A characterization of bipartite distance-regular graphs

翁志文

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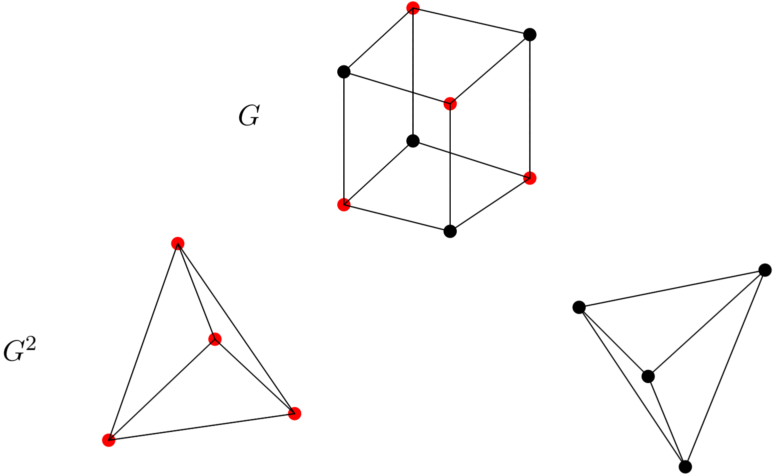
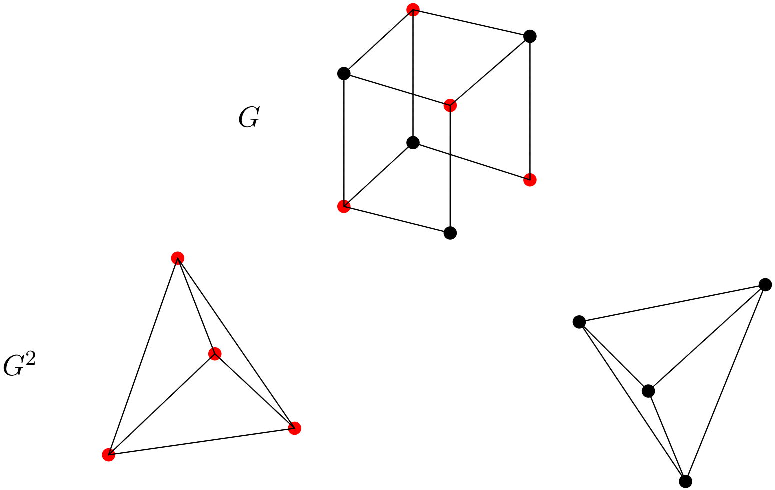
2013 年 5 月 11 日



If both parts of G^2 are isomorphic distance-regular graphs, can you conclude that the bipartite graph G is also distance-regular?



Maybe we need some regularity assumption on G.

A bipartite graph G with bipartition $X \cup Y$ is 2-partially distance-regular if G is regular, and

$$c_2 := |G_1(u) \cap G_1(v)|$$

is a constant for any two vertices $u, v \in X \cup Y$ at distance 2.

Note that the total graph of a symmetric BIBD is a bipartite 2-partially distance-regular graph of diameter 3.

Abstract

It is well-known that the halved graphs of a bipartite distance-regular graph are distance-regular. Examples are given to show that the converse does not hold. Thus, a natural question is to find out when the converse is true. In this talk we show that if the graph is connected bipartite 2-partially distance-regular with even spectral diameter then the converse mentioned above holds. This is a joint work with Guang-Siang Lee.

Keywords: Distance-regular graph, Distance matrices, Predistance polynomials, Spectral diameter

2000 MSC: 05E30, 05C50

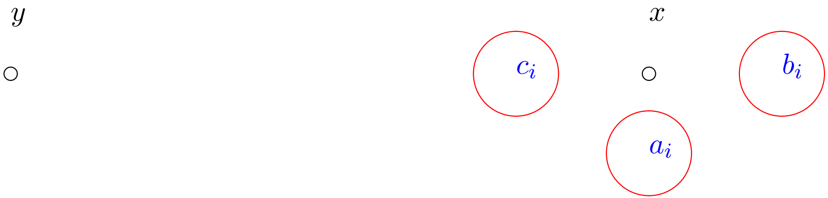
Distance-regular graphs

A graph G with diameter D is distance-regular if and only if for $i \leq D$,

$$\begin{aligned} c_i &:= |G_1(x) \cap G_{i-1}(y)|, \\ a_i &:= |G_1(x) \cap G_i(y)|, \\ b_i &:= |G_1(x) \cap G_{i+1}(y)| \end{aligned}$$

are constants subject to all vertices x, y with $\partial(x, y) = i$.

