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碩 士 論 文

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Hadamard Operations on Totally Nonnegative Matrices

研 究 生：林志嘉

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研 究 生：林志嘉　　Student：Chih-Chia Lin

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學生：林志嘉　　　　　　　　　　　　　　　　　指導教授：翁志文教授

國立交通大學應用數學學系（研究所）碩士班

摘　　　要

本論文我們關心的是全非負矩陣和全正矩陣的性質。一個其任意子矩陣的行列式值皆為非負或皆為正的非負矩陣稱之為全非負矩陣或為全正矩陣。這樣的矩陣在數學領域裡扮演著相當重要的角色。此篇論文主要的目的在於介紹非負矩陣和正矩陣的基本性質及收集一些關於全非負矩陣和全正矩陣的已知性質，並重新回憶哈達馬乘積和半正定性質的關聯。此外，我們也將討論關於永恆全正的性質。

Hadamard Operations on Totally Nonnegative Matrices

Student：Chih-Chia Lin　　　　Advisors：Dr. Chih-Wen Weng

Department（Institute）of Applied Mathematics
National Chiao Tung University

ABSTRACT

In this thesis we are concerned with the properties of totally nonnegative (resp. positive) matrices. An m-by-n entry-wise nonnegative matrix is totally nonnegative (resp. positive) if the determinant of any square submatrix is nonnegative (resp. positive). A totally nonnegative matrix plays an important role in various mathematical branches. The primary propose here is to introduce the basic properties of nonnegative (resp. positive) matrix and to collect the known results in matrix theory with the totally nonnegative (resp. positive) property involved. We will recall the relation between Hadamard product and the positive semidefinite property, and study the relation between Hadamard product and the totally nonnegative (resp. totally positive) property. Furthermore, we also discuss the eventually totally positive property.

誌謝

本篇論文的完成，首先要感謝我的指導教授 — 翁志文教授。在老師悉心的指導下我學習到做數學所需的細膩和敏銳，也明白了不是每件事情都如其表面上所看到的那個樣子。這樣子的成長是我在還沒有歷過前所無法去想像或是嘗試去描敘的。這之中我更親自體會到那份追求學問和渴望得到答案的真誠靈魂會在當我又邁向前一步時綻放開來。

其次要感謝的是黃喻培學長。學長提供了許多新的想法給我，並且設身處地的把我的論文當作自已的論文那般地要求。記憶猶新的是在口試前幾個禮拜的夜晚，不論天晴天雨，學長總是義無反顧的陪我到三更半夜只為了能幫助我完成論文。我們可以為了一個數學問題反覆演練修改，也可以為了一個人生規劃的話題辯論整晚，像場飛翔競賽般，時而周旋時而倏地衝刺，而那些論文中的問題既使我們都不知道最後會不會有終點，但不可否認的我們都因為可以更向目標靠近而真摯的感到開心。

很感謝陪我一起走過的研究生活的夥伴，有你們在才有一場又一場精彩的球賽，有你們在才有那些說不完的無理頭笑話，有你們在才有一個又一個圍坐著喝酒聊天的夜晚。特別感謝兩位和我一起往未來努力著的兄弟，陳紹琦、傅景祥，我相信我們都將會成為很優秀的老師。另外我想感謝最後這一年在我身邊最特別的一份鼓勵和陪伴，讓我的生活充滿精彩也感受到溫暖。

最後我特別感謝我的家人，無論我在人生的旅途上做了什麼樣的決定，家人總是百分之百的支持我，做為我強而有力的後盾。完成了這個階段，我知道我還會走得更遠，見識更大的世界，我會做好準備迎接新的挑戰。

Contents

1 Introduction

A matrix is totally nonnegative (resp. positive) if the determinant of any square submatrix is nonnegative (resp. positive). For example, a Vandermonde matrix whose rows are positive and increasing is a totally positive matrix. Hence it is safe to say that everywhere a Vandermonde matrix is applied, so is a totally positive matrix.

In the 1930s, F.R. Gantmacher and M.G. Krein developed the theory of totally nonnegative matrices connecting with vibrations of mechanical systems. Then, important instances of totally nonnegative matrices are provided by green functions of certain differential operators. Besides, the theory of total nonnegativity connecting with the variation diminishing properties of matrices was developed by I.J. Schoenberg, which leads to spline theory.

Then, the following researcher was S. Karlin who started to demonstrate in multitudinous publications in the late 1950s on the subject the breath of application and depth of mathematical importance of total nonnegativity. Affected by Karlin, many authors get the motivation to explore the application of total nonnegativity to approximation theory, analysis, statistics, biology, and geometric modeling [9].

In this paper, we organize the basic properties of totally nonnegative and totally positive matrices. The fundamental material is mainly obtained from [3, 10, 6, 1]. This paper is divided into eight sections. Section 1 is introductory. Section 2 contains the definitions and notations used in this paper. In the section 3, we investigate the properties of the matrices whose principal minors are nonnegative and prove the Schur Product Theorem. In the section 4, we give some notations and examples for the Hadamard product of two totally nonnegative matrices. In the section 5, we describe some property of *Hadamard core* and find which matrices is in *Hadamard core*. In the section 6, we describe the core for bidiagonal and tridiagonal matrices. In the section 7, we present relations between Hadamard power and determinant for a nonnegative matrix. In the last section, we set up and prove some theorems about eventually totally nonnegative property and eventually totally positive property. Furthermore, we characterize the relation between totally nonnegative property and eventually totally positive property. Most of the results in this paper can be found in [4, 5].

