Spectral Excess Theorem and its Applications

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Part 0

Introduction

Distance-regularity

- Let G be a connected graph with vertex set V and diameter D.
- For $0 \leq i \leq D$ and two vertices $u, v \in V$ at distance i, set

$$c_i(u,v) := |G_1(v) \cap G_{i-1}(u)|,$$
$$a_i(u,v) := |G_1(v) \cap G_i(u)|, \text{ and}$$
$$b_i(u,v) := |G_1(v) \cap G_{i+1}(u)|.$$

- These parameters are well-defined if they are independent of the choice of u, v. In this case we use the symbols c_i, a_i and b_i for short.
- A connected graph G with diameter D is called distance-regular if the above-mentioned parameters are well-defined.

- Assume that adjacency matrix A has $d+1$ distinct eigenvalues $\lambda_0 > \lambda_1 > \ldots > \lambda_d$ with corresponding multiplicities $m_0 = 1$, m_1, …, m_d.

- The spectrum of G is denoted by the multiset

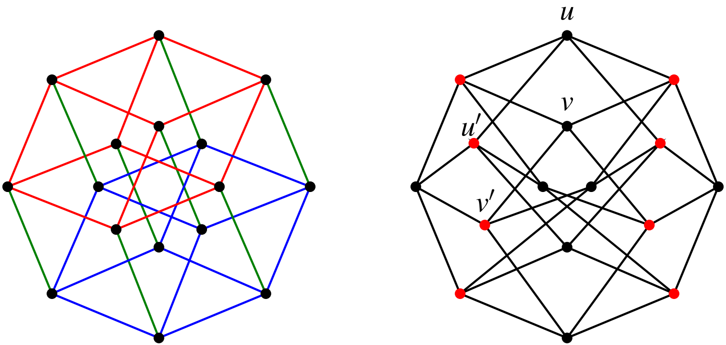
$$\text{sp } G = \{\lambda_0^{m_0}, \lambda_1^{m_1}, \ldots, \lambda_d^{m_d}\}.$$

- The parameter d is called the spectral diameter of G.

  Note that $D \leq d$.

Question: Is the distance-regularity of a graph determined by its spectrum?

Answer: In general, the answer is negative.



The Hamming 4-cube and the Hoffman graph
(distance-regular) (c_2 is not well-defined)

We have known that the distance-regularity of a graph is in general not determined by its spectrum.

Question: Under what additional conditions, the answer is positive?

Answer: The spectral excess theorem.

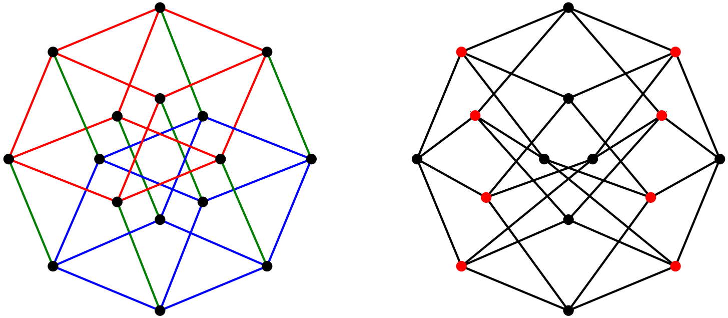
The spectral excess theorem gives a quasi-spectral characterization for a regular graph to be distance-regular.

Spectral excess theorem (Fiol and Garriga, 1997)

Let G be a regular graph with $d+1$ distinct eigenvalues. Then, $\overline{k}_d \leq p_d(\lambda_0)$, and equality is attained if and only if G is distance-regular. $\square$

- $\overline{k}_d$: average excess (combinatorial aspect) – the mean of the numbers of vertices at distance d from each vertex
- $p_d(\lambda_0)$: spectral excess (algebraic aspect) – a number which can be computed from the spectrum

Therefore, besides the spectrum, a simple combinatorial property suffices for a regular graph to be distance-regular.



