Bidiagonal triples and the quantum group $U_q(sl_2)$

Chih-wen Weng

(joint work with Paul Terwilliger)

Department of Applied Mathematics,

National Chiao Tung University,

Taiwan

The Basis of $sl_2(\mathbb{K})$

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

h, e, f satisfy

$$[h, e] = 2e, \qquad [h, f] = -2f, \qquad [e, f] = h.$$

Another Basis of $sl_2(\mathbb{K})$

$$A = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, C = \begin{pmatrix} -1 & 0 \\ -2 & 1 \end{pmatrix}.$$

A, B, C satisfy

$$[A, B] = -2A - 2B, [B, C] = -2B - 2C,$$

$$[C, A] = -2C - 2A.$$

Bidiagonal Matrices

Let X denote a square matrix. We say X is **upper bidiagonal** whenever both (i) each nonzero entry of X is on the diagonal or superdiagonal; (ii) each entry on the superdiagonal of X is nonzero. We say X is **lower bidiagonal** whenever the transpose of X is upper bidiagonal.

Examples

$$A = \begin{pmatrix} q^{-3} & q^{3}-q^{-3} & 0 & 0 \\ 0 & q^{-1} & q^{3}-q^{-1} & 0 \\ 0 & 0 & q & q^{3}-q \\ 0 & 0 & 0 & q^{3} \end{pmatrix},$$

$$C = \begin{pmatrix} q^{-3} & 0 & 0 & 0 \\ q^{-3}-q^{-1} & q^{-1} & 0 & 0 \\ 0 & q^{-3}-q & q & 0 \\ 0 & 0 & q^{-3}-q^{3} & q^{3} \end{pmatrix},$$

where $q^2 \neq 1, q^4 \neq 1, q^6 \neq 1.$

Bidiagonal Triple

Let $\mathbb{K}$ be an algebraically closed field with characteristic 0. Let $\mathbb{V}$ denote a vector space over $\mathbb{K}$ with finite positive dimension. By a **bidiagonal triple** on $\mathbb{V}$ we mean a sequence of linear transformations $A, B, C : \mathbb{V} \to \mathbb{V}$ that satisfy the following three conditions:

Bidiagonal Triple

(i) There exists a basis for $\mathbb{V}$ with respect to which the matrices representing A, B, C are upper bidiagonal, diagonal, and lower bidiagonal, respectively.

(ii) There exists a basis for $\mathbb{V}$ with respect to which the matrices representing B, C, A are upper bidiagonal, diagonal, and lower bidiagonal, respectively .

(iii) There exists a basis for $\mathbb{V}$ with respect to which the matrices representing C, A, B are upper bidiagonal, diagonal, and lower bidiagonal, respectively.

Example

A, C as before and

$$B = \begin{pmatrix} q^3 & 0 & 0 & 0 \\ 0 & q & 0 & 0 \\ 0 & 0 & q^{-1} & 0 \\ 0 & 0 & 0 & q^{-3} \end{pmatrix}.$$

Then A, B, C is a bidiagonal triple.

Proof

$$P = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & q^{-2} & -1-q^{-2} & 1 \\ -q^{-6} & q^{-2}+q^{-4}+q^{-6} & -1-q^{-2}-q^{-4} & 1 \end{pmatrix}$$

Then

$$P^{-1}BP = A, P^{-1}CP = B, P^{-1}AP = C.$$

More Examples

(i) A is upper bidiagonal with entries $A_{ii} = q^{2i-n}$ for $0 \leq i \leq n$ and $A_{i,i+1} = q^n - q^{2i-n}$ for $0 \leq i \leq n-1$.

(ii) B is diagonal with $B_{ii} = q^{n-2i}$ for $0 \leq i \leq n$.

(iii) C is lower bidiagonal with entries $C_{ii} = q^{2i-n}$ for $0 \leq i \leq n$ and $C_{i,i-1} = q^{-n} - q^{2i-n}$ for $1 \leq i \leq n$.

Then the sequence A, B, C is a bidiagonal triple on $\mathbb{K}^{n+1}$ (with base q).

More Examples

(i) A is upper bidiagonal with entries $A_{ii} = 2i - n$ for $0 \leq i \leq n$ and $A_{i,i+1} = 2n - 2i$ for $0 \leq i \leq n - 1$.

(ii) B is diagonal with $B_{ii} = n - 2i$ for $0 \leq i \leq n$.

(iii) C is lower bidiagonal with entries $C_{ii} = 2i - n$ for $0 \leq i \leq n$ and $C_{i,i-1} = -2i$ for $1 \leq i \leq n$.

Then the sequence A, B, C is a bidiagonal triple on $\mathbb{K}^{n+1}$ (with base $q = 1$).

Normalized Bidiagonal Triples

We refer all of the above mentioned bidiagonal triples as **normalized bidisgonal triples with base** q**.**

Lemma

Let A, B, C denote a bidiagonal triple on $\mathbb{V}$. Let $\alpha^{\pm}, \beta^{\pm}, \gamma^{\pm}$ denote scalars in $\mathbb{K}$ with $\alpha^{+}, \beta^{+}, \gamma^{+}$ nonzero. Then the sequence

$$\alpha^{+}A + \alpha^{-}I, \quad \beta^{+}B + \beta^{-}I, \quad \gamma^{+}C + \gamma^{-}I$$

is a bidiagonal triple on $\mathbb{V}$.