2 Preliminaries

A nonnegative matrix is a matrix with all its entries nonnegative. For some reference of nonnegative matrices, please see [2, 7]. The *Hadamard product* of two m-by-n matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ is denoted and defined by $A \circ B = [a_{ij}b_{ij}]$, i.e., an entry-wise product.

Suppose A,B,C are matrices of the same size and λ is a scalar. Then

$$\begin{aligned} A \circ B &= B \circ A, \\ A \circ (B \circ C) &= (A \circ B) \circ C, \\ A \circ (B + C) &= A \circ B + A \circ C, \\ \lambda(A \circ B) &= (\lambda A) \circ B = A \circ (\lambda B). \end{aligned}$$

The t-th Hadamard power of $A = [a_{ij}]$ for any $t \geq 0$ and $A \geq 0$ (entry-wise nonnegative) is defined by $A^{(t)} = [a_{ij}^t]$.

The Hadamard product and Hadamard power play a central role within matrix analysis and its applications. Here we assume that all matrices are entry-wise nonnegative.

For an m-by-n matrix A, $\alpha \subseteq \{1, 2, \ldots, m\}$, and $\beta \subseteq \{1, 2, \ldots, n\}$, the submatrix of A lying in rows with indices in α and the columns with indices in β (in the arithmetic order). Let $A[\alpha \mid \beta]$ be the submatrix of A. Furthermore, let $A(\alpha \mid \beta)$ be the submatrix obtained from A by deleting the rows indexed by α and columns indexed by β. The principal submatrix $A[\alpha \mid \alpha]$ is abbreviated to $A[\alpha]$, and the complementary principal submatrix $A(\alpha \mid \alpha)$ is abbreviated to $A(\alpha)$. Moreover, we denote $A[\alpha \mid \cdot]$ as $A[\alpha \mid \{1, 2, \ldots, n\}]$ and $A[\cdot \mid \beta]$ as $A[\{1, 2, \ldots, m\} \mid \beta]$.

A *minor* of a matrix A is the determinant of the submatrix $A[\alpha \mid \beta]$, where $A[\alpha \mid \beta]$ is a square matrix. The determinant of $A[\alpha \mid \alpha]$ is called a *principal minor*. If the indices of α and β are contiguous, then the determinant of $A[\alpha \mid \beta]$ is called a *contiguous minor*.

An m-by-n matrix A is called totally nonnegative (resp. totally positive), if all minors of A are nonnegative(resp. positive). We use TN (resp. TP) to denote the set of all totally nonnegative (resp. totally positive) matrices. And we use TN_k (resp. TP_k) to denote the set of matrices whose j-by-j minors are all nonegative (resp. positive) for $1 \leq j \leq k$. In particular, TN_1 is the set of matrices with nonnegative entry. We use TN_2^+ to denote the set of all square TN_2 matrices whose 2-by-2 principal minors based upon consecutive indices are positive. Hence $TP \subseteq TP_2 \subseteq TN_2^+$.

3 Schur Product Theorem

Here we want to recall the *Schur Product Theorem* (see[8]) and its proof. The theorem relates positive semidefinite matrices to the Hadamard product, which plays a central role in the analysis of the determinants of matrices. Here we assume that all matrices have real nonnegative entries.

Now we introduce some notations. The transpose of a matrix A is denoted by A^T. A square matrix is called *symmetric* if it is equal to its transpose. That is, $A = A^T$.

Definition 3.1. If A is an n-by-n symmetric matrix and $\mathcal{Q}(x) = x^T Ax \geq 0$ for any non-zero vector $x \in \mathbb{R}^n$, then A is a *positive semidefinite matrix.* Note that all eigenvalues of a positive semidefinite matrix A are nonnegative and all principal minors of A are nonnegative.

Here we collect some properties about positive semidefinite matrices to prove the Schur Product Theorem.

Lemma 3.2. *Any n-by-n rank one positive semidefinite matrix A can be written as the form $A = xx^T$ where x is a column n-vector.*

Proof. Let A be an n-by-n rank one positive semidefinite matrix. Since A is positive semidefinite, there exists an orthogonal matrix U made up of orthonormalized eigenvectors of A such that

$$\begin{aligned} A &= U \begin{bmatrix} \lambda_1 & & & 0 \\ & 0 & & \\ & & \ddots & \\ 0 & & & 0 \end{bmatrix} U^T \\ &= \lambda_1 u_1 u_1^T, \end{aligned}$$

where u_1 is the first column of U. We choose $x = \sqrt{\lambda_1} u_1$,and then $A = xx^T$. $\square$

Lemma 3.3. *If B is an n-by-n positive semidefinite matrix with rank r, then there exists r positive semidefinite rank one matrices $B_1, B_2, \ldots, B_r$ such that $B = B_1 + B_2 + \cdots + B_r$.*

Proof. Let B be an n-by-n rank r positive semidefinite matrix. Since B is positive semidefinite, there exists an orthogonal matrix U made up of orthonormalized eigenvectors of B such that

$$
\begin{aligned}
B &= U \begin{bmatrix} \lambda_1 & & & & & 0 \\ & \ddots & & & & \\ & & \lambda_r & & & \\ & & & 0 & & \\ & & & & \ddots & \\ 0 & & & & & 0 \end{bmatrix} U^T \\
&= \lambda_1 u_1 u_1^T + \cdots + \lambda_r u_r u_r^T,
\end{aligned}
$$

where u_i is the i-th column of U and $\lambda_k > 0$ for $1 \leq k \leq r$. We choose $B_i = \lambda_i u_i u_i^T$ then $B = B_1 + B_2 + \cdots + B_r$. Note that λ_i is the only positive eigenvalue of B_i.