$\partial(x, y) = i$



Note that $a_i + b_i + c_i = b_0$ and $k := b_0$ is the valency of G.

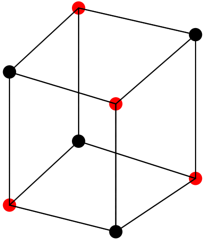
Distance matrices

The matrices that we are concerned are square matrices with rows and columns indexed by the vertex set VG. For each i let A_i be a 01-matrix with entries

$$(A_i)_{xy} = \begin{cases} 1, & \text{if } \partial(x,y) = i; \\ 0, & \text{else.} \end{cases}$$

A_i is called i-th distance matrix, and $A = A_1$ is also called the adjacency matrix of Γ. Note $A_0 = I$ and $A_{-1} = A_{D+1} = 0$.

G



$$A_2 = \begin{pmatrix} 0&1&1&1&0&0&0&0\\ 1&0&1&1&0&0&0&0\\ 1&1&0&1&0&0&0&0\\ 1&1&1&0&0&0&0&0\\ 0&0&0&0&0&1&1&1\\ 0&0&0&0&1&0&1&1\\ 0&0&0&0&1&1&0&1\\ 0&0&0&0&1&1&1&0 \end{pmatrix},$$

$A_0 = I,$

$$A_1 = \begin{pmatrix} 0&0&0&0&1&1&1&0\\ 0&0&0&0&1&1&0&1\\ 0&0&0&0&1&0&1&1\\ 0&0&0&0&0&1&1&1\\ 1&1&1&0&0&0&0&0\\ 1&1&0&1&0&0&0&0\\ 1&0&1&1&0&0&0&0\\ 0&1&1&1&0&0&0&0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0 \end{pmatrix}.$$

Three-term recurrence relation of distance matrices

G is distance-regular if and only if

$$AA_i = b_{i-1}A_{i-1} + a_iA_i + c_{i+1}A_{i+1} \qquad 0 \le i \le D,$$

where $c_{D+1} := 1$.

Proof.

$$(AA_i)_{xy} = \begin{cases} b_{i-1}, & \text{if } \partial(x,y) = i-1; \\ a_i, & \text{if } \partial(x,y) = i; \\ c_{i+1}, & \text{if } \partial(x,y) = i+1. \end{cases}$$

□

利用方程式描述組合性質

Orthogonal polynomials

In last page we show

$$AA_i = b_{i-1}A_{i-1} + a_iA_i + c_{i+1}A_{i+1} \qquad 0 \le i \le D.$$

Consider polynomials $f_0(x) := 1$, $f_1(x) := x$ and $f_i(x)$ is defined recursively using

$$xf_i(x) = b_{i-1}f_{i-1}(x) + a_if_i(x) + c_{i+1}f_{i+1}(x) \qquad 2 \le i \le D.$$

Note that $A_i = f_i(A)$, $f_{D+1}(A) = A_{D+1} = 0$, and $f_i(x)$ has degree i.

The polynomials $f_0(x) = 1$, $f_1(x) = x$, …, $f_D(x)$ are orthogonal with respect to the inner product defined by

$$\langle f_i(x), f_j(x)\rangle_\triangle := \frac{\mathrm{tr}(f_i(A)f_j(A))}{n} = \frac{\mathrm{tr}(A_iA_j)}{n}.$$

Indeed the converse is also true, so we have the following

Theorem

1. G *is distance-regular*
2. *there exist a sequence of polynomial* $f_0(x) = 1,\ f_1(x) = x,\ \ldots,\ f_D(x)$ *such that* $\deg(f_i) = i$ *and* $A_i = f_i(A)$.