The Hamming 4-cube and the Hoffman graph

$(\bar{k}_d = 1 = p_d(\lambda_0))$ $(\bar{k}_d = 1/2 < 1 = p_d(\lambda_0))$

Spectral excess theorem (Fiol and Garriga, 1997)

Let G be a regular graph with $d+1$ distinct eigenvalues. Then, $\overline{k}_d \leq p_d(\lambda_0)$, and equality is attained if and only if G is distance-regular. $\square$

- An example will be given to demonstrates that this theorem cannot directly apply to nonregular graphs.
- Thus, a 'weighted' version of the spectral excess theorem is given in order to make it applicable to nonregular graphs.

Using spectral excess theorem, van Dam and Haemers proved the following odd-girth theorem for regular graphs.

Odd-girth theorem (van Dam and Haemers, 2011)

A connected regular graph with $d+1$ distinct eigenvalues and odd-girth $2d+1$ is distance-regular. □

- In the same paper, they posed the question to determine whether the regularity assumption can be removed.
- Moreover, they showed that the answer is affirmative for the case $d+1=3$, and claimed to have proofs for the cases $d+1 \in \{4,5\}$.
- For an application of the 'weighted' spectral excess theorem, we show that the regularity assumption is indeed not necessary.

We then apply this line of study to the class of bipartite graphs.

- The distance-2 graph G^2 of G is the graph whose vertex set is the same as of G, and two vertices are adjacent in G^2 if they are of distance 2 in G.
- For a connected bipartite graph, the halved graphs are the two connected components of its distance-2 graph.
- For an integer $h \leq d$, we say that G is weighted h-punctually distance-regular if $\widetilde{A}_h = p_h(A)$.

Proposition (BCN, Proposition 4.2.2, p.141)

The halved graphs of a bipartite distance-regular graph are distance-regular. □

Problem (The converse statement)

Suppose that G is a connected bipartite graph, and both halved graphs are distance-regular. Can we say that G is distance-regular? If not, what additional conditions do we need?

Answer:

- Three examples will be given to show that the converse does not hold, that is, a connected bipartite graph whose halved graphs are distance-regular may not be distance-regular.
- We will give a quasi-spectral characterization of a connected bipartite weighted 2-punctually distance-regular graph whose halved graphs are distance-regular.
- In the case the spectral diameter is even we show that the graph characterized above is distance-regular.

Part I

A spectral excess theorem for nonregular graphs

Notations

- Let G be a connected graph of order n and diameter D.
- Assume that adjacency matrix A has $d+1$ distinct eigenvalues $\lambda_0 > \lambda_1 > \ldots > \lambda_d$ with corresponding multiplicities $1 = m_0$, m_1, $\ldots$, m_d.
- The spectrum of G is denoted by the multiset

$$\text{sp } G = \{\lambda_0^{m_0}, \lambda_1^{m_1}, \ldots, \lambda_d^{m_d}\}.$$

- The parameter d is called the spectral diameter of G.

  Note that $D \leq d$.

Two kinds of inner products

- Consider the vector space $\mathbb{R}_d[x]$ consisting of all real polynomials of degree at most d with the inner product

$$\langle p,q\rangle_G := \mathrm{tr}(p(A)q(A))/n = \sum_{u,v}(p(A)\circ q(A))_{uv}/n,$$

  for $p,q\in\mathbb{R}_d[x]$, where $\circ$ is the entrywise product of matrices.

- For any two $n\times n$ symmetric matrices M,N over $\mathbb{R}$, define the inner product

$$\langle M,N\rangle := \frac{1}{n}\sum_{i,j}(M\circ N)_{ij},$$

  where " $\circ$ " is the entrywise product of matrices.

- Thus $\langle p,q\rangle_G = \langle p(A),q(A)\rangle$ for $p,q\in\mathbb{R}_d[x]$.

Two kinds of polynomials

- By the Gram–Schmidt procedure, there exist polynomials $p_0, p_1, \ldots, p_d$ in $\mathbb{R}_d[x]$ satisfying

$$\deg\, p_i = i \qquad \text{and} \qquad \langle p_i, p_j \rangle_G = \delta_{ij} p_i(\lambda_0)$$

  for $0 \le i, j \le d$, where $\delta_{ij} = 1$ if $i = j$, and 0 otherwise.