Let D_k be the diagonal matrix whose only nonzero entry is $[D_k]_{kk} = \lambda_k$, $B_k = UD_kU^T$. Since $\lambda_k > 0$ for all $1 \leq k \leq r$, then D_k is positive semidefinite for all $1 \leq k \leq r$. Now consider $x^T B_k x$ for any $x \in \mathbb{R}^n$, then

$$
\begin{aligned}
x^T B_k x &= x^T U D_k U^T x \\
&= (U^T x)^T D_k (U^T x) \\
&\geq 0.
\end{aligned}
$$

This is true for all $1 \leq k \leq r$ and so B_k is positive semidefinite for all k. $\square$

Note that the converse of Lemma 3.3 also holds.

Lemma 3.4. *If B is an n-by-n rank r matrix and suppose $B = B_1 + B_2 + \cdots + B_r$, where B_i's are rank one positive semidefinite n-by-n matrices. Then B itself is positive semidefinite.*

Proof. By the Definition 3.1 and note that for all $x \in \mathbb{R}^n$,

$$\begin{aligned} x^T B x &= x^T(B_1 + B_2 + \cdots + B_r)x \\ &= x^T B_1 x + x^T B_2 x + \cdots + x^T B_r x \\ &\geq 0. \end{aligned}$$

□

Theorem 3.5. *(Schur Product Theorem) Suppose A and B are two n-by-n positive semidefinite matrices. Then the Hadamard product of A and B is also positive semidefinte.*

Proof. Let A and B be two n-by-n positive semidefinite matrices. There are three cases to consider. First, suppose rank(B)=0. This implies that $B = \mathcal{O}$, and therefore $A \circ B = \mathcal{O}$, which is clearly positive semidefinite. Second, suppose rank(B)=1. Then B can be written as the form $B = xx^T$ for some vector $x \in \mathbb{R}^n$. Then $[A \circ B]_{ij}$=$[A]_{ij}[B]_{ij}$=$[A]_{ij}[x]_i[x^T]_j$=$[D_x A D_x]_{ij}$, where D_x is the diagonal matrix with x in the diagonal. Now we prove that $D_x A D_x$ is positive semidefinite. Note that, for any $v \in \mathbb{R}^n$

$$\begin{aligned} v^T D_x A D_x v &= v^T D_x^T A D_x v \\ &= (D_x v)^T A (D_x v) \\ &\geq 0, \end{aligned}$$

since A is positive semidefinite. This proves that $D_x A D_x$ is positive semidefinite. Finally, suppose that rank(B)=r for some $1 < r \leq n$. Then $B =$

$B_1 + B_2 + \cdots + B_r$, where the rank of B_i is one for $1 \leq i \leq r$. Then $A \circ B = A \circ (B_1 + B_2 + \cdots + B_r) = A \circ B_1 + A \circ B_2 + \cdots + A \circ B_r$. Since each $A \circ B_i$ is positive semidefinite, then $A \circ B$ is positive semidefinite by Lemma 3.4.

Therefore, if A and B are two n-by-n positive semidefinite matrices, then the Hadamard product of A and B is also positive semidefinte. $\square$

4 The Closeness of Hadamard Product and Hadamard Power

In this section, we prove the closeness of TN_2(resp. TP_2) under Hadamard product and Hadamard power. Furthermore, we also give some counterexamples to show TN_k and TP_k are not generally closed under Hadamard product and Hadamard power for $k \geq 3$.

Theorem 4.1. *If $A, B \in TN_2$ (resp. TP_2), then $A \circ B \in TN_2$ (resp. TP_2) and $A^{(t)} \in TN_2$ (resp. TP_2) for all $t > 0$.*

Proof. Let

$$A_1 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

and

$$B_1 = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

be arbitrary 2-by-2 submatrices in the same site of A and B respectivity. Here $a, b, c, d, e, f, g, h \geq 0$ and $det(A_1) \geq 0$ (i.e. $ad \geq bc$), $det B_1 \geq 0$(i.e.

$eh \geq fg$). We consider the matrices

$$A_1 \circ B_1 = \begin{bmatrix} ae & bf \\ cg & dh \end{bmatrix}$$

and

$$A_1^{(t)} = \begin{bmatrix} a^t & b^t \\ c^t & d^t \end{bmatrix}.$$

Then

$$\begin{aligned} det(A_1 \circ B_1) &= aedh - bfcg \\ &= (ad)(eh) - (bc)(fg) \\ &\geq 0 \end{aligned}$$

and

$$\begin{aligned} det(A_1^{(t)}) &= a^t d^t - b^t c^t \\ &= (ad)^t - (bc)^t \\ &\geq 0. \end{aligned}$$

Since A_1, B_1 are arbitrary, so we have $A \circ B \in TN_2$ and $A^{(t)} \in TN_2$ for all $t > 0$. Similarly the TP_2 case also holds. $\square$

Note that, from the proof above, we know that if $A, B \in TN_2$, then $A \circ B \in TN_2$. But $A \circ B$ in TN_2 does not imply that A and B are both in TN_2.

Example 4.2. Let

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 5 & 3 \\ 2 & 2 \end{bmatrix}.$$

Then $A \notin TN$ and $B \in TN$, but

$$A \circ B = \begin{bmatrix} 10 & 9 \\ 2 & 2 \end{bmatrix}$$

is in TN.

We know from Lemma 4.1, if $A, B \in TN_2$ (resp. TP_2), then $A \circ B \in TN_2$ (resp. TP_2). However TN_k and TP_k are not generally closed under Hadamard product when $k \geq 3$.