□

The polynomial $f_0(x) = 1,\ f_1(x) = x,\ \ldots,\ f_D(x)$ are called the distance-polynomials of distance-regular graph G.

Preliminaries of spectral graph theory

Let G be a connected graph of order n and diameter D (not necessary to be distance-regular). Assume that adjacency matrix $A = A_1$ has $d + 1$ distinct eigenvalues $k = \lambda_0 > \lambda_1 > \ldots > \lambda_d$ with corresponding multiplicities $1 = m_0, m_1, \cdots, m_d$. Note that $D \leq d$ and

$$Z(x) := \prod_{i=0}^{d}(x - \lambda_i)$$

is the minimal polynomial of A, and d is called the spectral diameter of G.

Consider the vector space $\mathbb{R}_d[x] \cong \mathbb{R}[x]/\langle Z(x) \rangle$ with the inner product

$$\langle p(x), q(x) \rangle_\triangle := \mathrm{tr}(p(A)q(A))/n,$$

for $p(x), q(x) \in \mathbb{R}_d[x]$.

Gram-Schmidt process

There exists a unique sequence of polynomials

$$p_0(x), p_1(x), \ldots, p_d(x) \in \mathbb{R}_d[x],$$

called predistance polynomials of G, satisfying

$$\deg p_i(x) = i \qquad \text{and} \qquad \langle p_i(x), p_j(x)\rangle_\triangle = \delta_{ij} p_i(\lambda_0).$$

It turns out that

$$p_0(x) + p_1(x) + \cdots + p_d(x) = n \prod_{i=1}^{d} \frac{x-\lambda_i}{\lambda_0 - \lambda_i},$$

which is called the Hoffman polynomial of G. In particular if G is regular,

$$p_0(A) + p_1(A) + \cdots + p_d(A) = J,$$

the all ones matrix.

Lemma

The predistance polynomials satisfy a three-term recurrence:

$$xp_i(x) = c'_{i+1}p_{i+1}(x) + a'_i p_i(x) + b'_{i-1}p_{i-1}(x) \qquad 0 \le i \le d,$$

where c'_{i+1}, a'_i, $b'_{i-1} \in \mathbb{R}$ *with* $b'_{-1} = c'_{d+1} := 0$. □

Note that $p_0(x) = 1$, and if G is regular then $p_1(x) = x$.

Lemma (Spectrum Lemma 1)

Let G, G' be two distance-regular graphs. Then G and G' have the same intersection numbers if and only if G and G' have the same spectrum. Indeed $c_{i+1} = c'_{i+1}$, $a_i = a'_i$, $b_i = b'_{i-1}$, $f_i(x) = p_i(x)$.

Theorem

Suppose G is a regular graph with diameter D and spectral diameter d. Then the following (i)-(iii) are equivalent.

(i) *G is distance-regular;* *(Equivalently $A_i = p_i(A)$ for $0 \le i \le D$.)*

(ii) $A_d = p_d(A)$. *(This implies $d = D$.)*

Proof.

This follows by backward induction and using $p_{d+1}(A) = 0$,

$$A_0 + A_1 + \cdots + A_D = J = p_0(A) + p_1(A) + \cdots + p_d(A),$$

$$Ap_i(A) = c'_{i+1}p_{i+1}(A) + a'_i p_i(A) + b'_{i-1}p_{i-1}(A).$$

□

1. M.A. Fiol, E. Garriga and J.L.A. Yebra, Locally pseudo-distance-regular graphs, *J. Combin. Theory Ser. B* 68 (1996), 179–205.
2. E.R. van Dam, The spectral excess theorem for distance-regular graphs: a global (over)view, Electron. J. Combin. 15(1) (2008), R129.

