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在樹圖上之限亮點西格瑪遊戲與其對偶遊戲

Lit-only Sigma Game and its Dual Game on Tree

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摘　　　要

令G是一個簡單的連通圖，G的點集為{1, 2, … , n}。若將每一個頂點皆給定黑或是白其中一個顏色，便成G的一個配置。在每一個遊戲中的一個走法是將一個配置換成另一個配置。在此篇論文中給兩個特別的遊戲走法。第一個遊戲便是限亮點西格瑪遊戲，此遊戲包含了對應n個定點的n個走法，規則為：在配置u中點i若是黑色，則走法L_i將點i的鄰居的顏色黑白互換，而且不改變其他點(包括i)的顏色。第二個遊戲則是第一個遊戲的對偶遊戲，也包含了對應n個定點的n個走法，規則為：在配置u中點i的鄰居中若是有奇數個黑色點，則L_i^*便可以將點i的顏色黑白互換，而且不改變其他點的顏色。這兩種遊戲的關係在這篇論文中也會說明，另外，在這兩種遊戲之下的任何一個規則，我們可以利用它們對應的走法將配置的集合做出分割，並求出這些軌跡。我們稱一個包含超過一個元素的軌跡為“非簡單”的軌跡。若給定一些

前提，我們可以猜測在限亮點西格瑪遊戲的對偶遊戲之中將有兩個非簡單的軌跡。此外，我們知道若G是一個擁有完美配對的樹圖，則在限亮點西格瑪遊戲之中將有三種軌跡存在，最後也給出一個演算法以及利用其對偶遊戲的結果來描述這三種軌跡。

Lit-Only σ-Game and its Dual Game on Tree

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Lit-Only σ-Game and its Dual Game on Tree

ABSTRACT

Let G be a simple connected graph with n vertices {1, 2, … , n}. A configuration of G is an assignment of one of two colors, black or white, to each vertex of G. A move on the set of configurations of G is a function from the set to itself. Two different games with their own sets of moves are investigated in this thesis. The first one which is called the lit-only σ-game, contains n moves L_i corresponding to the vertices i. When the move L_i is applied to a configuration u, the color of a vertex j in u is changed if and only if i is a black vertex and j is a neighbor of i. The second one which is called the lit-only dual σ-game, has n moves L_i^* corresponding to the vertices i. When the move L_i^* is applied to a configuration u, the color of a vertex j in u is changed if and only if i has odd number of black neighbors and j=i. The dual relation between these two games will be clarified. In each of the two games, the set of configurations is partitioned into orbits by the action of its moves. An orbit with more than one configuration is called a nontrivial orbit. When G is a tree with some minor assumptions, we conjecture that there are two nontrivial lit-only dual σ-game orbits. We prove the conjecture under certain assumptions. It is known that the lit-only σ-game on a tree with perfect matchings has three orbits. We give an algorithm to describe these three orbits by applying the results in its dual game.

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1 Introduction

Let G be a simple connected graph with vertex set $V(G) = \{1, 2 \ldots, n\}$ and edge set $E(G)$. A *configuration* of G is an assignment of one of two colors, black or white, to each vertex of G. And we call a configuration u *trivial* if all the vertices are white. In each game on G we have a rule on configurations to apply with and we call those steps *moves*. For convenience, we use the set F_2^n of column vectors over $F_2 := \{0, 1\}$ to denote the set of configurations. The i-th entry u_i of a configuration u is 1 if and only if the vertex i is black on this configuration. An *orbit* $\mathcal{O}$ in a game is a subset of configurations such that for any two configurations $u, v \in \mathcal{O}$, there exists a sequence of moves that u can reach v by applying these moves in order. And we call a orbit *trivial* if and only if it has only one element. Our goal is to decrease the number of black vertices by applying several moves.