Example 4.3. Let

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 2 & 2 & 3 \\ 3 & 3 & 5 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 2 & 2 & 3 \\ 1 & 2 & 3 \\ 0 & 3 & 5 \end{bmatrix}$$

be in TN. Note that

$$A \circ B = \begin{bmatrix} 4 & 2 & 0 \\ 2 & 4 & 9 \\ 0 & 9 & 25 \end{bmatrix}$$

is not in TN, since $\det(A \circ B) = -24 < 0$. Similiarly (see [5]), let

$$A = \begin{bmatrix} 1 & 11 & 22 & 20 \\ 6 & 67 & 139 & 140 \\ 16 & 182 & 395 & 445 \\ 12 & 138 & 309 & 376 \end{bmatrix}.$$

Then A can be checked to be in TP. But $\det(A^{(2)}) = -114904113$, so $A^{(2)}$ is not in TP.

5 Hadamard Core

In previous section, we see that the set TN is not closed under Hadamard product. Our interest here is to characterize a subset of TN which is closed under Hadamard product.

Definition 5.1. For TN and TN_k, the corresponding *Hadamard cores* are defined as follows:

$$CTN := \{A \in TN \ : \ B \in TN \Rightarrow A \circ B \in TN\}$$

and

$$CTN_k := \{A \in TN_k \ : \ B \in TN_k \Rightarrow A \circ B \in TN_k\}.$$

Let $\alpha \geq 0$ and $J_{m \times n}$ be the m-by-n matrix with all entries 1. Then αI_n and $\alpha J_{m \times n}$ are clearly in CTN. We discuss the properties of the Hadamard core further.

Lemma 5.2. $TN_2 = CTN_2$.

Proof. Suppose $A \in TN_2$. Since the TN_2 is closed under Hadamard product, by Lemma 4.1 we have $A \circ B \in TN_2$ for all $B \in TN_2$. It follows that A is in CTN_2. For the converse, if $A \in CTN_2$ then from the Definition 5.1 we know $A \in TN_2$. □

Lemma 5.3. *[4] If $A \in CTN$ and $B \in CTN$, then $A \circ B \in CTN$.*

Proof. Let C be any m-by-n TN matrix. Then $B \circ C \in TN$ since $B \in CTN$. Hence $A \circ (B \circ C) \in TN$. Since $A \circ (B \circ C) = (A \circ B) \circ C$, We have that $A \circ B \in CTN$. □

Lemma 5.4. *If $A \in CTN$, then $A^{(t)} \in CTN$ for all $t \in \mathbb{N}$.*

Proof. If $A \in CTN$, then $A \circ B \in TN$ for any $B \in TN$. We prove this by mathematical induction. If $t = 1$, this is trivial true. Suppose $t = k$ holds. Consider the case $t = k + 1$. $A^{(k+1)} \circ B = (A^{(k)} \circ A) \circ B = A^{(k)} \circ (A \circ B)$. Since $A \in CTN$ and by the induction hypothesis, $A^{(k+1)} \circ B \in TN$. We conclude that $A^{(t)} \in CTN$ for all $t \in \mathbb{N}$. □

In the following Lemma, we show that the property of being in the concerned sets or not is not influenced by multiplicating a positive diagonal matrix.

Lemma 5.5. *Multiplicating by positive diagonal matrices at the left or right side does not affect the property of a matrix A being in TN, TP, or CTN.*

Proof. Let D, E be two positive diagonal matrices. Consider the matrix $DAE[\alpha \mid \beta]$ where $\alpha = \{a_1, \cdots, a_k\}$, $\beta = \{b_1, \cdots, b_k\}$. If $A \in TN$, then

$\det(DAE[\alpha \mid \beta])$=$(\Pi_{i=1}^{k} D_{a_i a_i} E_{b_i b_i})$ $\cdot\det A[\alpha \mid \beta] \geq 0$. Since $DAE[\alpha \mid \beta]$ is arbitrary, DAE is in TN. If $A \notin TN$, there exists a minor $A[\alpha' \mid \beta'] < 0$ where $\alpha' = \{a_1', \cdots, a_k'\}$, $\beta' = \{b_1', \cdots, b_k'\}$. Thus $\det(DAE[\alpha' \mid \beta'])$=$(\Pi_{i=1}^{k} D_{a_i' a_i'} E_{b_i' b_i'})$ $\cdot\det A[\alpha' \mid \beta'] < 0$. Similarly, the TP case also holds.

If $A \in CTN$, we know the Hadamard product of A and B is in TN for $B \in TN$. Consider the matrix $(DA) \circ B$, $(DA) \circ B = A \circ (DB)$ where $DB \in TN$. It is followed by that $A \circ (DB) \in TN$ and $DA \in CTN$. Similarly, multiplicating by positive diagonal matrices on the right side does not affect the property of being in CTN.

$\square$

For $x = [x_i] \in \mathbb{R}^n$, let diag$(x)$ denote the n-by-n diagonal matrix such that $[\text{diag}(x)]_{ii} = x_i$ and $[\text{diag}(x)]_{ij} = 0$ for $i \neq j$. Next, we discuss the rank one totally nonnegative matrices.

Theorem 5.6. *Let A be an n-by-n rank one matrix. Then the following are equivalent.*

(i) $A \in TN$;

(ii) $A = xy^T$ *for some nonnegative column vectors;*

(iii) $A \in CTN$;

(iv) $A \in TN_1$.

Proof. Let A_i denote the i-th row of A. Since rank(A)=1, there exists a $k \in \{1, 2, \cdots, n\}$ such that $A_i = c_i A_k$ where c_i's are constants. In particular, $c_k = 1$. Let $y^T = A_k$ and x be the column vector such that $x_i = c_i$. Then $A = xy^T$.