A graph G is 2-partially distance-regular if and only if G is k-regular and

$$A_2 = p_2(A) = c_2'^{-1}(A(A - a_1' I) - kI);$$

which is equivalent to

$$|\Gamma_1(x) \cap \Gamma_1(y)| = \begin{cases} a_1', & \text{if } \partial(x,y) = 1; \\ c_2', & \text{if } \partial(x,y) = 2. \end{cases}$$

If G is bipartite 2-partially distance-regular then

$$A_2 = p_2(A) = c_2'^{-1}(A^2 - kI).$$

A detour to freshmen linear linear algebra

Lemma

Let N be an $n \times m$ matrix. Then there exists a one-one correspondence between the nonzero eigenvalues of NN^T and N^TN.

Proof.

Suppose μ is a nonzero eigenvalue of NN^T with corresponding eigenvector u. Then $NN^Tu = \mu u \neq 0$. In particular $N^Tu \neq 0$. Since $N^TNN^Tu = \mu N^Tu$, N^Tu is an eigenvector of N^TN corresponding to the eigenvalue μ. Suppose μ has multiplicity m as an eigenvalue of NN^T. Let $u_1, u_2, \ldots, u_m$ be the corresponding orthogonal eigenvectors. If $c_1N^Tu_1 + \cdots + c_mN^Tu_m = 0$ then

$$0 = N(c_1N^Tu_1 + \cdots + c_mN^Tu_m) = \mu(c_1u_1 + \cdots + c_mu_m),$$

and hence $c_1 = c_2 = \cdots = c_m = 0$. This proves that the multiplicity of μ in NN^T is no larger than that in N^TN. Similarly for the other side, so the two multiplicities are the same. □

Lemma (Spectrum Lemma 2)

If $G = (X, Y)$ is a connected regular bipartite graph with $A_2 = p_2(A)$, then the halved graphs G^X and G^Y have the same spectrum.

Proof.

Let X_1 and Y_1 be adjacency matrices of G^X and G^Y respectively. First note that

$$A = \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix}$$

for some square matrix B. Then

$$\begin{pmatrix} X_1 & 0 \\ 0 & Y_1 \end{pmatrix} = A_2 = p_2(A) = aA^2 + bI = \begin{pmatrix} aBB^T + bI & 0 \\ 0 & aB^TB + bI \end{pmatrix}$$

for some real numbers a, b with $a \neq 0$. Since (from linear algebra) BB^T and B^TB have the same characteristic polynomial, G^X and G^Y have the same spectrum. □

The spectrum of a bipartite graph is symmetric to the 0.

Lemma

Let M be bipartite. Then λ is an eigenvalue of M iff $-\lambda$ is an eigenvalue of M. Moreover λ and $-\lambda$ has the same geometry multiplicity.

Proof.

Observe in block form product

$$\begin{pmatrix} \mathbf{0} & M_{21} \\ M_{12} & \mathbf{0} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$$

iff

$$\begin{pmatrix} \mathbf{0} & M_{21} \\ M_{12} & \mathbf{0} \end{pmatrix} \begin{pmatrix} -x \\ y \end{pmatrix} = -\lambda \begin{pmatrix} -x \\ y \end{pmatrix}.$$

□

Lemma (Diameter Lemma)

Let $G = (X, Y)$ be a connected regular bipartite graph with diameter D, even spectral diameter d and $A_2 = f(A)$ for some polynomial $f(x) \in \mathbb{R}[x]$ of degree 2. Suppose one of G^X and G^Y has spectral diameter d' equal to its diameter. Then $D = d = 2d'$.

Proof.

Let $f(x) = ax^2 + bx + c$ for some real numbers a, b, c with $a \neq 0$. Since G is bipartite, by comparing the uv-entry with $\partial(u, v) = 1$ of both sides of $A_2 = f(A) = aA^2 + bA + cI$, we have $b = 0$, and thus $A_2 = aA^2 + cI$. If λ is an eigenvalue of A with eigenvector u then $a\lambda^2 + c$ is an eigenvalue of A_2 with the same eigenvector u. Since G is bipartite and d is even, A has $d + 1$ distinct eigenvalues including one 0 eigenvalue by the symmetric spectrum property. Then A_2 has $d/2 + 1$ distinct eigenvalues, which implies that both G^X and G^Y have $d/2 + 1$ distinct eigenvalues, and hence have diameter at most $d/2$. Thus $d \geq D \geq 2(d/2) = d$, and hence $D = d$. □

Theorem

Let G be a connected regular bipartite graph with bipartition $X \cup Y$ and even spectral diameter d. Assume G is 2-partially distance-regular and both of the halved graphs G^X and G^Y are distance-regular. Then G is distance-regular.

