國 立 交 通 大 學

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碩士論文

特徵值插值定理對無三角形圖之探討

The study of triangle-free graphs
by interlacing theorem for eigenvalues

研 究 生：蘇慧文　　　　　　　　Student：Hui-Wen Su

指導教授：翁志文 教授　　　　　　Advisor：Chih-Wen Weng

中華民國一百零一年六月

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碩士論文

A Thesis

Submitted to Department of Applied Mathematics

College of Science

National Chiao Tung University

in Partial Fulfillment of the Requirements

for the Degree of Master

In

Applied Mathematics

June 2012

Hsinchu, Taiwan, Republic of China

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摘　　　要

　　這篇論文利用特徵值插值定理探討無三角形的圖，進而以圖的特徵值及圍長，刻劃參數為$(k^2+1, k, 0, 1)$之強正則圖。

The study of triangle-free graphs
by interlacing theorem for eigenvalues

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Abstract

The thesis applies interlacing theorem for eigenvalues to study graphs without triangle. We give a characterization of strongly regular graph $\mathrm{srg}(k^2+1, k, 0, 1)$ in terns of eigenvalues and the girth of a graph.

Keywords：Adjacency matrix, Interlacing theorem, Eigenvalues, Quotient matrix.

誌謝

　　首先由衷的感謝我的指導教授翁志文老師，對於駑鈍的我老師始終耐心的指導，不時的討論並指引我研究的方向，老師對學問的嚴謹態度更是我們所要學習的典範。本論文的完成另外亦得感謝系上諸多師長在這兩年內的指導，若沒有師長們點滴的教導，就沒有今日的我。

　　在完成論文的這段期間內，發生了很多突發狀況，在繳初稿的前一天，我的筆記型電腦故障了，硬碟資料險些報銷，感謝資料救援的先生幫了我一個大忙，禍不單行，緊接著我得到了腸胃炎，隔了兩天因為胃炎藥物過敏，引起急性蕁麻疹，論文險些來不及，但總算完成了，這期間很感謝老師的體諒，以及朋友們的加油打氣。

　　這兩年的校園生活，研究室裡共事的點點滴滴，回想碩一每逢作業考試，幾乎快夜宿研究室的革命情感，大家共同砥礪(共同墮落?)都好像昨日才剛發生，因為有你們生活多采多姿。感謝家安、光祥、函恩學長們不厭其煩的指出我研究上的問題，以及逸軒、明淇學長在我人生迷惘時為我解惑，感謝如汶同學的相互扶持，恭喜我們順利走過這兩年，也謝謝稟鈞學弟，在微積分助教課程上的協助，時時體諒我。畢業只是暫時的離別，願大家一起邁向美好的前程，並期待下一次的相遇。對於男友盛楷這兩年在背後的默默支持，發自內心地感謝他的體諒、包容，雖然分隔兩地，但依舊沒有忘記聯繫，若沒有他的鼓勵這兩年的生活定是很不一樣的光景。

　　最後，謹以此文獻給我摯愛的母親以及家人，因為你們的支持，讓我能無慮的完成這兩年碩士的學業，並沒有因為性別而要求我女子無才便是德，把一切榮耀歸給上帝，感謝主。

<div style="text-align: right">

研究生：蘇慧文

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中華民國一百零一年六月

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Contents

Chapter 1

Introduction

This research is about the eigenvalues or the spectrum of the adjacency matrix of a graph. The adjacency matrix of a graph and its eigenvalues can be used in several areas, for examples mathematical research, physical interpretation, chemical and so on. It was investigated very much in the past. Spectrum of a graph have appeared frequently in the mathematical literature since a few fundamental papers, e.g. L. Collatz and U.Sinogowitz [8]. Theoretical chemists were also interested in graph spectra, although they used different terminology.

Spectral graph theory is the study of properties of a graph in relationship to the characteristic polynomial, eigenvalues, and eigenvectors of matrices associated to the graph, such as its adjacency matrix or Laplacian matrix. And spectral graph theory emerged in the 1950s and 1960s. Besides graph theoretic research on the relationship between structural and spectral properties of graphs, another major source was research in quantum chemistry (e.g. energy of graphs), but the connections between these two lines of work were not discovered until much later [5].

Research papers on the bounds for whose second largest eigenvalue of the adjaceny matrix of a graph are a lot. In [10], considering the graphs whose second largest eigenvalue at most 1, the author has almost summed up most of regular graphs with eigenvalues corresponding

to this range. And he was almost determing graphs with the second largest eigenvalue less than or equal to 1. Even some undetermined graphs also have the second largest eigenvalue is not more than 1, he gave examples to illustrate.

The main tool we used is interlacing theorem for eigenvalues. Eigenvalue interlacing theorem has been applied to graphs in many papers. For instance, Brouwer and Mesner [1] used it to prove that the connectivity of a strongly regular graph equals its degrees and in Brouwer and Haemers [2] eigenvalue interlacing is a basic tool for their proof of the uniqueness of Gewirtz graph. To obtain our results we use quotient matrices of the adjacency matrix of a graph with respect to some partition of the vertices.

Finally, the graphs that we are interested in is a connected simple graph. A simple graph is an undirected graph without loops and multiple edges. The adjacency matrix of a simple graph is a $(0, 1)$-matrix with zeros on its diagonal. If the graph is undirected, the adjacency matrix is symmetric.
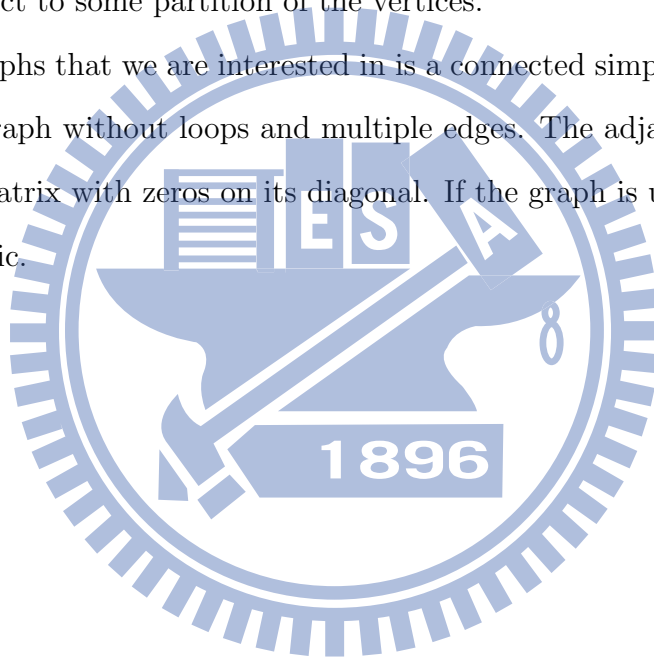
Chapter 2

Preliminaries

In this chapter we introduce basic definitions and theorems which will be used throughout this thesis.