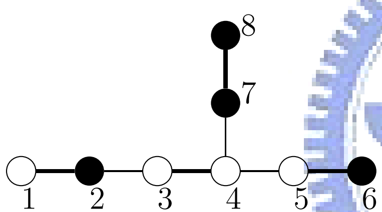
Here we consider in two different games, lit-only σ-game and it's dual game which is called Reeder's game. The lit-only σ-game is a variation of σ-game which was investigated from 1989 [4]. The Reeder's game was appeared in the 2005 paper [3] of M. Reeder. Although the two games seem different ostensibly, there are many connections between them.

2 Lit-only σ-game

A move L_i in the lit-only σ-game is defined as follows: If a vertex i is in black color in the configuration u, then when L_i applies to u, the colors of all neighbors of i will be changed but keep the colors of other vertices including i unchanged. On the other hand, if i is in white color in u then L_i does nothing about the configuration. And it is the reason we called the game lit-only σ-game. Let the $n \times n$ matrix A be the adjacency matrix of the given graph G. Note that $e_i^T u$ is the parity of u_i. where $\{e_i\}$ is the standard basis of F_2^n, that is, for $1 \leq i \leq n$, the i-th entry of e_i is 1 and the other entries are 0. We have that

$$L_i(u) = u + (e_i^T u)Ae_i = u + Ae_ie_i^T u = (I_n + Ae_ie_i^T)u, \tag{2.1}$$

where I_n is the $n \times n$ identity matrix. Note that for any vertex i and any configuration u, $L_i(L_i(u)) = u$. Here we have an example.



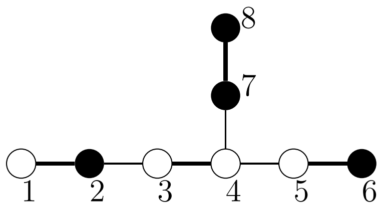
$$u = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad L_7 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

Figure 1: A configuration u and a move L_7 in the lit-only σ-game.

Now we consider in the Reeder's game. A move L_i^* is defined as follows: If the vertex i in the configuration u' has odd number of black neighbors then the move L_i^* changes the color of i and keeps other vertices unchanged. And if vertex i has even number of neighbors in black color, the move L_i^* does nothing. Note that $e_i^T Au'$ is the parity of the number of black neighbors of i in u'. Like in lit-only σ game, we also use matrices to represent the moves and then

$$\begin{aligned} L_i^*(u') &= u' + \left(e_i^T Au'\right) e_i = u' + e_ie_i^T Au' \\ &= (I_n + e_ie_i^T A)u' = (I_n + Ae_ie_i^T)^T u' \\ &= L_i^T(u'), \end{aligned} \tag{2.2}$$

and $L_i^*(L_i^*(u')) = u'$. Here is an example.



$$u' = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad L_7^* = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Figure 2: A configuration u' and a move L_7^* in the lit-only dual σ-game.

By (2.1) and (2.2) we notice that each of the two matrices representations of the moves in different games, respectively, is the transport of the other one.

We have one more relation between the two moves on G.

Proposition 2.1. *For a given graph G, let A be the adjacency matrix of G. Then $L_iA = AL_i^*$, i.e. the following diagram commutes:*

$$\begin{array}{ccc} u' & \xrightarrow{L_i^*} & L_i^*(u') \\ A\downarrow & \circlearrowright & \downarrow A \\ u = Au' & \xrightarrow{L_i} & L_i(u) \end{array}.$$

Proof. Note that

$$\begin{aligned} & AL_i^* \\ = \;& A(I_n + e_ie_i^TA) \\ = \;& (I_n + Ae_ie_i^T)A \\ = \;& L_iA \end{aligned}$$

and the proposition follows. ∎

3 Lit-only dual σ-game on a tree

Throughout this section let G be a tree with the vertex set $\{1, 2, \ldots, n\}$. We study the lit-only dual σ-game on G in this section. Let $u \in F_2^n$ be a configuration. Then u is *moveable* if there exists a vertex with odd black neighbors, i.e. $Au \neq 0$. Let B_u denote the subset of vertices consisting of black vertices in u, i.e. $B_u := \{i \mid u_i = 1\}$, and let $c(B_u)$ denote the number of components in the subgraph induced by B_u. Recall that an *independent set* of G is a subset of vertices in which each pair of vertices are not adjacent. For a subset S of vertices we denote $N[S]$ as the set of closed neighbors of S, i.e. $N[S] := S \cup \{a \mid \{a, s\} \in E(G), \text{ for some } s \in S\}$.