Proof.

Since G is 2-partially distance-regular, $A_2 = c_2'^{-1}(A^2 - \lambda_0 I) = f(A)$, where $f(x) = c_2'^{-1}(x^2 - \lambda_0)$ is a polynomial of degree 2. By Spectrum Lemma 2, G^X and G^Y have the same spectrum, and by Spectrum Lemma 1 both G^X and G^Y have the same intersection numbers and the same diameter d'; indeed we have $D = d = 2d'$ by Diameter Lemma. Thus G^X and G^Y have the same (pre)distance-polynomials f_i, $0 \leq i \leq d/2$. Note that

$$A_{2i} = \begin{pmatrix} X_i & 0 \\ 0 & Y_i \end{pmatrix} = \begin{pmatrix} f_i(X_1) & 0 \\ 0 & f_i(Y_1) \end{pmatrix} = f_i(A_2) = g_{2i}(A), \quad 0 \leq i \leq d/2,$$

and X_i and Y_i are i-th distance matrices of G^X and G^Y respectively, and g_{2i} is even of degree $2i$. In particular $A_d = g_d(A)$ is a polynomial of A with degree d. It remains to show that $g_d = p_d$. Since G is regular,

$$A_d J = g_d(A)J = g_d(\lambda_0)J.$$

Then each row of A_d has exactly $g_d(\lambda_0)$ ones. Note that $||g_d||^2 = \langle g_d(A), g_d(A)\rangle = \langle A_d, A_d\rangle = g_d(\lambda_0)$. For every polynomial $h \in \mathbb{R}_{d-1}[x]$,

$$\langle g_d, h\rangle = \langle A_d, h(A)\rangle = 0.$$

By the uniqueness of the predistance polynomials, it follows that $g_d = p_d$.

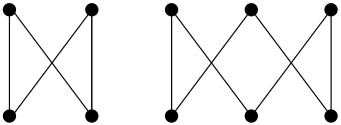
□

The assumption 2-partially distance-regular is necessary

The following example gives a regular bipartite graph G with $G^X = G^Y$ being a clique and even spectral diameter, but G is not 2-partially distance-regular.

Example

Let $G = K_{5,5} - C_4 - C_6$ be a regular graph obtained by deleting a C_4 and a C_6 from $K_{5,5}$. We have sp $G = \{3^1, 2^1, 1^2, 0^2, (-1)^2, (-2)^1, (-3)^1\}$, $D = 3 < 6 = d$ and $G^2 = 2K_5$.



$C_4 + C_6$

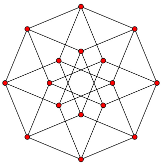
The assumption 2-partially distance-regular is necessary

Example

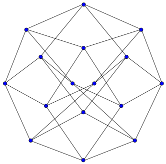
Let G be the Hoffman graph, which is a cospectral graph of 4-cube obtained from 4-cune by applying GM-swithching of edges. Then sp $G = \{4^1, 2^4, 0^6, (-2)^4, (-4)^1\}$, $D = d = 4$, and

$$A_i = p_i(A) \qquad \text{iff} \qquad i \in \{0, 1, 3\}.$$

Note that G^2 is the disjoint union of K_8 and $K_{2,2,2,2} (= K_8 - 4K_2)$, which are both distance-regular (sp $K_{2,2,2,2} = \{6^1, 0^4, (-2)^3\}$).