  These polynomials are called the predistance polynomials of G.

- The polynomial

$$H := n \prod_{i=1}^{d} \frac{x - \lambda_i}{\lambda_0 - \lambda_i}$$

  is called the Hoffman polynomial of G.

- The sum of all predistance polynomials gives the Hoffman polynomial, i.e.,

$$H = p_0 + p_1 + \cdots + p_d.$$

- average excess (combinatorial aspect): the mean of the numbers of vertices at distance d from each vertex
- spectral excess (algebraic aspect): a number which can be computed from the spectrum

The parameter

$$\overline{k}_d := \frac{1}{n} \sum_{u \in V} |G_d(u)|$$

is called the average excess of G, and the parameter $p_d(\lambda_0)$ is called the spectral excess of G.

Lemma (Fiol and Garriga, 1997)

The spectral excess $p_d(\lambda_0)$ can be expressed in terms of the spectrum, which is

$$p_d(\lambda_0) = \frac{n}{\pi_0^2}\left(\sum_{i=0}^{d}\frac{1}{m_i\pi_i^2}\right)^{-1},$$

where $\pi_i = \prod_{j\neq i}|\lambda_i - \lambda_j|$ for $0 \le i \le d$. □

Spectral excess theorem (Fiol and Garriga, 1997)

Let G be a regular graph with $d+1$ distinct eigenvalues. Then, $\overline{k}_d \le p_d(\lambda_0)$, and equality is attained if and only if G is distance-regular. □

- For any two $n \times n$ symmetric matrices M, N over $\mathbb{R}$, let

$$\mathrm{Proj}_N M := \frac{\langle N, M \rangle}{\langle N, N \rangle} N$$

denote the projection of M onto $\mathrm{Span}\{N\}$.

- Let A_i be the distance-i matrix, i.e., an $n \times n$ matrix with rows and columns indexed by the vertex set V such that

$$(A_i)_{uv} = \begin{cases} 1, & \text{if } \partial(u,v) = i; \\ 0, & \text{else.} \end{cases}$$

In particular, $A_0 = I$ and $A_1 = A$.

Spectral excess theorem (Fiol and Garriga, 1997)

Let G be a regular graph with $d+1$ distinct eigenvalues. Then, $\overline{k}_d \leq p_d(\lambda_0)$, and equality is attained if and only if G is distance-regular.

Sketch of proof (an elegant proof, Fiol, Gago and Garriga, 2010)

- Show that $\mathrm{Proj}_{A_d} p_d(A) = A_d$, which implies that $\overline{k}_d \leq p_d(\lambda_0)$.
- Observe that equality holds if and only if $A_d = p_d(A)$.
- Show that $A_d = p_d(A)$ if and only if $A_i = p_i(A)$ for $0 \leq i \leq d$, that is, G is distance-regular. □

The following example shows that the regularity assumption of G in the spectrum excess theorem is necessary, that is, spectral excess theorem cannot directly apply to nonregular graphs.

Example

Let G be a path of three vertices. Note that $D = d = 2$, $\overline{k}_2 = 2/3$ and $p_2(\lambda_0) = 1/2$. This shows that $\overline{k}_d \leq p_d(\lambda_0)$ does not hold for nonregular graphs.

Preparation for the 'weighted' version

- Let α be the eigenvector of A (usually called the Perron vector) corresponding to λ_0 such that $\alpha^t\alpha = n$ and all entries of α are positive. Note that $\alpha = (1, 1, \ldots, 1)^t$ iff G is regular.

- For a vertex u, let α_u be the entry corresponding to u in α.

  The matrix $\widetilde{A}_i$ with rows and columns indexed by the vertex set V such that

$$(\widetilde{A}_i)_{uv} = \begin{cases} \alpha_u\alpha_v & \text{if } \partial(u,v) = i, \\ 0 & \text{otherwise} \end{cases}$$

  is called the weighted distance-i matrix of G.

  Note that $\widetilde{A}_i := 0$ if $i > D$.

- Recall that $\alpha = (1, 1, \ldots, 1)^t$ iff G is regular.