Lemma 3.1. *For any configuration u, each component of B_u can be reduced to a black vertex by a sequence of lit-only dual σ-game moves on G. Formally, for any $u \in F_2^n$ there exists $v \in F_2^n$ such that u, v are in the same orbit, B_v is an independent set and $c(B_v) = c(B_u)$.*

Proof. Since G is a tree, we know that each connected component of B_u is also a tree. Start from a component which has vertices more than 2 and select a leaf i of the component. Since i has only one neighbor in black color, we can use the move L_i^* to get a new configuration $w = u + e_i$. We change the color of the vertex i by L_i^* without change the number $c(B_u)$ then we know that u, w are in the same orbit and $c(B_u) = c(B_w)$. Repeat this process we finally have a configuration v which is in the same orbit with u and $c(B_u) = c(B_v)$ and each connected component of B_v has only one black vertex, i.e. B_v is an independent set. □

Lemma 3.2. *Let u, v be two nontrivial configurations such that $c(B_u)$ and $c(B_v)$ have different parities. Then u and v are in different lit-only dual σ-game orbits.*

Proof. Suppose there are two moveable configurations u, v such that u, v are in the same orbit. That is, u can reach v by applying several moves. If there is a move L_i^* changes $c(B_u)$, i.e. $c(B_u) \neq c(B_{L_i^*(u)})$, we know that L_i^* separates a connected component of B_u or combines several connected components into one. By the definition of moves of Reeder's game, we know i has odd number of black neighbors. Then the move L_i^* separates one component into odd number of components or combines odd number of components into one. So each one of these moves can not change the parity of $c(B_u)$ for any configuration u to another one. In other words, if two configurations u, v in the same orbit, then $c(B_u), c(B_v)$ have the same parity. □

The special case when G is a path is easy to settle.

Proposition 3.3. *Let G be a path and let u and v be two moveable configurations. Then u and v are in the same lit-only dual σ-game orbit if and only if $c(B_u) = c(B_v)$.*

Proof. Each vertex in G has at most two neighbors since G is a path. For a configuration w, we know that any move L_i^* can not change the B_w by the definition of moves of Reeder's game. If two moveable configurations u, v are in the same orbit then $c(B_u), c(B_v)$ must be equal since $c(B_u)$ will hold by any move L_i^*.

On the opposite side, we assume $G = \{1, 2, \ldots, n\}$ and two moveable configurations u, v with $c(B_u) = c(B_v) = k$. By Lemma 3.1 there exists two configurations u', v' such that u', v' are in the same orbit with u, v, respectively. And $B_{u'}, B_{v'}$ are independent sets with $B_{u'} = \{i_1, i_2, \ldots, i_k\}, B_{v'} = \{j_1, j_2, \ldots, j_k\}$.

We use these moves $L^*_{i_1-1}, L^*_{i_1-2}, \ldots, L^*_1, L^*_{i_1}, L^*_{i_1-1}, \ldots, L^*_2$ in turn to shift the black vertex i_1 of u to the vertex 1. And we shift those black vertices $i_2, i_3, \ldots, i_k$ to the vertices $3, 5, \ldots, 2k-1$ similarly. Then we get a new configuration w such that u', w are in the same orbit and $B_w = \{1, 3, 5, \ldots, 2k-1\}$.

If we use the same method to shift these black vertices of v' then we can get the same configuration w. So that we know that v', w are in the same orbit and u', v' are in the same orbit, that is, u, v are in the same orbit. Then the proposition follows. □

Definition 3.4. We call a graph G a *binary star*, and defined by $D(n; r, s)$, if all the leaves of G are adjacent to one of the endpoints of path P_n.