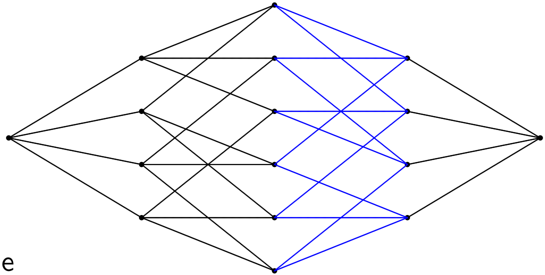
The 4-cube.



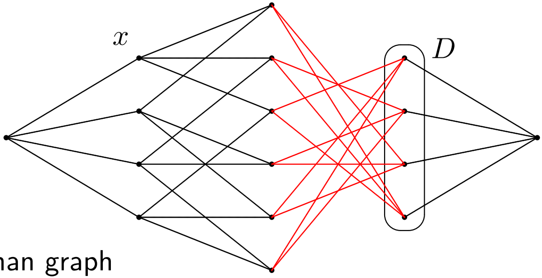
The Hoffman graph.

Copy from http://en.wikipedia.org/wiki/Hoffman_graph

Another drawing of 4-cube and Hoffman graph



The 4-cube



The Hoffman graph

The assumption even spectral diameter is necessary

The following example gives a bipartite 2-partially distance-regular graph G with $D = d = 5$ such that G^X, G^Y are distance-regular graphs with spectrum $\{6^1, 1^4, (-2)^5\}$ (the complement of petersen graph), but G is not distance-regular.

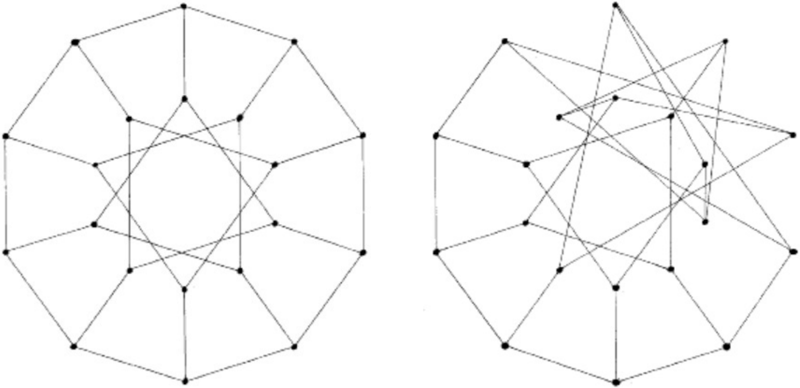
Example

Consider the regular bipartite graphs G on 20 vertices obtained from the Desargues graph (the bipartite double of the Petersen graph) by the GM-switching. One can check (by Maple) that $D = d = 5$, sp $G = \{3^1, 2^4, 1^5, (-1)^5, (-2)^4, (-3)^1\}$, and

$$A_i = p_i(A) \qquad \text{iff} \qquad i \in \{0, 1, 2, 4\}.$$

Then G is not distance-regular.

Desargues graph and its cospectral mate



For those who think that a combinatorial theorem should be stated and proved in a combinatorial way might want to solve the following problem which replace the spectral diameter d by the diameter D in the of main theorem.

Problem

Let G be a connected regular bipartite graph with bipartition $X \cup Y$ and even diameter D. Assume G is 2-partially distance-regular and both of the halved graphs G^X and G^Y are distance-regular with the same set of intersection numbers. Then G is distance-regular.

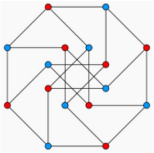
Unfortunately, we have an counterexample for the previous problem.

Example

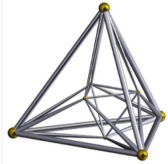
Consider the Möbius-Kantor graph G. One can check (by Maple) that $D = 4 < 5 = d$, and

$$A_i = p_i(A) \qquad \text{iff} \qquad i \in \{0, 1, 2, 4\}.$$

Note that $G^2 = 2X$, where X is the 16-cell graph, which is distance-regular with sp $X = \{6^1, 0^4, (-2)^3\}$.



Möbius-Kantor graph



16-cell graph

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Thanks for your attention.