$(i) \Rightarrow (ii)$ Since $A \in TN$, then all entries of A are nonnegative. We can further assume that all entries of x and y are nonnegative.

$(ii) \Rightarrow (iii)$ Let D=diag(x), E=diag(y) and J the all 1's n-by-n matrix. Note that $A = DJE$. Hence the (iii) follows since $J \in CTN$ and by Lemma 5.5.

$(iii) \Rightarrow (iv)$ If $A \in CTN$, it is clearly $A \in TN_1$.

$(iv) \Rightarrow (i)$ Since rank(A)=1, then $\det A[\alpha \mid \beta] = 0$ when $|\alpha| = |\beta| \geq 2$. It is clearly that $A \in TN$.

$\square$

Theorem 5.7. *Let A be an n-by-n rank two matrix. Then the following are equivalent.*

(i) $A \in TN$;

(ii) $A \in TN_2$;

(iii) $A \in CTN_2$.

Proof. $(i) \Rightarrow (ii)$ It trivially holds by the definition of TN.

$(ii) \Rightarrow (i)$ Since $A \in TN_2$, A is nonnegative and all 2-by-2 minors of A are nonnegative. To show that $A \in TN$, it suffices to check all the k-by-k minors of A for $3 \leq k \leq n$. Consider $A[\alpha \mid \beta]$ for $\alpha, \beta \subseteq \{1, 2, \cdots, n\}$ and $|\alpha| = |\beta| = k \geq 3$. Since rank$(A) = 2$, rank$(A[\alpha \mid \beta]) \leq 2$. It follows that $\det(A[\alpha \mid \beta]) = 0$. Thus we can conclude that $A \in TN$.

$(ii) \Longleftrightarrow (iii)$ This equivalence holds by Lemma 5.2. $\square$

6 The Hadamard Core for Bidiagonal and Tridiagonal Matrices

In this section, we describe the closeness of bidiagonal and tridiagonal matrices under Hadamard product. Furthermore, we show that the positive semidefinite tridiagonal matrices are in TN.

Definition 6.1. Let U be an n-by-n matrix with non-zero entries along the main diagonal and the super-diagonal. This kind of matrices are called *upper bidiagonal*. That is, U is a matrix of the form

$$U = \begin{bmatrix} a_1 & b_1 & & & & 0 \\ & a_2 & b_2 & & & \\ & & a_3 & b_3 & & \\ & & & \ddots & \ddots & \\ & & & & a_{n-1} & b_{n-1} \\ 0 & & & & & a_n \end{bmatrix}_{n\times n}.$$

Let L be an n-by-n matrix with non-zero entries along the main diagonal and the sub-diagonal. This kind of matrices are called *lower bidiagonal*. That is, L is a matrix of the form

$$L = \begin{bmatrix} a_1 & & & & & 0 \\ b_1 & a_2 & & & & \\ & b_2 & a_3 & & & \\ & & \ddots & \ddots & & \\ & & & b_{n-2} & a_{n-1} & \\ 0 & & & & b_{n-1} & a_n \end{bmatrix}_{n\times n}.$$

If an n-by-n matrix A is *upper bidiagonal* or *lower bidiagonal* matrix, then A is *bidiagonal matrix*.

Definition 6.2. A square matrix $T = [t_{ij}]$ is called *tridiagonal* if $t_{ij} = 0$ whenever $|i - j| > 1$. Thus a *tridiagonal matrix* has the form

$$T = \begin{bmatrix} b_1 & c_1 & & & & 0 \\ a_1 & b_2 & c_2 & & & \\ & a_2 & b_3 & c_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & a_{n-2} & b_{n-1} & c_{n-1} \\ 0 & & & & a_{n-1} & b_n \end{bmatrix}_{n\times n}.$$

The following theorem describes the relation of tridiagonal matrix and totally nonnegative property.

Theorem 6.3. *[4] Let T be an n-by-n nonnegative tridiagonal matrix. Then T is totally nonnegative if and only if T has nonnegative principal minors.*

Proof. If T is totally nonnegative, then by the definition of totally nonnegative property, T is an entry-wise nonnegative matrix with nonnegative principal minors. For the converse, suppose T is an n-by-n tridiagonal matrix with nonnegative principal minors. We want to show that all non-principal minors of T are also nonnegative. We prove this by mathematical induction. Let $T' = [t'_{ij}]$ and $T' = T[\alpha \mid \beta]$ with $\alpha \neq \beta$. If $n = 1$, this is trivially true. Suppose $n < k$ holds. Consider the case $n = k$. Let $\alpha = \{a_1, a_2, \cdots, a_k\}$ and $\beta = \{b_1, b_2, \cdots, b_k\}$ be the indices of rows and columns in T. There are three cases to consider. If $a_1 < b_1$, then $t'_{1i} = 0$ for all $2 \leq i \leq k$. Thus $\det(T')$ $= t'_{11} \cdot det(T'(1)) \geq 0$ by the induction hypothesis. If $a_1 > b_1$, then $t'_{i1} = 0$

for all $2 \leq i \leq k$. Thus $\det(T^{'}) = t^{'}_{11} \cdot det(T^{'}(1)) \geq 0$ by the induction hypothesis. If $a_i = b_i$ for $1 \leq i \leq l$ and $a_{l+1} \neq b_{l+1}$ for some $1 \leq l \leq k-1$, then either $T[\{a_1, \cdots, a_l\} \mid \{b_{l+1}, \cdots, b_k\}]$ or $T[\{a_{l+1}, \cdots, a_k\} \mid \{b_1, \cdots, b_{l+1}\}]$ is zero matrix. The matrix is a block triangular matrix and by the induction hypothesis, $\det[T^{'}] \geq 0$. We complete this proof.