2.1 Adjacent matrix

In mathematics and computer science, an adjacency matrix is a mean of representing which vertices (or nodes) of a graph are adjacent to which other vertices.

Definition 2.1.1. Let G be a graph with vertex set $V(G) = \{v_1, v_2, ..., v_n\}$ and $E(G) = \{e_1, \cdots, e_m\}$. **The adjacency matrix** of graph G, written $A(G)$, is the $n \times n$ matrix defined as follows. The rows and the columns of $A(G)$ are indexed by $V(G)$. If $i \neq j$ then the (i, j)-entry of $A(G)$ is 0 for vertices i and j nonadjacent, and the (i, j)-entry is 1 for i and j adjacent. The (i, i)-entry of $A(G)$ is 0 for $i = 1, \ldots, n$. We often denote $A(G)$ simply by A.

Example 2.1.2. Consider the graph G shown below, and its adjacency matrix $A(G)$.

$$G: \qquad , \qquad A(G): \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix},$$
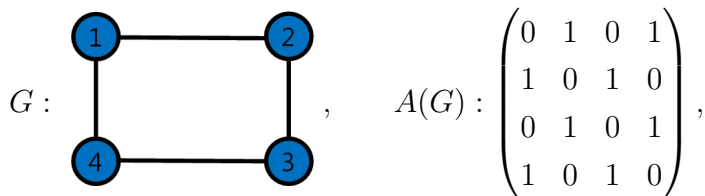
where coordinates are $1, 2, 3, 4$.

2.2 Eigenvalue and Eigenvector

Definition 2.2.1. Let A be an $n \times n$ matrix over $\mathbb{R}$. The number $\lambda \in \mathbb{C}$ is an **eigenvalue** of A if there exists a nonzero column vector u such that $Au = \lambda u$. The vector u is called an **eigenvector** of A associated with λ.

Example 2.2.2. If

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

then the eigenvalues of A are $\lambda_1 = 3, \lambda_2 = 2, \lambda_3 = 1$, and corresponding eigenvector

$$u_1 = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T, u_2 = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}^T, u_3 = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^T.$$

2.3 Quotient matrix

Computation of eigenvalues of a matrix can be very difficult. We now introduce a way, which enables us to obtain information about the eigenvalues of a matrix from a smaller matrix.

Definition 2.3.1. Consider $n \times n$ an symmetric matrix A and an $m \times m$ matrix U in block form

$$A = \begin{pmatrix} A_{1,1} & \cdots & A_{1,m} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,m} \end{pmatrix}, U := \begin{pmatrix} \mathbf{1}n_1 & & 0 \\ & \ddots & \\ 0 & & \mathbf{1}_{n_m} \end{pmatrix}$$

where $A_{i,j}$ is an $n_i \times n_j$ matrix, $n_1 + n_2 + \cdots + n_m = n$, and $\mathbf{1}_{n_i}$ is the all 1's column vector of size n_i. Let D be the $m \times m$ diagonal matrix $D := \mathrm{diag}((\sqrt{n_1})^{-1}, (\sqrt{n_2})^{-1}, \ldots, (\sqrt{n_m})^{-1})$. Hence $S := UD$ satisfies $S^T S = I_m$. Note that $N = S^T A S$ and $B := DND^{-1}$. The (i,j) entry $b_{i,j} = \mathbf{1}_{n_i}^T A_{i,j} \mathbf{1}_{n_j}/n_i$ of B is the average row sum of $A_{i,j}$. B is called a **quotient matrix** of A. Moreover B is an **equitable quotient matrix** of A if $A_{i,j}\mathbf{1}_{n_j} = B_{i,j}\mathbf{1}_{n_i}$. i.e. Each row-sum of A_{ij} has constant value $B_{i,j}$.

In the following example, one can see from the diagrams how a quotient matrix is obtained from a matrix.

Example 2.3.2.

$$\begin{pmatrix} 1 & 2 \\ 5 & 7 \end{pmatrix} \rightarrow \begin{pmatrix} 7.5 \end{pmatrix} \tag{2.1}$$

$$\left(\begin{array}{c|c|cc} 1 & 2 & 3 & 4 \\ \hline 5 & 6 & 8 & 1 \\ \hline 2 & 1 & 3 & 2 \\ 2 & 3 & 1 & 4 \end{array}\right) \rightarrow \begin{pmatrix} 1 & 2 & 7 \\ 5 & 6 & 9 \\ 2 & 2 & 5 \end{pmatrix} \tag{2.2}$$

The matrix that we are concerned with in this thesis is the adjacency matrix $A = A(G)$ of a graph G. The quotient matrix of $A(G)$ has combinatorial meaning. Let $\pi = C_1 \cup \cdots \cup C_k$ be a partition of the vertex set $V(G)$, and the matrix A is partitioned according to $\pi = C_1 \cup \cdots \cup C_k$ in block form such that

$$A = \begin{pmatrix} A_{1,1} & \cdots & A_{1,k} \\ \vdots & \ddots & \vdots \\ A_{k,1} & \cdots & A_{k,k} \end{pmatrix},$$

where $A_{i,j}$ denotes the block of A formed by rows in C_i and the columns in C_j. Let $b_{i,j}$ denote the average row-sum of $A_{i,j}$. If the average row-sums of each block $A_{i.j}$ is a constant $b_{i,j}$, then the partition $\pi = C_1 \cup \cdots \cup C_k$ is called **equitable**. The matrix $A^\pi := [b_{i,j}]$ is a quotient matrix B of A in the above definition.

We introduce a way to obtain a 3×3 quotient matrix B of a connected graph G. Pick a vertex $x \in V(G)$. Let $\Gamma_i := \{y \in V(G) | d(x,y) = i\}$ and $\Gamma_{\geq i} := \{y \in V(G) | d(x,y) \geq i\}$, where $d(x,y)$ is the distance between vertices x and y. According to the partition $\pi = \{x\} \cup \Gamma_1(x) \cup \Gamma_{\geq 2}(x)$ we have the following quotient matrix $B = A^\pi$ of the adjacency matrix $A(G)$.

