Triangle-free Distance-regular Graphs

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Main Results

Let Γ denote a distance-regular graph with $d \geq 3$. Assume Γ has intersection numbers $a_1 = 0$ and $a_2 \neq 0$. We prove the following (i)-(iii) are equivalent.

(i) Γ is Q-polynomial and contains no parallelograms of length 3;

(ii) Γ is Q-polynomial and contains no parallelograms of length i for $3 \leq i \leq d$;

(iii) Γ has classical parameters (d, b, α, β) with $b < -1$.

Main Results

Furthermore, suppose (i)-(iii) in the previous page hold we show that each of

$$b(b+1)^2(b+2)/c_2, (b-2)(b-1)b(b+1)/(2+2b-c_2)$$

is an integer and that $c_2 \leq b(b+1)$.

(Hermitian forms graph $Her_2(d)$ satisfies $a_1 = 0$, $a_2 \neq 0$ and $c_2 = b(b+1)$.)

A parallelogram of length i

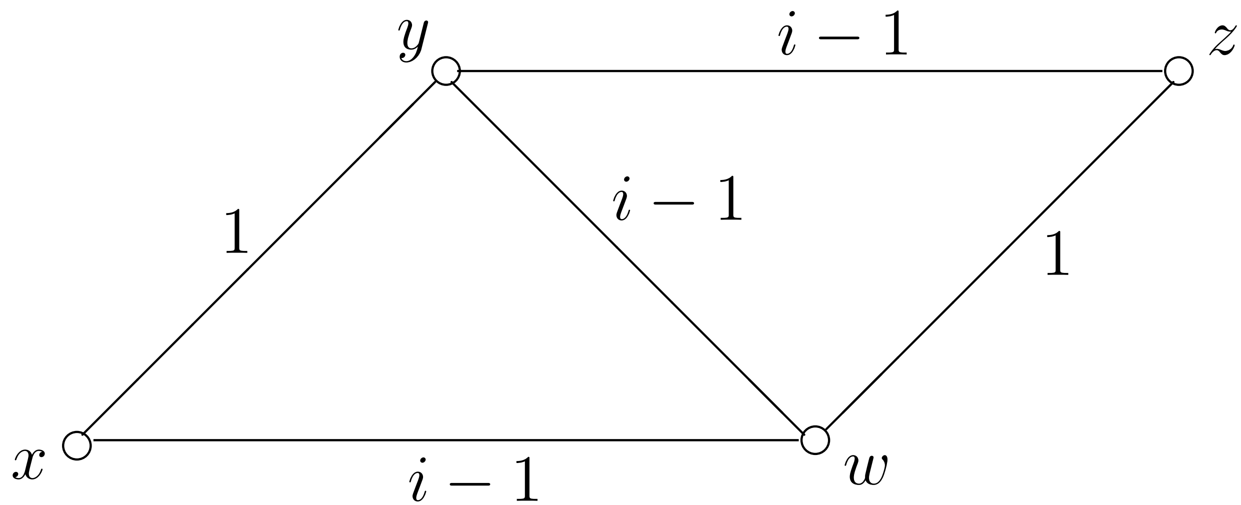

Figure 1: A parallelogram of length i.

Balanced set property in Q-poly. DRG

Theorem (Terwilliger, 1995) Assume Γ is Q-polynomial with respect to a primitive idempotent E, and let $\theta^*_0, \ldots, \theta^*_d$ denote the corresponding dual eigenvalues. Then For all integers $1 \leq h \leq d$, $0 \leq i, j \leq d$ and for all $x, y \in X$ such that $\partial(x, y) = h$,

$$\sum_{\substack{z \in X \\ \partial(x,z)=i \\ \partial(y,z)=j}} E\hat{z} - \sum_{\substack{z \in X \\ \partial(x,z)=j \\ \partial(y,z)=i}} E\hat{z} = p^h_{ij} \frac{\theta^*_i - \theta^*_j}{\theta^*_0 - \theta^*_h} (E\hat{x} - E\hat{y}).$$

DRG with classical parameters

Γ is said to have *classical parameters* (d, b, α, β) whenever the intersection numbers of Γ satisfy

$$c_i = \begin{bmatrix} i \\ 1 \end{bmatrix} \left(1 + \alpha \begin{bmatrix} i-1 \\ 1 \end{bmatrix} \right) \qquad \text{for } 0 \le i \le d, \tag{1}$$

$$b_i = \left(\begin{bmatrix} d \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix} \right) \left(\beta - \alpha \begin{bmatrix} i \\ 1 \end{bmatrix} \right) \qquad \text{for } 0 \le i \le d, \tag{2}$$

where

$$\begin{bmatrix} i \\ 1 \end{bmatrix} := 1 + b + b^2 + \cdots + b^{i-1}. \tag{3}$$

Being a special class of Q-poly DRGs

Let Γ denote a distance-regular graph with diameter $d \geq 3$. Choose $b \in \mathbb{R} \setminus \{0, -1\}$. Then the following (i), (ii) are equivalent.

(i) Γ is Q-polynomial with associated dual eigenvalues $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$ satisfying

$$\theta_i^* - \theta_0^* = (\theta_1^* - \theta_0^*) \begin{bmatrix} i \\ 1 \end{bmatrix} b^{1-i} \quad \text{for } 1 \leq i \leq d.$$

(ii) Γ has classical parameters (d, b, α, β) for some real constants α, β.