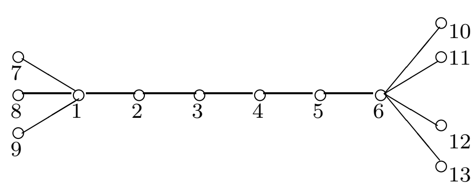


Figure 3: D(6; 3, 4).

Conjecture 3.5. *Assume G is a tree but not a binary star. Let u and v be two moveable configurations with $c(B_u) = c(B_v)$. Then u and v are in the same lit-only dual σ-game orbit.*

The following example, $D(5; 2, 0)$, is first found not have the conclusion of Conjecture 3.5 by Hau-wen Huang.

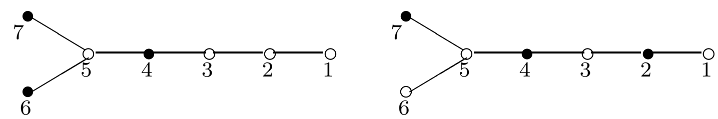


Figure 4: Two configurations which are not in the same lit-only dual σ-game orbit.

Lemma 3.6. *For any vertices i, j, the configurations e_i and e_j are in the same lit-only dual σ-game orbit. Moreover if S consists of the vertices in the path from i to j and u is a configuration with $N[B_u] \cap S = \emptyset$ then $u + e_i$ and $u + e_j$ are in the same lit-only dual σ-game orbit.*

Proof. Let the unique path from i to j be $i_0 i_1 i_2 \cdots i_k$ where $i_0 = i, i_k = j$ and $S = \{i_0, i_1, i_2, \ldots, i_k\}$. Since $N[B_u] \cap S = \emptyset$ these moves $L^*_{i_1}, L^*_{i_2}, \ldots, L^*_{i_k}$, $L^*_{i_0}, L^*_{i_1}, \ldots, L^*_{i_{k-1}}$ are doing nothing about u. And if we apply these moves in turn then we have

$$u + e_j = L^*_{i_{k-1}} L^*_{i_{k-2}} \cdots L^*_{i_0} L^*_{i_k} L^*_{i_{k-1}} \cdots L^*_{i_2} L^*_{i_1} (u + e_i),$$

that is, $u + e_i, u + e_j$ are in the same orbit. Let u be the configuration with no black vertices and for any i, j, e_i and e_j are in the same lit-only dual σ-game orbit. □

Conjecture 3.7. *Let G be a tree but not a binary star. Then the set F_2^n of configurations is partitioned into the following lit-only dual σ-game orbits:*

(i) the orbit $\{u\}$ of a single non-moveable configuration;

(ii) $\{u \in F_2^n \mid c(B_u) \neq 0$ is even.$\}$;

(iii) $\{u \in F_2^n \mid c(B_u)$ is odd.$\}$.

The following proves Conjecture 3.7 under the assumption that Conjecture 3.5 holds.

Proof. For any non-moveable configuration u we know that each move L_i^* does nothing on u. Then there are orbits of a single non-moveable configuration u.

There is a vertex i with at least three neighbors i_1, i_2, i_3 since G is not a path. We apply these moves $L_i^*, L_{i_3}^*, L_{i_2}^*, L_{i_1}^*$ on e_i and get a moveable configuration $v = L_i^* L_{i_3}^* L_{i_2}^* L_{i_1}^*(e_i)$ which $c(v) = 3$. Then we know that v, e_i are in the same orbit. By this method, in each process we have a moveable configuration u and find a vertex j with degree greater or equal to 3. First we shift black vertices out of $N[N[\{j\}]]$ and then shift one black vertex to j and then apply these moves $L_{j_1}^*, L_{j_2}^*, L_{j_3}^*, L_j^*$ and get a new moveable configuration v with $c(B_v) = c(B_u) + 2$.