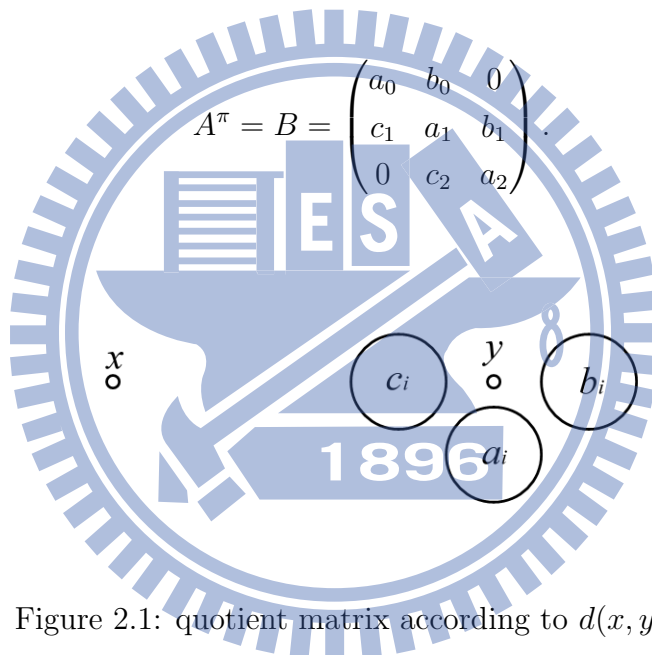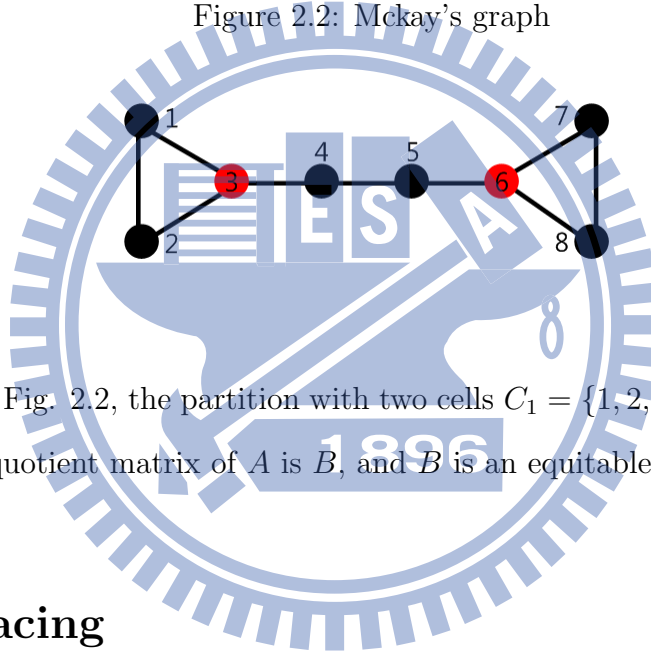$$A^\pi = B = \begin{pmatrix} a_0 & b_0 & 0 \\ c_1 & a_1 & b_1 \\ 0 & c_2 & a_2 \end{pmatrix}.$$

Figure 2.1: quotient matrix according to $d(x,y) = i$.

The following example, copied from [4], provides a different way to have quotient matrix.

Example 2.3.3. (Mckay's graph)

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 3 & 0 \end{pmatrix}.$$

Figure 2.2: Mckay's graph

In the graph of Fig. 2.2, the partition with two cells $C_1 = \{1, 2, 4, 5, 7, 8\}$ and $C_2 = \{3, 6\}$ is equitable. The quotient matrix of A is B, and B is an equitable quotient of A.

2.4 Interlacing

The eigenvalues of adjacency matrix $A(G)$ will be denoted by $\lambda_1, \cdots, \lambda_n$. Unless we indicate otherwise, we shall assume that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. The **eigenvalues** or the **spectrum** of a graph G is referred to the eigenvalues of $A(G)$.

Eigenvalues of graphs have application in chemistry, and it is called Huckel theory [5]. For instance, a (carbon) molecule is chemically stable if its underlying graph has half of its eigenvalues positive and half of its eigenvalues negative. The paper [9] consider the more general question how to make graphs (on an even number of vertices) with $\lambda_{\frac{1}{2}n} \geq 0$ and $\lambda_{\frac{1}{2}n+1} \leq 0$. Their method essentially uses interlacing.

Understanding the definition of interlacing and its properties is the main tool in Chapter $3, 4$ to estimate the eigenvalues of a graph.

Definition 2.4.1. For $m < n$, the sequence $\theta_1 \geq \theta_2 \geq \ldots \geq \theta_m$ is said to **interlace** the sequence $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ whenever

$$\lambda_i \geq \theta_i \geq \lambda_{n-m+i} \text{ for } 1 \leq i \leq m.$$

The above interlacing is **tight** if there exists an integer ℓ such that $\lambda_i = \theta_i$ for $1 \leq i \leq \ell$, and $\theta_i = \lambda_{n-m+i}$ for $\ell + 1 \leq i \leq m$.

Example 2.4.2. The sequence $6, 4, 3, 1$ interlaces the sequence $6, 5, 4, 3, 2, 1$. The interlacing is not tight since $5 > 4 > 3$. The sequence $6, 5, 2, 1$ interlaces the sequence $6, 5, 4, 3, 2, 1$ tightly.

The following theorem will be used in this thesis. See [3] for a proof.

Theorem 2.4.3. (Interlacing theorem) [3]

Let A be a symmetric matrix and B a quotient matrix of A. Then the eigenvalues of B interlace the eigenvalues of A.

Throughout the thesis for an $m \times m$ quotient matrix B of A, we use $\theta_1 \geq \theta_2 \geq \cdots \geq \theta_m$ to denote its eigenvalues. The following diagram illustrates this case $m = 3$.

$A : \lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_{n-2}, \lambda_{n-1}, \lambda_n.$
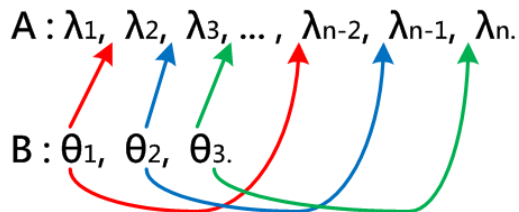
$B : \theta_1, \theta_2, \theta_3.$

Figure 2.3: the eigenvalues of B interlace the eigenvalues of A

The following example is about the adjacency matrix A of Petersen graph G and its quotient matrix B according to the partition $\pi = \{x\} \cup \Gamma_1(x) \cup \Gamma_2(x)$ for any $x \in V(G)$.

Example 2.4.4 (Petersen graph)**.**

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 3 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{pmatrix}.$$
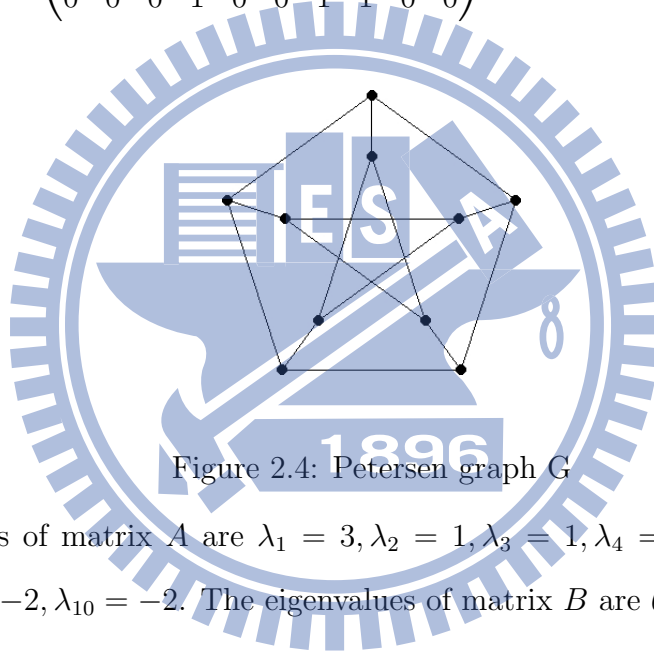

Figure 2.4: Petersen graph G

The eigenvalues of matrix A are $\lambda_1 = 3, \lambda_2 = 1, \lambda_3 = 1, \lambda_4 = 1, \lambda_5 = 1, \lambda_6 = 1, \lambda_7 = -2, \lambda_8 = -2, \lambda_9 = -2, \lambda_{10} = -2$. The eigenvalues of matrix B are $\theta_1 = 3, \theta_2 = 1, \theta_3 = -2$.