Affine Equivalence

Let A, B, C and $A^{'}, B^{'}, C^{'}$ denote two bidiagonal triples on $\mathbb{V}$. We say these two sequences are **affine equivalent** whenever

$$A^{'} = \alpha^{+}A + \alpha^{-}I, B^{'} = \beta^{+}B + \beta^{-}I, C^{'} = \gamma^{+}C + \gamma^{-}I$$

for some scalars $\alpha^{\pm}, \beta^{\pm}, \gamma^{\pm}$ $\mathbb{K}$ with $\alpha^{+}, \beta^{+}, \gamma^{+}$ nonzero.

Main Theorem

Each bidiagonal triple is affine equivalent to a normalized bidiagonal triple with base q.

Lie Algebra $sl_2(\mathbb{K})$

This algebra has a basis e, f, h satisfying

$$[h, e] = 2e, \qquad [h, f] = -2f, \qquad [e, f] = h,$$

where $[\,,\,]$ denotes the Lie bracket.

Irreducible sl_2-Modules

There exists a family

$$\mathbb{V}_n \qquad n = 0, 1, 2, \ldots \tag{1}$$

of finite dimensional irreducible sl_2-modules with the following properties. The module $\mathbb{V}_n$ has a basis $v_0, v_1, \ldots, v_n$ satisfying $hv_i = (n-2i)v_i$ for $0 \leq i \leq n$, $fv_i = (i+1)v_{i+1}$ for $0 \leq i \leq n-1$, $fv_d = 0$, $ev_i = (n-i+1)v_{i-1}$ for $1 \leq i \leq n$, $ev_0 = 0$.

Irreducible sl_2-Modules

Every irreducible sl_2-module of dimension $n+1$ is isomorphic to the $\mathbb{V}_n$ in previous slide.

Alternative Basis for sl_2

Set $x = -h + 2e, y = h, z = -h - 2f$ in sl_2. Then x, y, z is another basis of sl_2 satisfying

$$[x, y] = -2x - 2y, [y, z] = -2y - 2z, [z, x] = -2z - 2x.$$

Bidiagonal Triples and sl_2

The alternate basis x, y, z of sl_2 act on $\mathbb{V}_n$ as a bidiagonal triple.

$U_q(sl_2)$

Quantum algebra $U_q(sl_2)$ is the unital associative $\mathbb{K}$-algebra with generators e, f, k, k^{-1} and the following relations:

$$kk^{-1} = k^{-1}k = 1,$$
$$kek^{-1} = q^2 e, kfk^{-1} = q^{-2} f,$$
$$ef - fe = \frac{k - k^{-1}}{q - q^{-1}},$$

where $q \in \mathbb{K}$ is not a root of unity.

Alternative Presentation

The quantum algebra $U_q(sl_2)$ is isomorphic to the unital associative $\mathbb{K}$-algebra with generators x, y, z, z^{-1} and the following relations:

$$yy^{-1} = y^{-1}y = 1,$$

$$\frac{qxy - q^{-1}yx}{q - q^{-1}} = 1,$$

$$\frac{qyz - q^{-1}zy}{q - q^{-1}} = 1,$$

$$\frac{qzx - q^{-1}xz}{q - q^{-1}} = 1.$$

Proof

An isomorphism is given by:

$$\begin{aligned} y^{\pm 1} &\rightarrow k^{\pm 1}, \\ z &\rightarrow k^{-1} + f, \\ x &\rightarrow k^{-1} - q(q - q^{-1})^2 k^{-1} e. \end{aligned}$$

The inverse of this isomorphism is given by:

$$\begin{aligned} k^{\pm 1} &\rightarrow y^{\pm 1}, \\ f &\rightarrow z - y^{-1}, \\ e &\rightarrow \frac{1 - yx}{q(q - q^{-1})^2}. \end{aligned}$$

Irreducible $U_q(sl_2)$-Modules

There exists a family

$$\mathbb{V}_{\varepsilon,n} \qquad \varepsilon \in \{1,-1\}, \qquad n = 0,1,2\ldots$$

of finite dimensional irreducible $U_q(sl_2)$-modules with the following properties. The module $\mathbb{V}_{\varepsilon,n}$ has a basis $u_0, u_1, \ldots, u_n$ such that $ku_i = \varepsilon q^{n-2i} u_i$ for $0 \leq i \leq n$, $fu_i = [i+1]_q u_{i+1}$ for $0 \leq i \leq n-1$, $fu_n = 0$, $eu_i = \varepsilon[n-i+1]_q u_{i-1}$ for $1 \leq i \leq n$, $eu_0 = 0$.

Irreducible $U_q(sl_2)$-Modules

Every irreducible $U_q(sl_2)$-module of dimension $n+1$ is isomorphic to $\mathbb{V}_{-1,n}$ or $\mathbb{V}_{1,n}$.

Bidiagonal Triples and $U_q(sl_2)$

Let $\mathbb{V}_{\varepsilon,n}$ denote the finite dimensional irreducible $U_q(sl_2)$-module. Then the alternate generators $\varepsilon x, \varepsilon y, \varepsilon z$ act on $\mathbb{V}_{\varepsilon,n}$ as a bidiagonal triple.

Thank You