By Conjecture 3.5 and the previously method we know that for two moveable configurations u, v if $c(B_u), c(B_v)$ have the same parity then u, v are in the same orbit. And by lemma 3.2 we know that the set F_2^n of configurations is partitioned into the following lit-only dual σ-game orbits:

(i) Trivial orbits $\{u\}$ which u is a non-moveable configuration;

(ii) $\{u \in F_2^n \mid c(B_u) \neq 0 \text{ is even.}\}$;

(iii) $\{u \in F_2^n \mid c(B_u) \text{ is odd.}\}$.

□

4 Tree with perfect matching

Lemma 4.1. *Let G be a tree with perfect matching. Then the perfect matching is unique.*

Proof. Since G is a tree with perfect matching, we know that G has even vertices. So that we prove this lemma by induction on $|V(G)| = 2k$. For $k = 1$, we have that the perfect matching is unique. We assume the lemma holds for $1 \le k \le d-1$. Let G be a tree with perfect matching with $|V(G)| = 2d$. Since G is a tree, we can find a leaf i with neighbor j. If G has a perfect matching π then we know that $\{i, j\}$ belongs to π. Then we consider the graph $G' = G - \{i, j\}$ which is an union of connected components. Since G is a tree with perfect matching π then each component of G' is also a tree with even vertices. Moreover, we know that G has perfect matching π then each component of G' must have a perfect matching π' such that $\pi' \subset \pi$. Since the number of vertices of each component is less or equal to $2(d-1)$ we have that the perfect matching π' is unique by the assumption. Then G has an unique perfect matching π and the lemma follows. $\square$

5 Lit-only dual σ-game on a tree with perfect matching

In this section we collect a known result to support Conjecture 3.7 and then Conjecture 3.5. M. Reeder uses the property of quadratic form to prove the following theorem [3, page 33].

Theorem 5.1. *Let G be a tree with perfect matching but not a path. Then there are three lit-only dual σ-game orbits.*

Hau-wen Huang quotes the above theorem to describe the three orbits combinatorially [2].

Proposition 5.2. *Assume Conjecture 3.5 hold. Then the set F_2^n of configurations is partitioned into the following three orbits:* $\{0\}$, $\{u \mid c(B_u) \neq 0$ *is even.*$\}$, $\{u \mid c(B_u)$ *is odd.*$\}$.

Proof. G is a tree with perfect matching so that the adjacency matrix of G is invertible then there is only one non-moveable configuration $\{0\}$. And since a non-moveable configuration is an orbit, we have an orbit $\{0\}$. By Lemma 3.2 we know if two configurations u, v such that $c(B_u), c(B_v)$ have different parities then u, v are not in the same orbit. And by Theorem 5.1 we know that there are only three lit-only dual σ-game orbits. So that if two moveable configurations u, v such that $c(B_u), c(B_v)$ have the same parity then u, v must be in the same orbit otherwise the number of orbits is greater than 3. Then we have the three orbits: $\{0\}$, $\{u \mid c(B_u) \neq 0$ is even.$\}$, $\{u \mid c(B_u)$ is odd.$\}$. □

6 Combinatorial interpretation of A^{-1}

For the completeness, we shall provide a combinatorial proof of the following well-known theorem, See for example [1, page 21].

Theorem 6.1. *If G is a tree with perfect matching, then the adjacency matrix A of G is invertible.*

Proof. A graph with perfect matching must has even number of vertices, then we prove this by induction on the number of vertex set $|V(G)| = 2k$.

1. For $k = 1$, the adjacency matrix of G is $A(G) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ then we have $\det(A(G)) = -1$. Since the determinant of $A(G)$ is not 0, we know that $A(G)$ is invertible.