□

Corollary 6.4. *If T is an n-by-n entry-wise nonnegative bidiagonal matrix, then $T \in TN$.*

Proof. If T is an n-by-n entry-wise nonnegative bidiagonal matrix, then it is also a tridiagonal matrix with nonnegative principal minors. By Theorem 6.3, $T \in TN$. □

Theorem 6.5. *Let T be an n-by-n bidiagonal matrix. Then the following are equivalent.*

(i) $T \in TN_1$;

(ii) $T \in CTN$;

(iii) $T \in TN$.

Proof. $(i) \Rightarrow (ii)$ Let T be an n-by-n bidiagonal matrix and $T \in TN_1$. For any n-by-n matrix $B \in TN$, $T \circ B$ is still an n-by-n bidiagonal matrix and $T \circ B \in TN_1$. By Corollary 6.4, we can conclude that $T \in CTN$.

$(ii) \Rightarrow (iii)$ It follows trivially from the definition of Hadamard core.

$(iii) \Rightarrow (i)$ Since $TN \subseteq TN_1$, $T \in TN_1$.

□

We relate the totally nonnegative property in tridiagonal matrix via its eigenvalues.

Corollary 6.6. *Let T be a nonnegative symmetric tridiagonal matrix. If the eigenvalues of T are nonnegative, then $T \in TN$.*

Proof. Let T be a nonnegative symmetric tridiagonal matrix and $\lambda_1, \lambda_2, \cdots, \lambda_n$ be the eigenvalues of T. If $\lambda_i \geq 0$ for $1 \leq i \leq n$, then T is a positive semidefinite matrix. That is, T is a nonnegative tridiagonal matrix with nonnegative principal minors. By Theorem 6.3, T is totally nonnegative. $\square$

Here we give a counterexample to show that a nonnegative semidefinite matrix is not generally totally nonnegative.

Example 6.7. Let

$$A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}.$$

Note that A is a nonnegative semidefinite matrix, but $A \notin TN$ since $\det(A[\{1,2\} \mid \{2,3\}]) = -2$.

7 The Matrices Eventually of Positive Determinant

Here we consider the matrices whose sufficiently large Hadamard power have positive determinant. We list some sufficient conditions for these matrices.

Definition 7.1. An entry-wise nonnegative matirx A is called *eventaully of positive determinant* if there exists a number $T \geq 0$ such that $\det A^{(t)} > 0$ for all $t > T$.

Definition 7.2. We call a nonnegative square matrix $A = [a_{ij}]$ *normalized dominant* if $a_{ii} = 1$, for $i = 1, 2, \cdots, n$ and $0 \leq a_{ij} < 1$, for $i \neq j$.

Definition 7.3. Two nonnegative matrices A and B are *diagonally equivalent* if there exist positive diagonal matrices D, E such that $B = DAE$.

We want to show that a square matrix in TN_2^+ is eventually of positive determinant. This is obtained by Fallat and Johnson in [5]. We follow their idea but complete the details to rewrite the proof. We begin from the following Lemmas.

Lemma 7.4. *[5] If A is an n-by-n matrix and $A \in TN_2^+$, then A is diagonally equivalent to a normalized dominant matrix.*

Proof. Suppose the diagonal, the super- and sub- diagonal entries of A are all positive. Hence there exists a positive diagonal matrix D such that $B = DA$ and B has ones on its main diagonal. Let $F = [f_{ii}]$ be a positive diagonal matrix defined by choosing any positive number f_{11} and recursively applying

$$f^2_{(k+1),(k+1)} = f^2_{k,k} \frac{b_{k,(k+1)}}{b_{(k+1),k}}.$$

Let $C = [c_{ij}] = FBF^{-1}$. Then C has symmetric tridiagonal part and $c_{ii} = 1$ for all i. Since $C \in TN_2^+$, it follows that $0 < c_{i,i+1} < 1$, for each $i = 1, 2, \cdots, n-1$.

Claim: $c_{i,i+k} < 1$ for $k = 2, 3, \cdots, n-i$

We prove this by induction on k.

(1) For $k=2$, $\det(C[\{i,i+1\} \mid \{i+1,i+2\}])$=$c_{i,(i+1)}c_{(i+1),(i+2)} - c_{i,(i+2)}c_{(i+1),(i+1)} \geq 0$ since $C \in TN_2$

We know that $c_{(i+1),(i+1)} = 1$ and $0 \leq c_{i,(i+1)}, c_{(i+1),(i+2)} < 1$, hence we have $0 \leq c_{i,(i+2)} < 1$.

(2) Suppose $0 \leq c_{i,(i+k)} < 1$ for $2 \leq k < p$.

(3) For $k = p$. Consider $\det(C[\{i, i+p-1\} \mid \{i+p-1, i+p\}])$= $c_{i,(i+p-1)}c_{(i+p-1),(i+p)} - c_{i,(i+p)}c_{(i+p-1),(i+p-1)} \geq 0$. Since $0 \leq c_{i,(i+p-1)} < 1$ by the induction hypothesis, and we know that $0 \leq c_{(i+p-1),(i+p)} < 1$ and $c_{(i+p-1),(i+p-1)} = 1$, we have $0 \leq c_{i,(i+p)} < 1$.

In a word, all of the entries above the main diagonal are strictly less than one.

Similarly, consider the entries below the main diagonal of C. We can conclude that C is a normalized dominant matrix.

Suppose A has at least one zero in the super- or sub- diagonal entries of A. We verify this case by induction on n.