  If G is regular, then $\widetilde{A}_i$ is binary and hence the distance-i matrix A_i.

  Thus, $\widetilde{A}_i$ can be regarded as a 'weighted' version of A_i.

- Define $\widetilde{\delta}_i := \langle \widetilde{A}_i, \widetilde{A}_i \rangle$. If G is regular, then

$$\widetilde{\delta}_d = \langle \widetilde{A}_d, \widetilde{A}_d \rangle = \langle A_d, A_d \rangle = \frac{1}{n} \sum_{u \in VG} |G_d(u)| = \overline{k}_d.$$

  Hence, the parameter $\widetilde{\delta}_d$ can be viewed as a generalization of the average excess $\overline{k}_d$.

Spectral excess theorem and its 'weighted' version

Recall that

Spectral excess theorem (Fiol and Garriga, 1997)

Let G be a regular graph with $d+1$ distinct eigenvalues. Then, $\overline{k}_d \leq p_d(\lambda_0)$, and equality is attained if and only if G is distance-regular. □

Weighted spectral excess theorem (Lee and Weng, 2012)

Let G be a graph with $d+1$ distinct eigenvalues. Then, $\widetilde{\delta}_d \leq p_d(\lambda_0)$, and equality is attained if and only if G is distance-regular.

Weighted spectral excess theorem (Lee and Weng, 2012)

Let G be a graph with $d+1$ distinct eigenvalues. Then, $\widetilde{\delta}_d \leq p_d(\lambda_0)$, and equality is attained if and only if G is distance-regular.

Sketch of proof

- Show that $\mathrm{Proj}_{\widetilde{A}_d}(p_d(A)) = \widetilde{A}_d$, which implies that $\widetilde{\delta}_d \leq p_d(\lambda_0)$.
- Observe that equality holds if and only if $\widetilde{A}_d = p_d(A)$.
- Show that $\widetilde{A}_d = p_d(A)$ if and only if $\widetilde{A}_i = p_i(A)$ for $0 \leq i \leq d$.
- Show that $\widetilde{A}_0 = p_0(A)$ if and only if G is regular.
- Finally, we deduce that equality holds if and only if $A_i = p_i(A)$ for $0 \leq i \leq d$, that is, G is distance-regular. □

Revisiting the example that G is a path of three vertices.

Example

Let G be a path of three vertices.

- Note that $D = d = 2$, $\overline{k}_2 = 2/3$ and $p_2(\lambda_0) = 1/2$. This shows that $\overline{k}_d \leq p_d(\lambda_0)$ does not hold for nonregular graphs, that is, the regularity assumption in the spectrum excess theorem is necessary.

- Note that the Perron vector $\alpha = (\sqrt{3}/2, \sqrt{6}/2, \sqrt{3}/2)^t$, and

$$\widetilde{A}_2 = \begin{pmatrix} 0 & 0 & 3/4 \\ 0 & 0 & 0 \\ 3/4 & 0 & 0 \end{pmatrix}.$$

  Hence $\widetilde{\delta}_d = 3/8 < 1/2 = p_d(\lambda_0)$, that is, the weighted spectral excess theorem can apply to nonregular graphs.

An application: graphs with odd-girth $2d+1$

Odd-girth theorem (van Dam and Haemers, 2011)

A connected regular graph with $d+1$ distinct eigenvalues and odd-girth $2d+1$ is distance-regular. □

Odd-girth theorem (Lee and Weng, 2012)

A connected graph with $d+1$ distinct eigenvalues and odd-girth $2d+1$ is distance-regular.

Sketch of proof

Show that $\widetilde{\delta}_d = p_d(\lambda_0)$, and the result follows by weighted spectral excess theorem. □

Part II

A characterization of bipartite distance-regular graphs

- The distance-2 graph G^2 of G is the graph whose vertex set is the same as of G, and two vertices are adjacent in G^2 if they are of distance 2 in G.
- For a connected bipartite graph, the halved graphs are the two connected components of its distance-2 graph.
- For an integer $h \leq d$, we say that G is weighted h-punctually distance-regular if $\widetilde{A}_h = p_h(A)$.