2. Suppose for $k = n - 1$, it is true.

3. Let G is a tree with matching and $|V(G)| = 2(n-1)$. And another graph G with $V(G') = V(G) \cup \{2n-1, 2n\}$, and $E(G') = E(G) \cup \{\{i, 2n-1\}, \{2n-1, 2n\}\}$, for some $1 \le i \le 2n-2$. G' is also a tree with perfect matching. Then the $(2n) \times (2n)$ adjacency matrix $A(G')$ of G' is

$$\begin{pmatrix} & & & & & & 0 & 0 \\ & & & & & & \vdots & \\ & & & & & & 0 & 0 \\ & & & A(G) & & & 1 & \vdots \\ & & & & & & 0 & 0 \\ & & & & & & \vdots & 0 \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 1 \\ 0 & & & \cdots & & 0 & 1 & 0 \end{pmatrix}$$

And we get $\det(A(G')) = -\det(A(G)) \neq 0$

That is, a tree with perfect matching has an invertible adjacency matrix. $\square$

Definition 6.2. Let G be a tree with perfect matching π. A path $i_0 i_1 \ldots i_t$ of length t is *alternating* if t is odd and for $0 \le j \le t-1$,

$$i_j i_{j+1} \begin{cases} \in \pi, & \text{if } j \text{ is even;} \\ \notin \pi, & \text{if } j \text{ is odd.} \end{cases}$$

The following proposition gives a combinatorial interpretation of A^{-1}.

Proposition 6.3. $(A^{-1})_{ij} = \begin{cases} 1, & \textit{if the path from } i \textit{ to } j \textit{ is alternating;} \\ 0, & \textit{else.} \end{cases}$

Proof. The $n \times n$ matrix B is defined as : If the path from i to j is alternating then $B_{ij} = 1$, otherwise, $B_{ij} = 0$. And we want to show that $AB = I_n$. We have that

$$(AB)_{ij} = \sum_{k=i}^{n} A_{ik} B_{kj} = \sum_{k \in N[\{i\}]-\{i\}} B_{kj}.$$

In other words, $(AB)_{ij}$ stands for the number of neighbors k of i such that the paths from these neighbors k to j are alternating with odd length.

If $i = j$: Since G is a tree with perfect matching, for each vertex i there is only one neighbor k of i such that the path ik is an alternating path, and then $(AB)_{ii} = 1$.

If $i \neq j$: We assume $ik \in \pi$. If k is in the unique path from i to j, then there is not an alternating path from the neighbor of i to j and we have $(AB)_{ij} = 0$. If k is not in the unique path from i to j, and there is at most one neighbor $l \neq k$ of i such that the path from l to j is an alternating path, then the path from k to j is also an alternating path and we have $(AB)_{ij} = B_{kj} + B_{lj} = 0$ in F_2. If no such l exists, we know that $(AB)_{ij} = 0$.

Finally we have $AB = I_n$ and then $A^{-1} = B$. □

7 Lit-only σ-game on a tree with perfect matching

Here we have a relation between lit-only σ-game orbits and lit-only dual σ-game orbits on tree with perfect matching.

Proposition 7.1. *Let G be a tree with perfect matching, and $\mathcal{O}$ are $\mathcal{O}'$ are the sets of orbits in lit-only σ-game and lit-only dual σ-game respectively. Then $\mathcal{O} = \{AO' \mid O' \in \mathcal{O}'\}$ and $\mathcal{O}' = \{A^{-1}O \mid O \in \mathcal{O}\}$.*

Proof. Let u', v' be in the same orbit O' and $O' \in \mathcal{O}'$. If $v' = L^*_{i_1}(L^*_{i_2} \cdots L^*_{i_k}(u'))$, by proposition 2.1 we have

$$\begin{aligned} Av' &= A(L^*_{i_1}(L^*_{i_2} \cdots L^*_{i_k}(u'))) \\ &= L_{i_1}(A(L^*_{i_2} \cdots L^*_{i_k}(u'))) \\ &= \vdots \\ &= L_{i_1}(L_{i_2} \cdots L_{i_k}(Au')). \end{aligned}$$

So that Au', Av' are in the same orbit of lit-only σ game. And we know that if O' is an orbit in lit-only dual σ-game then AO' is an orbit in lit-only σ-game. We have that $\mathcal{O} = \{AO' \mid O' \in \mathcal{O}'\}$. Moreover, since G is a tree with perfect matching then A^{-1} exists so that we prove $\mathcal{O}' = \{A^{-1}O \mid O \in \mathcal{O}\}$ similarly. □

By using Proposition 7.1, for any configuration u we can know u is in which lit-only σ-game orbit by checking the lit-only dual σ-game orbit of $A^{-1}u$. And the following propositions are Hau-wen Huang's result [2].

