.

Through Example 3.16 and Example 3.17, we have a corollary about complete k-partite graph, which, as follow.

Corollary 3.18. *Let G be a complete k-partite graph($k \geq 2$). Then, $\ell_1(G) = M'(G)$ if and only if every part in G has the same number of vertices.*

Proof. Let $G = K^c_{n_1} \vee K^c_{n_2} \vee \cdots \vee K^c_{n_k}$, where $n_1 \le n_2 \le \cdots \le n_k$ is a non-decreasing sequence. By Theorem 3.12, we can calculate $\ell_1(G) = n$, where $n = \sum_{i=1}^{k} n_i$. Now, we start to compute $M'(G)$. First, for $1 \le i \le k$, we calculate $d_i = n - n_i$, $\lambda = \sum_{i=1}^{k-2} n_i$, $\mu = \sum_{i=1}^{k-1} n_i = n - n_k$, and $d_i m_i = \sum\limits_{v_j \sim v_i} d_j = \sum\limits_{\substack{j=1 \\ j \ne i}}^{k} n_j d_j = \sum\limits_{\substack{j=1 \\ j \ne i}}^{k} n_j(n - n_j) = n \sum\limits_{\substack{j=1 \\ j \ne i}}^{k} n_j - \sum\limits_{\substack{j=1 \\ j \ne i}}^{k} n_j^2$. We recall the result of Theorem 3.5 $M'(G) = \max\limits_{v_i \in V(G)} \left\{ \dfrac{2d_i - \lambda + \mu + \sqrt{4d_i m_i - 4(\lambda - \mu)d_i + (\lambda - \mu)^2 - 4\mu n}}{2} \right\}$. We let $\xi_i = \dfrac{2d_i - \lambda + \mu + \sqrt{4d_i m_i - 4(\lambda - \mu)d_i + (\lambda - \mu)^2 - 4\mu n}}{2}$. Therefore, $\xi_i = \dfrac{2(n - n_i) - (-n_{k-1}) + \sqrt{B}}{2} = n + \dfrac{\sqrt{B} - (2n_i - n_{k-1})}{2}$,where

$$
\begin{aligned}
B =& 4(n \sum_{\substack{j=1 \\ j \ne i}}^{k} n_j - \sum_{\substack{j=1 \\ j \ne i}}^{k} n_j^2) - 4(-n_{k-1})(n - n_i) + (-n_{k-1})^2 - 4(n - n_k)n \\
=& 4n(n - n_i) - 4 \sum_{\substack{j=1 \\ j \ne i}}^{k} n_j^2 + 4n \cdot n_{k-1} - 4n_{k-1}n_i + n_{k-1}^2 - 4n^2 + 4n \cdot n_k \\
=& -4 \sum_{\substack{j=1 \\ j \ne i}}^{k} n_j^2 - 4n_{k-1}n_i + n_{k-1}^2 + 4n(-n_i + n_{k-1} + n_k).
\end{aligned}
$$

In order to calculate ξ_i, we compare $\sqrt{B}$ and $(2n_i - n_{k-1})$, as follows. We calculate $B - (2n_i - n_{k-1})^2 = -4 \sum\limits_{\substack{j=1 \\ j \ne i}}^{n} n_j^2 - 4n_{k-1}n_i + n_{k-1}^2 + 4n(-n_i + n_{k-1} + n_k) - (4n_i^2 + n_{k-1}^2 - 4n_i n_{k-1}) = -4 \sum\limits_{j=1}^{k} n_j^2 + 4n(-n_i + n_{k-1} + n_k)$. We have two case to discuss above formula.

Case 1: When $i = k$, $B - (2n_i - n_{k-1})^2 = -4 \sum\limits_{j=1}^{k} n_j^2 + 4n(-n_k + n_{k-1} + n_k) = -4 \sum\limits_{j=1}^{k} n_j^2 + 4n \cdot n_{k-1} \ge -4n_k \sum\limits_{j=1}^{k} n_j + 4n \cdot n_{k-1} = n(n_k - n_{k-1}) \ge 0$.

Case 2: When $1 \le i \le k-1$, $B - (2n_i - n_{k-1})^2 \ge -4 \sum\limits_{j=1}^{k} n_j^2 + 4n(-n_{k-1} + n_{k-1} + n_k) \ge -4n_k \sum\limits_{j=1}^{k} n_j + 4n(n_k) = 0$.

On two case, we have the same conclusion, as follows.