We can check that $\lambda_1 = 3 \geq \theta_1 = 3 \geq \lambda_8 = -2, \lambda_2 = 1 \geq \theta_2 = 1 \geq \lambda_9 = -2,$ $\lambda_3 = 1 \geq \theta_3 = -2 \geq \lambda_{10} = -2$, so Theorem 2.4.3 holds. Indeed the interlacing of B on A is tight.

2.5 Diameter

The **diameter** of a graph G is the value $\max_{u,v} d(u,v)$ among vertices $u, v \in V(G)$. In other words, a graph's diameter is the largest number of vertices which must be traversed in

order to travel from one vertex to another when paths which backtrack, detour, or loop are excluded from consideration. The graphs in Fig. 2.5 on 10 vertices have diameters 3, 4, 5, and 7, respectively.
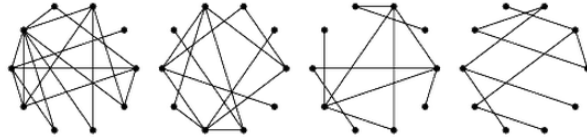

Figure 2.5: diameter

Definition 2.5.1. Let G be k-regular with v vertices. G is said to be an **strongly regular graph**(SRG) if there are integers λ and μ such that:

1. Every two vertices x, y with $d(x, y) = 1$ have λ common neighbours.

2. Every two nonadjacent vertices have μ common neighbours.

A graph of this kind is said to be an **srg**(v, k, λ, μ).

Chapter 3

K-regular graph

In this chapter we will study the bounds of eigenvalues on some k-regular graph with girth at least 5. A graph is said to be **regular** if all its vertices have the same degree. If the degree of each vertex of G is k, then G is said to be k-**regular**. Examples of regular graphs include cycles, complete graphs and so on. The **girth** of a graph is the length of a shortest cycle contained in the graph. If a graph does not contain any cycle, its girth is defined to be infinity. For example, the Petersen graph has girth 5. We fix the notation that A is adjacency matrix of G with eigenvalues λ_i in decreasing order and B is quotient matrix of A with eigenvalues θ_i in decreasing order.

3.1 Special case

Let G be a connected k-regular graph of order at least 3 and fix a vertex $x \in V(G)$. We give the following assumptions on G throughout the Sections 3.1 and 3.2.

Assumption：

1. G has no triangle;

2. Each vertex in $\Gamma_{\geq 2}(x)$ is adjacent to a unique vertex in $\Gamma_1(x)$. (Then $\Gamma_{\geq 2}(x) = \Gamma_2(x)$.)

11

Then according to the partition $\pi = \{x\} \cup \Gamma_1(x) \cup \Gamma_{\geq 2}(x)$, the quotient matrix A^π is

$$B = A^\pi = \begin{pmatrix} 0 & k & 0 \\ 1 & 0 & k-1 \\ 0 & 1 & k-1 \end{pmatrix},$$

and indeed A^π is an equtable quotient of A.

We can easily compute $\theta_1 = k, \theta_2 = (-1 + \sqrt{-3 + 4k})/2, \theta_3 = (-1 - \sqrt{-3 + 4k})/2$. Then by interlacing theorem in Theorem 2.4.3,

$$\lambda_2 \geq \theta_2 = \frac{-1 + \sqrt{-3 + 4k}}{2}, \theta_3 = \frac{-1 - \sqrt{-3 + 4k}}{2} \geq \lambda_n.$$

We are interested in the necessary and sufficient conditions for the following identity

$$\lambda_2 = \frac{-1 + \sqrt{-3 + 4k}}{2}.$$

By the following discussion, we might guess it be a strongly regular graph.

Lemma 3.1.1 ([3]). For an equitable partition, if u is an eigenvector of B for an eigenvalue λ, then Su is an eigenvector of A for the same eigenvalue λ, where S is defined in Definition 2.3.1.

So eigenvalues of the equitable quotient matrix of an adjacency matrix are also eigenvalues of the adjacency matrix.

Example 3.1.2. In special case with $k = 2$, $G = C_5$ is a cycle of five vertices as the following graph.
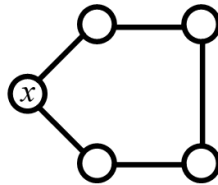
Hence the spectrum of C_5 is $2, (-\frac{1}{2} + \frac{1}{2}\sqrt{5}), (-\frac{1}{2} + \frac{1}{2}\sqrt{5}), (-\frac{1}{2} - \frac{1}{2}\sqrt{5}), (-\frac{1}{2} - \frac{1}{2}\sqrt{5})$.

Note that Spectrum $(A) = 2, (-\frac{1}{2} + \frac{1}{2}\sqrt{5})^2, (-\frac{1}{2} - \frac{1}{2}\sqrt{5})^2$.

$$B = \begin{pmatrix} 0 & 2 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Spectrum $(B) = 2, -\frac{1}{2} + \frac{1}{2}\sqrt{5}, -\frac{1}{2} - \frac{1}{2}\sqrt{5}$.

Note that B is a equitable quotient of A, and the eigenvalues of B interlace the eigenvalues of A tightly.

In special case with $k = 3$. We can obtain 2 cases. In the following, we discuss them respectively.