Proposition 7.2. *Let G be a tree with perfect matching but not a path. Then there are three lit-only σ-game orbits. Moreover, the three orbits are $\{0\}$, $\{Au \mid c(B_u) \neq 0$ is even.$\}$, $\{Au \mid c(B_u)$ is odd.$\}$.*

Proposition 7.3. *There exist distinct vertices i, j such that e_i and e_j are in different lit-only σ-game orbits.*

Then we know that in each orbit of lit-only σ-game there is a configuration with at most one black vertex.

8 Algorithm

Let G be a tree with perfect matching but not a path. By the above result 5.2, 7.1 we know that two configurations e_i and e_j are in the same lit-only σ-game orbit if and only if $c(B_{A^{-1}e_i})$ and $c(B_{A^{-1}e_j})$ have the same parity. We shall give the algorithm to determine which orbit the configuration e_i is belonging to.

Algorithm. For a configuration e_i is given, we want to find the corresponding configuration u' such that $e_i = Au'$.

Input Set $u' = 0$

Step 1 Start from the subset $X = \{i\}$ of $V(G)$.

Step 2 If the vertex j is in the same matching with i, set $u' := u' + e_j$, and $X := N[X]$.

Step 3 If vertex $k \in N[X] - X$ is adjacent to a black vertex in u', and the vertex l is in the same matching with k, then set $u' := u' + e_l$.

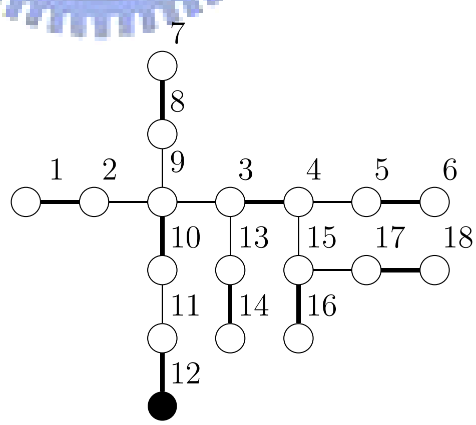
Step 4 Set $X := X \cup N[X]$.

Step 5 Repeat Step 3 and Step 4 until $X = V(G)$.

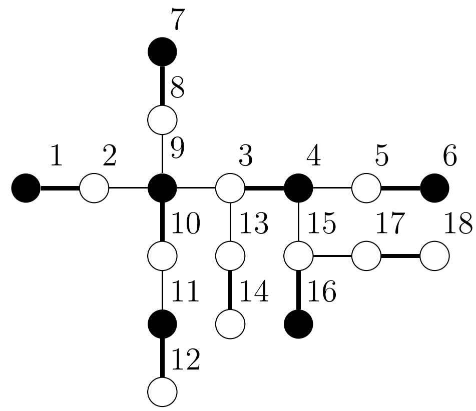
Output We get a configuration u'.

Here is an example.

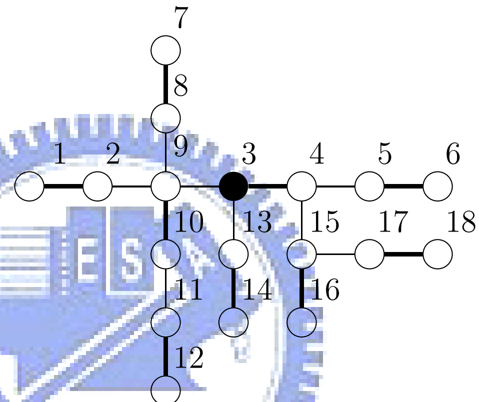
Example 8.1. G is shown and $u_1 = e_{12}$. And these thick edges are the matching of G.