1. $B - (2n_i - n_{k-1})^2 \geq 0$, it implies $\xi_i = n + \dfrac{\sqrt{B} - (2n_i - n_{k-1})}{2} \geq n + \dfrac{0}{2} = n.$

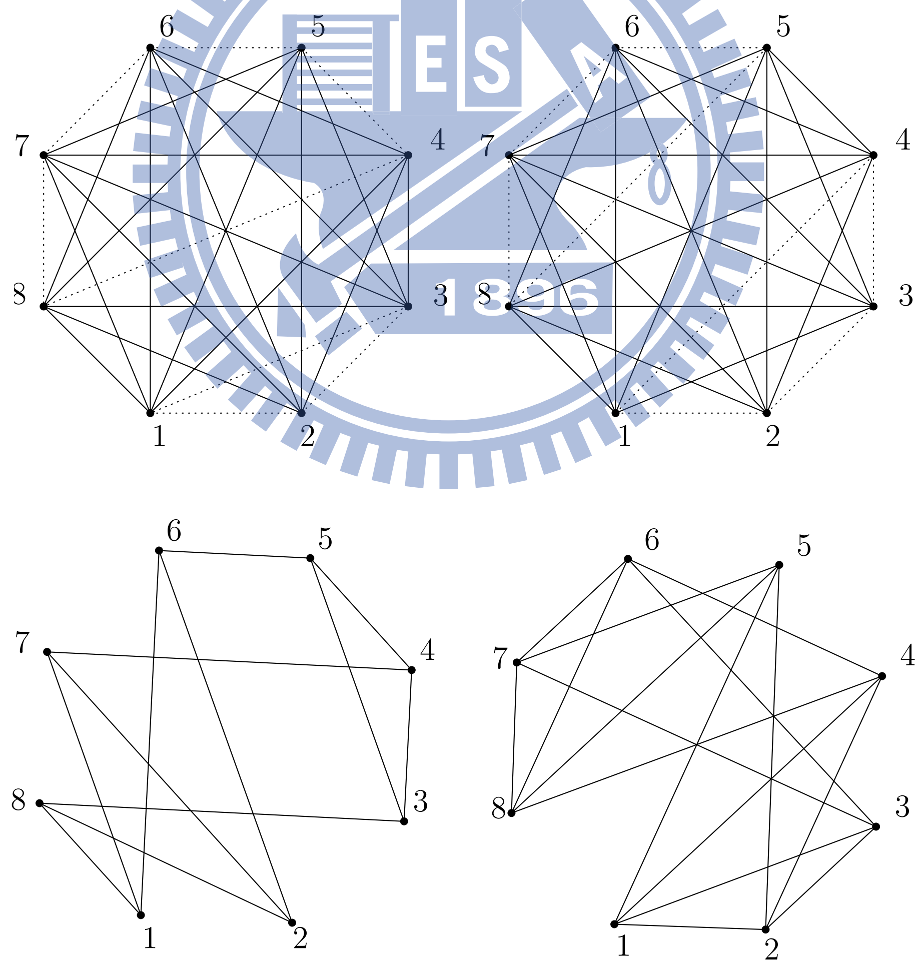
   Therefore, $M'(G) \geq n = \ell_1(G)$.

2. The equality holds on two case if and only if $n_1 = n_2 = \cdots = n_k$.

Hence, we complete the proof. $\square$

In Example 3.19, we have some graphs, which are not k-partite graph or strongly regular graph, but satisfy $\ell_1(G) = M'(G)$.

Example 3.19. In this example, we give four graphs which satisfy the equality in (3.3).



Therefore, we have some graphs, which satisfy the equality in (3.3), but they are not k-partite graph or strongly regular graph. Finally in this example, we note that some graphs, which satisfy the equality in (3.3), but the common neighbors of any two adjacent or nonadjacent vertices are not a fixed number.

Bibliography

[1] W.N. Anderson,, T.D. Morley, Eigenvalues of the Laplacian of a graph, *Linear Multilinear Algebra*, 18 (1985), 141-145.

[2] A.E. Brouwer and W. H. Haemers Spectra of Graphs, 2012: Springer-Verlag

[3] Miroslav Fiedler, Praha, Algebraic connectivity of graphs, *Czechoslovak Mathematical Journal* 23 (98) 1973, Praha.

[4] J.M. Guo, J. Li, W.C. Shiu, A note on the upper bounds for the Laplacian spectral radius of graphs, *Linear Algebra and its Applications*, 439 (2013), 1657-1661.

[5] J.S. Li, Y.L. Pan, De Caen's inequality and bounds on the largest Laplacian eigenvalue of a graph, *Linear Algebra and its Applications*, 328 (2001), 153-160.

[6] R. Merris, Laplacian graph eigenvectors, *Linear Algebra and its Applications* 278 (1998), 221-236.

[7] R. Merris, A note on Laplacian graph eigenvalues, *Linear Algebra and its Applications*, 285 (1998), 33-35.

[8] M. W. Newman, The Laplacian Spectrum of Graphs , University of Manitoba, Winnipeg, MB, Canada, 2000.

[9] O. Rojo, R. Soto, H. Rojo, , An always nontrivial upper bound for Laplacian graph eigenvalues, *Linear Algebra and its Applications*, 312 (2000), 155-159.

[10] L.S. Shi, Bounds on the (Laplacian) spectral radius of graphs, *Linear Algebra and its Applications*, 422 (2007), 755-770.

[11] X.D. Zhang, Two sharp upper bounds for the Laplacian eigenvalues, *Linear Algebra and its Application* 376 (2004), 207-213.

[12] X.D. Zhang, R. Luo, The Laplacian eigenvalues of mixed graphs, *Linear Algebra and its Application* 362 (2003), 109-119.