Use balanced set idea to count

Fix three vertices x, y, z such that

$$\partial(x,y) = 1, \quad \partial(y,z) = i-1, \quad \partial(x,z) = i.$$

Define

$$s_i = s_i(x,y,z) := |\Gamma_{i-1}(x) \cap \Gamma_{i-1}(y) \cap \Gamma_1(z)|,$$
$$\ell_i = \ell_i(x,y,z) := |\Gamma_i(x) \cap \Gamma_{i-1}(y) \cap \Gamma_1(z)|.$$

Then

$$\begin{aligned} s_i + \ell_i &= a_{i-1} \\ s_i\theta^*_{i-1} + \ell_i\theta^*_i &= a_{i-1}\theta^*_2 + a_{i-1}\frac{\theta^*_{i-1} - \theta^*_1}{\theta^*_0 - \theta^*_{i-1}}(\theta^*_1 - \theta^*_i) \end{aligned}$$

Solving s_i to obtain the first result

$$s_i = a_{i-1} \frac{(\theta^*_0 - \theta^*_{i-1})(\theta^*_2 - \theta^*_i) - (\theta^*_1 - \theta^*_{i-1})(\theta^*_1 - \theta^*_i)}{(\theta^*_0 - \theta^*_{i-1})(\theta^*_{i-1} - \theta^*_i)},$$

and

$s_3 = 0$ (no parallelogram of length 3)

iff $s_i = 0$ for $3 \leq i \leq d$

iff Γ has classical parameters (d, b, α, β) with $b < -1$.

Weak-geodetically closed subset of Γ

Recall that a sequence x, y, z of vertices of Γ are *geodetic* whenever

$$\partial(x, y) + \partial(y, z) \quad = \quad \partial(x, z).$$

Recall that a sequence x, y, z of vertices of Γ are *weak-geodetic* whenever

$$\partial(x, y) + \partial(y, z) \leq \partial(x, z) + 1.$$

Definition 0.1. A subset $\Omega \subseteq X$ is *weak-geodetically closed* if for any weak-geodetic sequence x, y, z of Γ,

$$x,\ z \in \Omega \Longrightarrow y \in \Omega.$$

The existence of w.g.c. subgraphs

Theorem(C. Weng, H. Suzuki, 1998) Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $d \geq 3$. Assume that the intersection numbers $a_1 = 0$ and $a_2 \neq 0$. Suppose that Γ contains no parallelograms of length 3. Then for each pair of vertices $v, w \in X$ at distance $\partial(v, w) = 2$, there exists a weak-geodetically closed subgraph Ω of diameter 2 in Γ containing v, w.

A strongly regular subgraph

Furthermore Ω is strongly regular with intersection numbers

$$\begin{aligned} a_i(\Omega) &= a_i(\Gamma), && (4) \\ c_i(\Omega) &= c_i(\Gamma), && (5) \\ b_i(\Omega) &= a_2(\Gamma) + c_2(\Gamma) - a_i(\Omega) - c_i(\Omega) && (6) \end{aligned}$$

for $0 \leq i \leq 2$.

Note that by assuming no parallelograms of length up to 4, $a_1 = 0$ and $a_2 \neq 0$, it is still open if there exists a weak geodetically closed subgraph of diameter 3.

The integral conditions

By using the integral conditions on the previous SRG, we find

Theorem Let Γ denote a distance-regular graph with classical parameters (d, b, α, β), where $d \geq 3$. Assume Γ has intersection numbers $a_1 = 0$ and $a_2 \neq 0$. Then

$$\frac{b(b+1)^2(b+2)}{c_2}, \tag{7}$$

$$\frac{(b-2)(b-1)b(b+1)}{2+2b-c_2} \tag{8}$$

are both integers.

A bound of intersection numbers

Proposition (Weng, 1998) Let Γ denote a distance-regular graph with diameter $d \geq 3$. Suppose there exists a weak-geodetically closed subgraph Ω of Γ with diameter 2. Then the intersection numbers of Γ satisfy the following inequality

$$a_3 \geq a_2(c_2 - 1) + a_1.$$

Apply the bound in the last page

We apply the bound in the last page to find

Corollary Let Γ denote a distance-regular graph with classical parameters (d, b, α, β), where $d \geq 3$. Suppose the intersection numbers $a_1 = 0$ and $a_2 \neq 0$. Then

$$c_2 \leq b^2 + b + 2.$$

In fact the bound can be improve to $c_2 \leq b^2 + b$ by using the integral condition to eliminate the case $c_2 = b^2 + b + 1$ and $c_2 = b^2 + b + 2$.

Examples

Example Hermitian forms graph $Her_2(d)$ is a distance-regular graph with classical parameters (d, b, α, β) with $b = -2$, $\alpha = -3$ and $\beta = -((-2)^d + 1)$, which satisfies $a_1 = 0$, $a_2 \neq 0$ and $c_2 = b(b+1)$.

Example Gerwitz graph is a distance-regular graph with diameter 2 and intersection numbers $a_1 = 0$, $c_2 = 2$, $k = 10$, which can be written as classical parameters (d, b, α, β) with $d = 2$, $b = -3$, $\alpha = -2$, $\beta = -5$, so we have $c_2 = \dfrac{(b+1)^2}{2}$.

Conjectures

Conjecture. (Gerwitz graph does not grow.) There is no distance-regular graph with classical parameters $(d, -3, -2, -\frac{1+(-3)^d}{2})$, where $d \geq 3$.

More generally,

Conjecture. There is no distance-regular graph with classical parameters (d, b, α, β), where $\alpha = (b-1)/2, \beta = -(1+b^d)/2$, and $-b$ is a power of an odd prime.

Thank You!!!