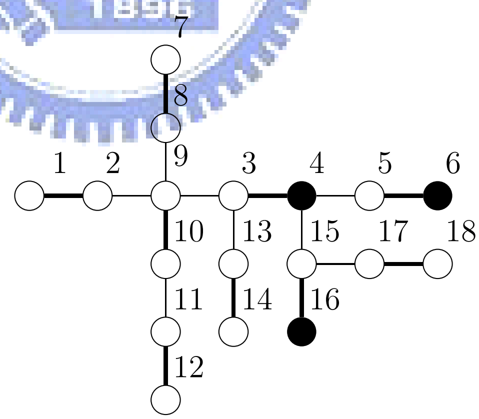
Start form G with all vertices in white. First we change $\{11\}$ to black, then $\{9\}$, and then $\{1,7,4\}$, and finally $\{6,16\}$. We have e'_{12} shown as



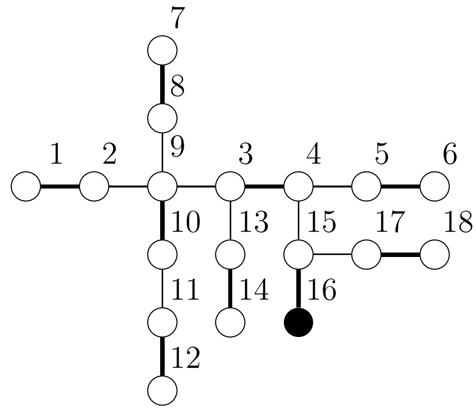
$u_2 = e_3$,



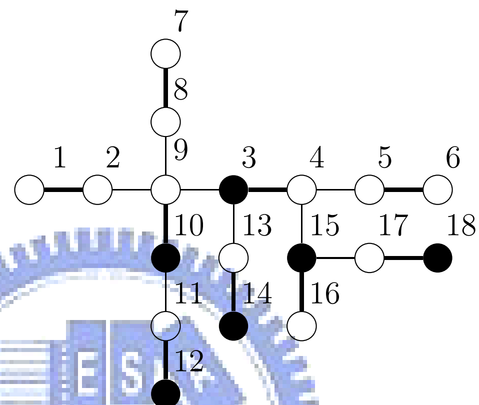
then we have e'_3



$u_3 = e_{16}$,



e'_{16} is shown



Since $c(e_{12}) = 7, c(e_3) = 3, c(e_{16}) = 6$, we know that e_{12}, e_3 are in the same orbit and e_{16} is not.

9 Conclusion

Let G be a tree with perfect matching but not a path. For any moveable configurations $u, v \in F_2^n$, there are two configurations $u', v' \in F_2^n$ such that $u' = A^{-1}u, v' = A^{-1}v$, and we know that u, v are in the same orbit in lit-only σ-game if and only if u', v' are in the same orbit in lit-only dual σ-game if and only if $c(B_{u'}), c(B_{v'})$ have the same parity. So we can know that whether two configurations u, v are in the same orbit or not by checking the parities of $c(B_{A^{-1}u}), c(B_{A^{-1}v})$.

Moreover, for a moveable configuration u in lit-only σ-game there is a moveable configuration $u' = A^{-1}u$ in lit-only dual σ-game and by Proposition 7.3 we know that: Whether $B_{u'}$ is odd or even, there is a configuration e_i which has only one vertex in black color in the same orbit with u. And by applying the algorithm, we can find these e_i's which are in the same orbit with u.

That is: Given a tree G with perfect matching but not a path, and any initial configuration u with at least one vertex in black color, then we can reach a configuration e_i with a single vertex i in black color by applying several moves in lit-only σ-game. Moreover, we know the single black vertex appearing at which vertex of G.

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