Proposition (BCN, Proposition 4.2.2, p.141)

The halved graphs of a bipartite distance-regular graph are distance-regular. □

Problem (The converse statement)

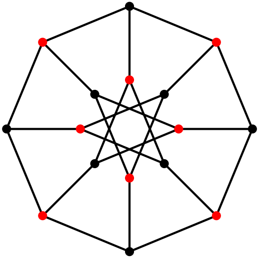
Suppose that G is a connected bipartite graph, and both halved graphs are distance-regular. Can we say that G is distance-regular? If not, what additional conditions do we need?

Three counterexamples

Example 1 (weighted 2-punctually distance-regular & odd spectral diameter)

The Möbius–Kantor graph, with spectrum $\{3^1, \sqrt{3}^4, 1^3, (-1)^3, (-\sqrt{3})^4, (-3)^1\}$.

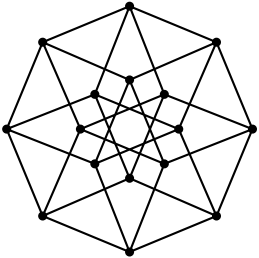
- $D = 4 < 5 = d$,
- $\widetilde{A}_i = p_i(A)$ for $i \in \{0, 1, 2, 4\}$, and
- both halved graphs the complete multipartite graphs $K_{2,2,2,2}$ (with spectrum $\{6^1, 0^4, (-2)^3\}$), which are distance-regular.

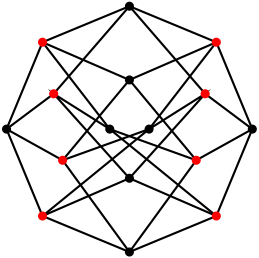


Example 2 (not weighted 2-punctually distance-regular & even spectral diameter)

Consider the Hoffman graph with spectrum $\{4^1, 2^4, 0^6, (-2)^4, (-4)^1\}$, which is cospectral to the Hamming 4-cube but not distance-regular.

- $D = d = 4$,
- $\widetilde{A}_i = p_i(A)$ for $i \in \{0, 1, 3\}$ ($i \neq 2$), and
- its two halved graphs are the complete graph K_8 and the complete multipartite graph $K_{2,2,2,2}$, which are both distance-regular.

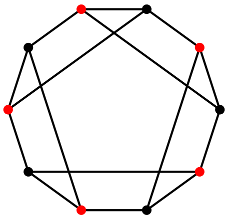




Example 3 (not weighted 2-punctually distance-regular & odd spectral diameter)

Consider the graph obtained by deleting a 10-cycle from the complete bipartite graph $K_{5,5}$, with spectrum
$\{3^1, ((\sqrt{5}+1)/2)^2, ((\sqrt{5}-1)/2)^2, ((-\sqrt{5}+1)/2)^2, ((-\sqrt{5}-1)/2)^2, (-3)^1\}$.

- $D = 3 < 5 = d$,
- $\widetilde{A}_i = p_i(A)$ for $i \in \{0,1\}$ $(i \neq 2)$, and
- both halves graphs are the complete graphs K_5, which are distance-regular.



The remaining case

We have considered three counterexamples.

Example 1 (weighted 2-punctually distance-regular & odd spectral diameter)
Example 2 (not weighted 2-punctually distance-regular & even spectral diameter)
Example 3 (not weighted 2-punctually distance-regular & odd spectral diameter)

Note that the remaining case is that

G is weighted 2-punctually distance-regular with even spectral diameter.

Question: How about the remaining case?

Answer: Under these additional conditions, the converse statement is true.

Theorem (Lee and Weng, 2014)

Suppose that G is a connected bipartite graph, and both halved graphs are distance-regular. If G is weighted 2-punctually distance-regular with even spectral diameter, then G is distance-regular.

Sketch of proof

- By the weighted 2-punctually distance-regularity assumption,
  - G is regular, and
  - both halved graphs have the same spectrum, and thus have the same (pre)distance-polynomials.
- By the above results and the even spectral diameter assumption,
  - $\widetilde{\delta}_d = p_d(\lambda_0)$, and the result follows by (weighted) spectral excess theorem. □

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Thank you for your listening!