Example 3.1.3. In this example, we talk about the graph of case 1 and indeed the graph is Petersen graph.
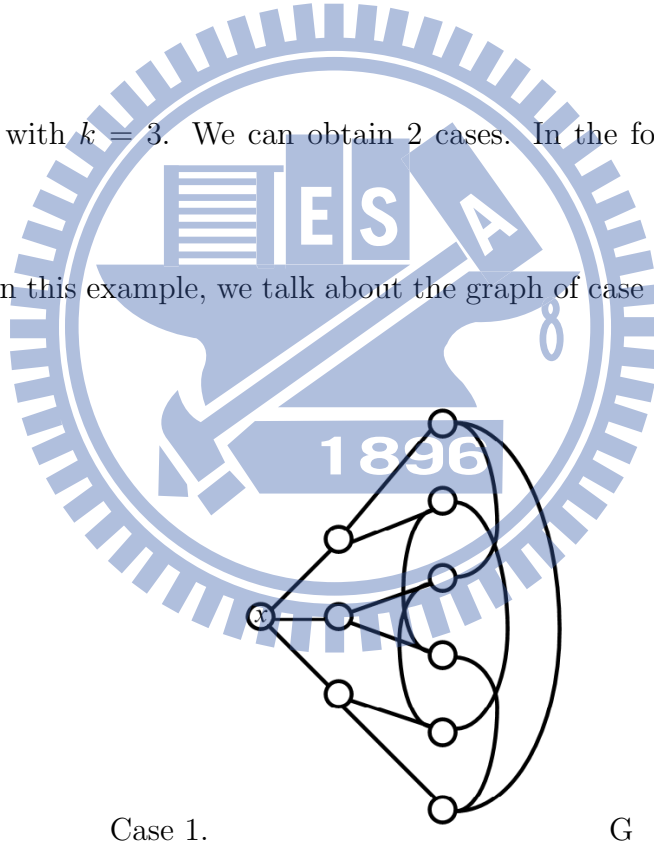
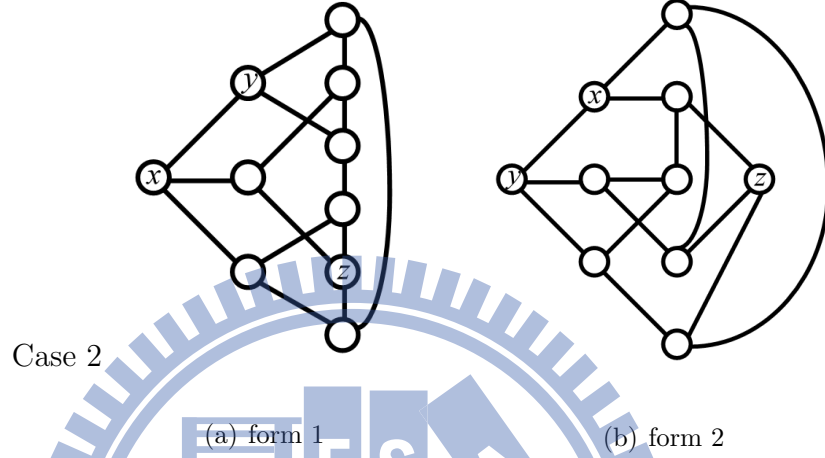Case 1.   G

Note that Spectrum $(A) = 3, 1^5, (-2)^4$. The quotient matrix of G is

$$B = \begin{pmatrix} 0 & 3 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{pmatrix},$$

and Spectrum $(B) = 3, 1, -2$.

13

In next example, we consider another example of 3-regular graph.

Example 3.1.4. The graph form 1 and form 2 are the same graph with different partitions of the vertex set.

Case 2

(a) form 1   (b) form 2

Let A be the adjacency matrix of the graph in case 2. Then we obtain Spectrum$(A) = 3, -\frac{1}{2} + \frac{1}{2}\sqrt{17}, 1^3, 0, -1, (-2)^2, -\frac{1}{2} - \frac{1}{2}\sqrt{17}$. We choose these partitions $\pi_1 = \{x\} \cup \Gamma_1(x) \cup \Gamma_2(x)$ and $\pi_2 = \{y\} \cup \Gamma_1(y) \cup \Gamma_2(y) \cup \Gamma_3(y)$ respectively, and two quotient matrices corresponding respectively as follow:

$$B_1 = A^{\pi_1} = \begin{pmatrix} 0 & 3 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{pmatrix}, B_2 = A^{\pi_2} = \begin{pmatrix} 0 & 3 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 6/5 & 6/5 & 3/5 \\ 0 & 0 & 3 & 0 \end{pmatrix}.$$

Note that Spectrum$(B_1) = 3, 1, -2$. We check these eigenvalues $3 \geq 3 \geq -2$, $(-1 + \sqrt{17})/2 \geq 1 \geq -2$, $1 \geq -2 \geq (-1 - \sqrt{17})/2$.

Note that Spectrum$(B_2) = 3, 1.145, -0.7, -2.24$. We check that $3 \geq 3 \geq -1$, $(-1 + \sqrt{17})/2 \geq 1.145 \geq -2$, $0 \geq -0.7 \geq -2$, $-1 \geq -2.24 \geq (-1 - \sqrt{17})/2$, so Theorem 2.4.3 also holds.

It is easy to see 1.145 is a better lower bound of λ_2 than 1, and -0.7 is a better lower bound of λ_3 than -2. In the part of upper bounds, 1.145 is a better upper bound of λ_8 than 3, -0.7 is a better upper bound of λ_9 than 1, and -2.24 is a better upper bound of λ_{10} than -2.

From the above example, the bounds obtained from eigenvalues of 4×4 quotient matrices could be better than the bounds from 3×3 quotient matrices, but we do not discuss it here. Our focus is 3×3 quotient matrices. Note that the two graphs in Example 3.1.3 and Example 3.1.4 are all the 3-regular graphs satisfying Assumption 3.1.

There are many graphs with $k = 4$. We only study one of them in the following example.

Example 3.1.5. Here we only pick a case to explain. The graph has the largest vertices with $d(y, z) = 4$ in special case with $k = 4$, where $y, z \in V(G)$
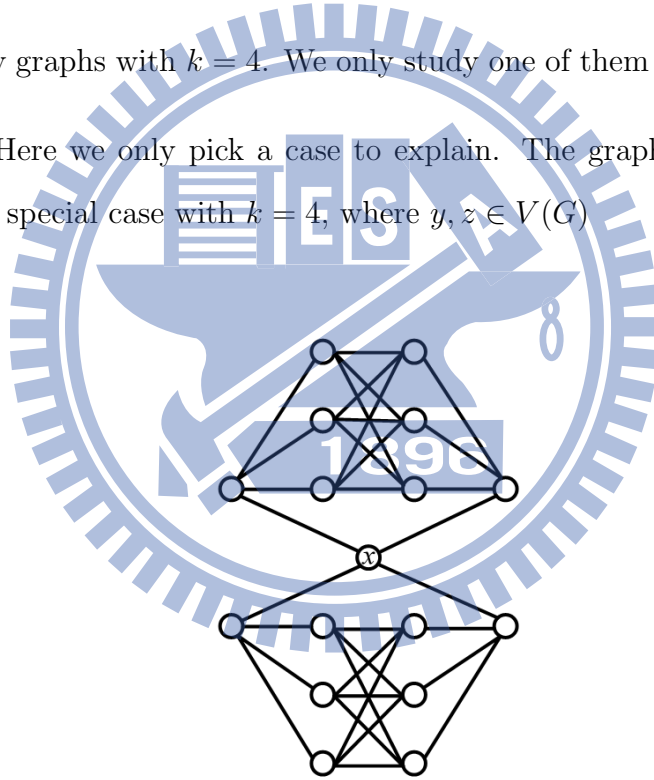
.