(1) For $n = 1$, it is trivial. For $n = 2$, suppose A is lower-triangular. Then $A = \begin{bmatrix} a & 0 \\ b & c \end{bmatrix}$. Let $D = \begin{bmatrix} \frac{1}{a} & 0 \\ 0 & \epsilon \end{bmatrix}$ and $E = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{c\epsilon} \end{bmatrix}$ for some sufficiently small $\epsilon > 0$. Then $DAE = \begin{bmatrix} 1 & 0 \\ \epsilon b & 1 \end{bmatrix}$ is a normalized dominant matrix. Similarly, the lower-triangular case also holds.

(2) Suppose that the statement holds when $n < m$.

(3) Consider the case of $n = m$. Since $A \in TN_2^+$ is an m-by-m matrix with at least one zero in the super- or sub- diagonal. Suppose the zero entry occurs in the super-diagonal of A, that is $a_{i,i+1} = 0$ for some i with $1 \leq i \leq m-1$. Since $A \in TN_2^+$, all of the main diagonal entries of A are positive. For any $i+2 \leq j \leq m$, consider the submatrix $A[\{i, i+1\} \mid \{i+1, j\}]$. Then $\det(A[\{i, i+1\} \mid \{i+1, j\}]) = a_{i,i+1}a_{i+1,j} - a_{i,j}a_{i+1,i+1} \geq 0$. Since $a_{i,i+1} = 0$ and $a_{i+1,i+1} > 0$, we can conclude that $a_{i,j} = 0$. Furthermore, for any $1 \leq k \leq i$, $i+1 \leq l \leq m$, consider the submatrix $A[\{k, i\} \mid \{i, l\}]$. Then $\det(A[\{k, i\} \mid \{i, l\}]) = a_{kl}a_{il} - a_{kl}a_{ii} \geq 0$, since $a_{il} = 0$ and $a_{ii} > 0$, we can conclude that $a_{kl} = 0$.

In a word, A must be in the form $A = \begin{bmatrix} A_1 & 0 \\ A_2 & A_3 \end{bmatrix}$, where A_1 is i-by-i and A_3 is $(m-i)$-by-$(m-i)$, and both A_1 and A_3 are in TN_2^+.

By induction, both A_1 and A_3 are diagonally equivalent to a normalized dominant matrix. Let $B_1 = D_1A_1E_1, B_3 = D_3A_3E_3$ be two normalized dominant matrices where D_1, E_1, D_3, E_3 are all diagonal matrices. Then

$$\begin{bmatrix} D_1 & 0 \\ 0 & D_3 \end{bmatrix} \begin{bmatrix} A_1 & 0 \\ A_2 & A_3 \end{bmatrix} \begin{bmatrix} E_1 & 0 \\ 0 & E_3 \end{bmatrix} = \begin{bmatrix} B_1 & 0 \\ D_3A_2E_1 & B_3 \end{bmatrix}.$$

For proper small $\epsilon > 0$,

$$\begin{bmatrix} I & 0 \\ 0 & \epsilon I \end{bmatrix} \begin{bmatrix} B_1 & 0 \\ D_3A_2E_1 & B_3 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \epsilon^{-1} I \end{bmatrix} = \begin{bmatrix} B_1 & 0 \\ \epsilon D_3A_2E_1 & B_3 \end{bmatrix}.$$

Since $\epsilon > 0$ and small, so the entries of $\epsilon D_3A_2E_1$ are positive and less than one. By choosing $D = \begin{bmatrix} D_1 & 0 \\ 0 & \epsilon D_3 \end{bmatrix}$, $E = \begin{bmatrix} E_1 & 0 \\ 0 & \epsilon^{-1}E_3 \end{bmatrix}$, DAE

is normalized dominant. Similarly, the case that the zero entry occurs in the sub-diagonal also holds.

□

Lemma 7.5. *[5] If A is an n-by-n normalized dominant matrix, then A is eventually of positive determinant.*

Proof. Since A is an n-by-n normalized dominant matrix, the off-diagonal entries of $A^{(t)}$ tend to 0 as t increases, and so $\det(A^{(t)})$ tend to 1 as t increases. Hence there exists a $T > 0$ such that $\det(A^{(t)}) > 0$ for all $t \geq T$. □

Diagonally equivalence ensures the invariance of being eventually of positive determinant.

Lemma 7.6. *If A is an n-by-n matrix that is diagonally equivalent to a eventually of positive determinant matrix, then A is eventually of positive determinant.*

Proof. There exists positive diagonal matrices D, E,and $T > 0$ such that $B = DAE$, and there exists a $T \geq 0$ such that $det(B^{(t)}) > 0$ for all $t > T$. Note that

$$\begin{aligned} det(B^{(t)}) &= det((DAE)^{(t)}) \\ &= det(D^{(t)}A^{(t)}E^{(t)}) \\ &= det(D^{(t)})det(A^{(t)})det(E^{(t)}), \end{aligned}$$

and we have

$$\frac{det(B^{(t)})}{det(D^{(t)})det(E^{(t)})} = det(A^{(t)}).$$

Since $\det(B^{(t)})$, $\det(D^{(t)})$ and $\det(E^{(t)})$ are all positive, then $det(A^{(t)})$ is positive. So we conclude that A is eventually of positive determinant. $\square$

Theorem 7.7.

Proof. By Lemma 7.4, A is diagonally equivalent to a normalized dominant matrix B. Additionally, by Lemma 7.5 and Lemma 7.6, A is eventually of positive determinant. $\square$

8 Eventually Totally Nonnegative Matrices and Eventually Totally Positive Matrices

The purpose of this section is to set up and characterize the eventually totally positive property. We show that totally positive property is closely related to eventually totally positive property.