Figure 3.1: graph G from special case with $k = 4$

We calculate the eigenvalues of the adjacency matrix of graph in Figure 3.1 and its quotient matrix according to the partition $\pi = \{x\} \cup \Gamma_1(x) \cup \Gamma_{\geq 2}(x)$. We obtain the following result:

$\text{Spectrum}(A) = 4, 3.791, 1.303, (-\frac{3}{2}+\frac{1}{2}\sqrt{21})^2 = 0.791^2, 0^8, -0.791, -2.303, (-\frac{3}{2}-\frac{1}{2}\sqrt{21})^2 =$

-3.791^2, and Spectrum$(B) = 4, -\frac{1}{2} + \frac{1}{2}\sqrt{13} = 1.303, -\frac{1}{2} - \frac{1}{2}\sqrt{13} = -2.303$. Then we also check Theorem 2.4.3.

$$\lambda_1 = 4 \geq \theta_1 = 4 \geq \lambda_{15} = -2.303,$$

$$\lambda_2 = 3.791 \geq \theta_2 = 1.303 \geq \lambda_{16} = -3.791,$$

$$\lambda_3 = 1.303 \geq \theta_3 = -2.303 \geq \lambda_{17} = -3.791.$$

3.2 Diameters and eigenvalues

Under the Assumption 3.1, we have $d \leq 4$ no matter what k is. We consider $d = 2$ and $3 \leq d \leq 4$ separately. The following theorem characterizes the case $d = 2$.

Lemma 3.2.1. [7] If an srg$(k^2 + 1, k, 0, 1)$ exists, then $k = 1, 2, 3, 7, 57$.

Theorem 3.2.2. G is a connected k-regular graph with diameter 2 satisfying the Assumption 3.1 if and only if G is a strongly regular graph srg$(k^2 + 1, k, 0, 1)$, where $k = 2, 3, 7, 57$.

Proof. Let G be a srg$(k^2 + 1, k, 0, 1)$. Clealy G is k-regular satisfying the Assumption 3.1. Since $\mu \neq 0$, G is connected with diameter 2. On the other hand, let G be a connected k-regular graph with $d = 2$ satisfying the Assumption 3.1. The number of vertices is $|\{x\} \cup \Gamma_1(x) \cup \Gamma_{\geq 2}(x)| = 1 + k + (k - 1)k = k^2 + 1$. G is no triangle, so $\lambda = 0$. Finally, we need to check that $\mu = 1$. Pick any vertex $y \in V(G)/\{x\}$, then check each vertex in $\Gamma_2(y)$ is adjacent to a unique vertex in $\Gamma_1(y)$. Suppose there are two vertices $u_1, u_2 \in \Gamma_1(y)$ adjacent to $v_1 \in \Gamma_2(y)$. The remaining edges in $\Gamma_1(y) = k(k - 1) - 2 = k^2 - k - 2$, and the number of vertices in $\Gamma_2(y)/\{y\} = k^2 - k - 1$. It means there is at least one vertex in $\Gamma_2(y)$ not adjacnet to $\Gamma_1(y)$. Contradict to $d = 2$. So, every two vertices x, y with $d(x, y) = 2$ have $\mu = 1$ common neighbours. $\square$

The following theorem gives the relation between the diameter of a graph and its eigenvalues.

Theorem 3.2.3. [[6] Chung, 1989] Let G be a k-regular graph on $n \geq 3$ vertices with diameter d. Let $\lambda = \max_{i>1}|\lambda_i|$. Then $d \leq \lceil \frac{\log(n-1)}{\log(k/\lambda)} \rceil$.

Consider $d \geq 3$. By Theorem 3.2.3, we have a bound of $max_{i>1}|\lambda_i|$ as follows.

Theorem 3.2.4. Let G be a connected k-regular graph of order at least 3 with diameter d at least 3 satisfying the Assumption 3.1, and the 3×3 quotient matrix of the adjacency matrix is
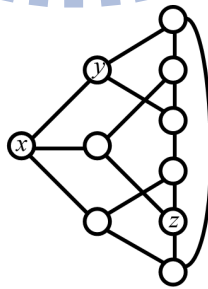
$$\begin{pmatrix} 0 & k & 0 \\ 1 & 0 & k-1 \\ 0 & 1 & k-1 \end{pmatrix}.$$

Then $\lambda > 1$, where $\lambda = \max_{i>1}|\lambda_i|$.

Proof. Apply $3 \leq d$ and $n = k^2 + 1$ to Theorem 3.2.3. We obtain $3 \leq \lceil \frac{\log(k^2)}{\log(k/\lambda)} \rceil$. Using $2 < \frac{\log(k^2)}{\log(k/\lambda)}$ can receive $\lambda > 1$, so the proof is complete. $\square$

We give an example to check the previous theorem.

Example 3.2.5. G is consistent with Assumption 3.1.

The eigenvalue $\max_{i>1}|\lambda_i|$ of adjacency matrix of graph G is $(-1 - \sqrt{17})/2$. We check that

$$|\lambda_{10}| = |\frac{(-1 - \sqrt{17})}{2}| = 2.5616 > 1.$$

3.3 General case with k-regular

In this section, we want to use Theorem 2.4.3 on general case with k-regular. Let G be a connected k-regular graph of order v with the following assumptions on G.

Assumption : The girth of G is at least 5;

Fix a vertex $x \in V(G)$. Then according to the partition $\pi = \{x\} \cup \Gamma_1(x) \cup \Gamma_{\geq 2}(x)$, the quotient matrix A^π is

$$B = A^\pi = \begin{pmatrix} 0 & k & 0 \\ 1 & 0 & k-1 \\ 0 & \dfrac{k(k-1)}{v-k-1} & \dfrac{vk-2k^2}{v-k-1} \end{pmatrix}. \tag{3.1}$$

We shall compute the eigenvalues of B.

Lemma 3.3.1. Let G be a k-regular graph of order $k^2 + 1$ satisfying Assumption 3.3. Then G has diameter 2.

Proof. Pick $x \in V(G)$. Note that $k^2 + 1 = v = |\{x\} \cup \Gamma_1(x) \cup \Gamma_2(x)|$. Hence G has diameter 2. $\qquad\square$

Lemma 3.3.2. If the girth of a k-regular graph G is at least 5, then

$$v \geq k^2 + 1,$$

where v is the number of vertices.

Proof. When the girth is at least 5, $v \geq |\{x\} \cup \Gamma_1(x) \cup \Gamma_2(x)|$ for any vertex $x \in V(G)$. Let G be a k-regular graph. Thus

$$v \geq 1 + k + (k-1)k = k^2 + 1.$$

$\qquad\square$

The entries of B follow from $|\Gamma_1| = k, |\Gamma_{\geq 2}| = v - k - 1$, and then

$$k - \frac{k(k-1)}{v-k-1} = \frac{vk - 2k^2}{v-k-1}.$$

Let v be an eignevector corresponding to eigenvalue λ, using $Bv = \lambda v$ to solve eigenvalues.

$$\begin{pmatrix} 0 & k & 0 \\ 1 & 0 & k-1 \\ 0 & \frac{k(k-1)}{v-k-1} & \frac{vk-2k^2}{v-k-1} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

We can obtain the following equations:

$$\begin{cases} bk = \lambda a, \\ a + (k-1)c = \lambda b, \\ \frac{k(k-1)}{v-k-1}b + \frac{vk-2k^2}{v-k-1}c = \lambda c. \end{cases}$$

Assume $a = k$. Then $b = \lambda$ and $c = \frac{\lambda^2 - k}{k-1}$. Substituting into the third equation,

$$\frac{k(k-1)}{v-k-1} \cdot \lambda + \frac{vk-2k^2}{v-k-1} \cdot \frac{\lambda^2 - k}{k-1} = \lambda \cdot \frac{\lambda^2 - k}{k-1}.$$

Multiply by $(v-k-1)(k-1)$,

$$(v-k-1)\lambda^3 + k(2k-v)\lambda^2 + (-k^3 + 3k^2 - kv)\lambda + k^2(v-2k) = 0. \tag{3.2}$$

Because G is k-regular, its largest eigenvalue is k. (note : three eignvalues of matric B are $\theta_1 \geq \theta_2 \geq \theta_3$)

Since B has a eigenvalue k, we need to find another two eigenvalues of B in terms of v and k. Note that Equation 3.2 becomes

$$(\lambda - k)[(v-k-1)\lambda^2 + (k^2 - k)\lambda + (2k^2 - kv)] = 0.$$

Then we can obtain $\lambda = k$ and

$$\lambda = \frac{-(k^2 - k) \pm \sqrt{(k^2 - k)^2 - 4(v-k-1)(2k^2 - kv)}}{2(v-k-1)}.$$

19

So, $\theta_1 = k, \theta_2 = \dfrac{-(k^2-k) + \sqrt{(k^2-k)^2 - 4(v-k-1)(2k^2-kv)}}{2(v-k-1)}$ and

$$\theta_3 = \frac{-(k^2-k) - \sqrt{(k^2-k)^2 - 4(v-k-1)(2k^2-kv)}}{2(v-k-1)}.$$

Lemma 3.3.3. If G is an $\mathrm{srg}(k^2+1, k, 0, 1)$ then G has exactly three distinct eigenvalues $\lambda_1 = k$, $\lambda_2 = \dfrac{-1+\sqrt{4k-3}}{2}$, $\lambda_n = \dfrac{-1-\sqrt{4k-3}}{2}$.

Proof. Note that B in (3.1) with $v = k^2+1$ is an equitable quotient of A. Then by Lemma 3.1.1, $\theta_1 = k$, $\theta_2 = \dfrac{-(k^2-k) + \sqrt{(k^2-k)^2 - 4(v-k-1)(2k^2-kv)}}{2(v-k-1)}$,

$\theta_3 = \dfrac{-(k^2-k) - \sqrt{(k^2-k)^2 - 4(v-k-1)(2k^2-kv)}}{2(v-k-1)}$ are eigenvalues of A. It is well-known A has exactly three eigenvalues [4]. $\qquad\square$

Lemma 3.3.4. If graph is girth ≥ 5 and k-regular of order v. $(k \geq 2)$

$$B = \begin{pmatrix} 0 & k & 0 \\ 1 & 0 & k-1 \\ 0 & \dfrac{k(k-1)}{v-k-1} & \dfrac{vk-2k^2}{v-k-1} \end{pmatrix},$$

then the following (i), (ii) holds.

(i) $\theta_2(v) \geq \dfrac{-1+\sqrt{4k-3}}{2}$ with equality iff $v = k^2+1$.

(ii) $\theta_3(v) \leq \dfrac{-(k^2-k)}{2(v-k-1)} - \dfrac{\sqrt{4k-3}}{2}$.

Proof. Because girth of graph G is at least 5, G has no cycle of length 4. Then $|\Gamma_{\geq 2}| \geq k(k-1)$. If not, it has a cycle of length 4. Contradict to girth at least 5. We can get an inequality $v \geq k^2 + 1 = k(k-1) + k + 1$.

We want to claim

$$\theta_2 = \frac{-(k^2-k) + \sqrt{(k^2-k)^2 - 4(v-k-1)(2k^2-kv)}}{2(v-k-1)} \geq \frac{-1+\sqrt{4k-3}}{2}. \qquad (3.3)$$

We have $v \geq k^2+1$ and $k^2+1 \geq \dfrac{2k^2+k+3}{3}$, for all $k \geq 2$. Note that

$$3(v - \frac{2k^2+k+3}{3})^2 - \frac{1}{3}(k^4 - 2k^3 + k^2) \geq 3(k^2 + 1 - \frac{2k^2+k+3}{3})^2 - \frac{1}{3}(k^4 - 2k^3 + k^2) = 0.$$

20

Let

$$f(v) = 3(v - \frac{2k^2 + k + 3}{3})^2 - \frac{1}{3}(k^4 - 2k^3 + k^2) \geq 0. \tag{3.4}$$

Simplifying $f(v) = 3v^2 - 2(2k^2 + k + 3)v + k^4 + 2k^3 + 4k^2 + 2k + 3$. We split into two parts of left and right sides of the Inequality 3.4, and we can obtain

$$4kv^2 - 4k(3k + 1)v + (k^4 + 6k^3 + 9k^2) \geq (4k - 3)v^2 - 2(4k - 3)(k + 1)v + (4k - 3)(k + 1)^2.$$

Simplifying again,

$$(k^2 - k)^2 - 4(v - k - 1)(2k^2 - kv) \geq (4k - 3)(v - k - 1)^2 \geq 0.$$

Taking square root on both sides,

$$\sqrt{(k^2 - k)^2 - 4(v - k - 1)(2k^2 - kv)} \geq \sqrt{4k - 3}(v - k - 1). \tag{3.5}$$

Because $(k^2 + 1 - v) \leq 0$, for all $v \geq k^2 + 1$, we can get

$$\sqrt{4k - 3}(v - k - 1) \geq \sqrt{4k - 3}(v - k - 1) + (k^2 + 1 - v)$$
$$= \sqrt{4k - 3}(v - k - 1) + (k + 1 - v) + (k^2 - k)$$
$$= (-1 + \sqrt{4k - 3})(v - k - 1) + (k^2 - k).$$

Then

$$\sqrt{(k^2 - k)^2 - 4(v - k - 1)(2k^2 - kv)} \geq (-1 + \sqrt{4k - 3})(v - k - 1) + (k^2 - k).$$

Since $(v - k - 1) > 0$, as $k \geq 2$. We divide by $2(v - k - 1)$ on the both sides to obtain

$$\frac{-(k^2 - k) + \sqrt{(k^2 - k)^2 - 4(v - k - 1)(2k^2 - kv)}}{2(v - k - 1)} \geq \frac{-1 + \sqrt{4k - 3}}{2}.$$