Definition 8.1. An entry-wise nonnegative matirx A is called *eventually totally nonnegative* (resp. *eventually totally positive*), if there exists a number $T \geq 0$ such that $A^{(t)} \in TN$(resp. TP) for all $t > T$.

To check whether a matrix is in TP_2 or not, it suffices to check the 2-by-2 contiguous minors. This is also brought up by Fallat and Johnson in [5]. We rewrite the proof.

Lemma 8.2. *Let A be an 2-by-n or n-by-2 entry-wise positive matrix such that all the 2-by-2 contiguous minors are positive. Then $A \in TP_2$.*

Proof. Consider the 2-by-n matrix A. There exists a positive diagonal matrix D such that $AD = B$ is in the form

$$B = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \end{bmatrix}.$$

Since multiplied by a positive diagonal matrix does not affect the positive 2-by-2 contiguous minors property, we have that $x_{i+1} > x_i$, for $i = 1, 2, \ldots, n-1$. Hence it follows that B is in TP_2, and by Lemma 5.5, A is in TP_2. Similarly, the n-by-2 case also holds. □

Applying this Lemma, the following Theorem can be obtained.

Theorem 8.3. *[5] Let A be an m-by-n entry-wise positive matrix such that all the 2-by-2 contiguous minors of A are positive. Then $A \in TP_2$.*

Proof. Consider the 2-by-2 matrix $A[\{a_1, a_2\} \mid \{b_1, b_2\}]$. We want to show that the corresponding minor is positive. Consider the two consecutive rows $A[\{i, i+1\} \mid \cdot]$ in A. Then it is in TP_2 by Lemma 8.2. In particular, $\det(A[\{i, i+1\} \mid \{b_1, b_2\}]) > 0$. Thus $A[\cdot \mid \{b_1, b_2\}]$ is an n-by-2 matrix whose consecutive minors are positive. By applying Lemma 8.2 again, we have that $A[\cdot \mid \{b_1, b_2\}]$ is also in TP_2. In particular, $\det(A[\{a_1, a_2\} \mid \{b_1, b_2\}]) > 0$. Since the choice of $\{a_1, a_2\}$, $\{b_1, b_2\}$ is arbitrary, then we conclude that $A \in TP_2$.

The following is a counterexample about TN_2 case.

Example 8.4. Let

$$A = \begin{bmatrix} 4 & 2 & 0 & 5 \\ 2 & 1 & 0 & 1 \\ 1 & 2 & 0 & 7 \end{bmatrix}.$$

Then all the 2-by-2 contiguous minors of A are nonnegative. Since $\det(A[\{1,2\} \mid \{2,4\}])= -3 < 0$, $A \notin TN_2$.

$\square$

The following well known Theorem shows that being of positive 2-by-2 contiguous minors is a sufficient condition for being eventually totally positive. See [5] for detail.

Theorem 8.5. *([5]) Suppose A is an m-by-n matrix. Then the following statements are equivalent:*

(i) A is eventually totally positive.

(ii) $A \in TP_2$.

(iii) A is entry-wise positive and all 2-by-2 contiguous minors of A are positive.

Proof. $(i) \Rightarrow (ii)$ Suppose A is eventually totally positive then clearly A must have positive entries. Since the 2-by-2 minor in row $\{i,j\}$ and columns $\{p,q\}$ in $A^{(t)}$ is positive for some t, we have $a_{ip}^{(t)}a_{jq}^{(t)} > a_{iq}^{(t)}a_{jp}^{(t)}$, by taking t-th roots, this minor of A is positive. Since the 2-by-2 minor is arbitrary, then $A \in TP_2$.

$(ii) \Rightarrow (iii)$ Since $A \in TP_2$, then A is entry-wise positive and all 2-by-2 minors of A are positive, so the 2-by-2 contiguous minors of A are positive.

$(iii) \Rightarrow (i)$ Since A is entry-wise positive and all 2-by-2 contiguous minors of A are positive, then $A[\alpha \mid \beta] \in TN_2^+$. By Theorem 7.7, $A[\alpha \mid \beta]$ is eventually of positive determinant. We conclude that A is eventually TP.

$\square$

Example 8.6. Let A be the following square matrix

$$A = \begin{bmatrix} 8 & 3 & 2 & 1 \\ 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 \\ 1 & 2 & 3 & 7 \end{bmatrix}.$$

Note that $A \in TN_2$ but $A \notin TN$ since $\det(A[\{2,3,4\}]) = -1$.

Corollary 8.7. *If $A \in TP$, then A is eventually totally positive.*

Proof. If A is in TP, then A must be in TP_2. By the Theorem 8.5, A is in eventually TP. $\square$

But A is eventually totally positive does not imply that A is in TP.

Example 8.8. Let

$$A = \begin{bmatrix} 1 & 3^{1/2} & \epsilon \\ 3^{1/2} & 4 & 2 \\ \epsilon & 1 & 1 \end{bmatrix}.$$

where ϵ is a sufficiently small positive number. Then A is not in TP since $\det(A) = -1 + 3^{3/2}\epsilon - 4\epsilon^2 < 0$. Consider the t-th power of matrix A,

$$A^{(t)} = \begin{bmatrix} 1 & 3^{t/2} & \epsilon^t \\ 3^{t/2} & 4^t & 2^t \\ \epsilon^t & 1 & 1 \end{bmatrix}.$$

The 2-by-2 minors of $A^{(t)}$ are trivially positive and $\det(A^{(t)}) = 4^t - 2^t - 3^t + 6^t\epsilon^{2t} - 4^t\epsilon^{2t} > 0$ for any $t \geq 2$. Thus A is eventually totally positive.

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