Finally, substitute $v = k^2 + 1$ into $\theta_2 = \frac{-(k^2 - k) + \sqrt{(k^2 - k)^2 - 4(v - k - 1)(2k^2 - kv)}}{2(v - k - 1)}$,

then we can get $\theta_2 = \frac{-1 + \sqrt{4k - 3}}{2}$. Solving $\frac{-(k^2 - k) + \sqrt{(k^2 - k)^2 - 4(v - k - 1)(2k^2 - kv)}}{2(v - k - 1)} =$

$\dfrac{-1+\sqrt{4k-3}}{2}$, we can obtain $v = k^2 + 1$. So, the Inequality 3.3 holds iff $v = k^2 + 1$. From Inequality 3.5, we obtain

$$\frac{-(k^2-k)-\sqrt{4k-3}(v-k-1)}{2(v-k-1)} \geq \theta_3 = \frac{-(k^2-k)-\sqrt{(k^2-k)^2-4(v-k-1)(2k^2-kv)}}{2(v-k-1)}.$$

The proof is complete.

$\square$

Theorem 3.3.5. Let G be a k-regular graph ($k \geq 2$) of order v with girth at least 5 and let the eigenvalues of adjacency matrix be

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_v.$$

Then

$$\lambda_2 \geq \frac{-1+\sqrt{4k-3}}{2},$$

$$\lambda_v \leq \frac{-(k^2-k)}{2(v-k-1)} - \frac{\sqrt{4k-3}}{2}.$$

Moreover, the following are equivalent.

(i) $\lambda_2 = \dfrac{-1+\sqrt{4k-3}}{2}$;

(ii) G is $\mathrm{srg}(k^2+1, k, 0, 1)$, where $k = 2, 3, 7, 57$;

(iii) $d = 2$;

(iv) $v = k^2 + 1$.

Proof. By above lemma and Theorem 2.4.3.

$$\lambda_2 \geq \theta_2 \geq \frac{-1+\sqrt{4k-3}}{2}, \tag{3.6}$$

$$\lambda_n \leq \theta_3 \leq \frac{-(k^2-k)}{2(v-k-1)} - \frac{\sqrt{4k-3}}{2}. \tag{3.7}$$

22

Then we can obtain

$$\lambda_2 \geq \frac{-1 + \sqrt{4k - 3}}{2},$$

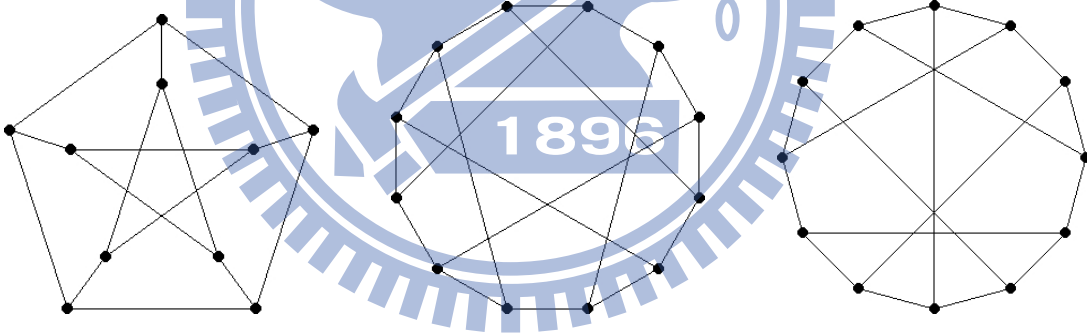$$\lambda_v \leq \frac{-(k^2 - k)}{2(v - k - 1)} - \frac{\sqrt{4k - 3}}{2}.$$

$(ii) \Rightarrow (i)$ This is clear by Lemma 3.3.3.

$(iii) \Rightarrow (ii)$ Note that the Assumption 3.3 holds for some vertex x. So, (ii) follows from Theorem 3.2.2.

$(iv) \Rightarrow (iii)$ Note that Assumption 3.1 holds for some vertex x. Therefore (iii) follows from Lemma 3.3.1.
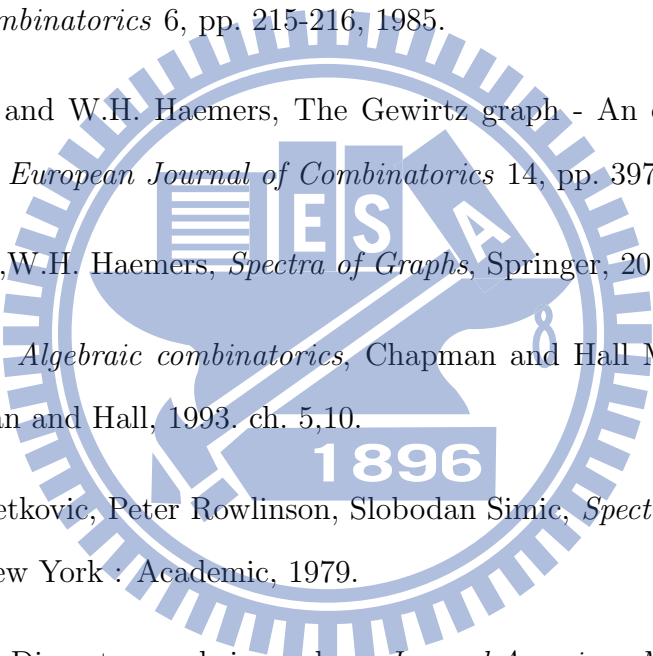
$(i) \Rightarrow (iv)$ From (3.6) and Lemma 3.3.4. □

Example 3.3.6. These graphs are k-regular and girth ≥ 5. $v = 10, 12, 12$ and $k = 3$.

$\lambda_2(\text{left}) = 1$, $\lambda_2(\text{center}) = 1.56$, $\lambda_2(\text{right}) = 1.53$. These eigenvalues are all $\geq \dfrac{-1 + \sqrt{4k - 3}}{2} = 1$. Note that $\lambda_{10}(\text{left}) = -2 \leq \theta_3(\text{left}) = -2 \leq \dfrac{-(k^2 - k)}{2(v - k - 1)} - \dfrac{\sqrt{4k - 3}}{2} = -2$, $\lambda_{12}(\text{center}) = -2.56 \leq \theta_3(\text{center}) = -1.92 \leq \dfrac{-(k^2 - k)}{2(v - k - 1)} - \dfrac{\sqrt{4k - 3}}{2} = -1.875$, and $\lambda_{12}(\text{right}) = -2.3 \leq \theta_3(\text{right}) = -1.92 \leq -1.875$. Moreover, the graph on the left side is srg$(10, 3, 0, 1)$, its diameter is 2 and $\lambda_2 = \dfrac{-1 + \sqrt{4k - 3}}{2} = 1$.

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