Reeder's puzzle on a tree

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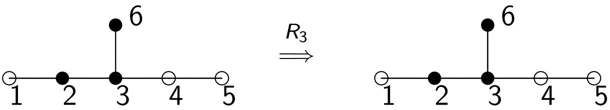
6
R_3
$\Longrightarrow$
6
1 2 3 4 5
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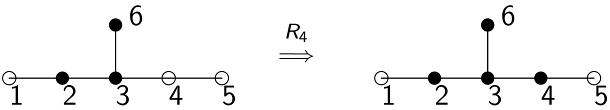
6
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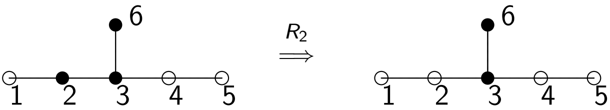
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6. Hence R_i corresponds to the move by selecting vertex i in the Reeder's puzzle.

A quadratic form

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1. Let $q : F_2^n \to F_2$ be the function defined by

$$q(u) := \sum_{1 \leq i \leq n} u_i^2 + \sum_{\substack{i<j \\ i \sim j}} u_i u_j$$

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4. If G is a tree then $q(u)$ is the parity of the number of connected components in the subgraphs induced on the black vertices of u.

The invariant $q(u)$

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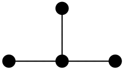
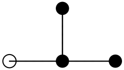
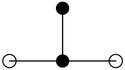
2. Then $q(R_i u) = q(u)$.
3. The invariant property of q has great importance in determining the orbits of the action of the group $< R_1, R_2, \ldots, R_n >$ on F_2^n.

$q(u) = 1$



$\Longrightarrow$





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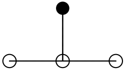

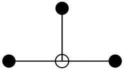

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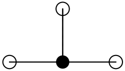

Non-degenerate quadratic form q

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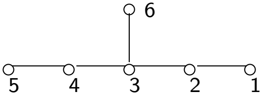
3. A special case of a tree G with non-degenerate q is when G has (unique) perfect matching.

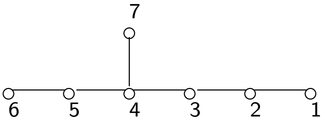
Trees with non-degenerate quadratic form q

A_n $(n \not\equiv 3 \pmod 4)$

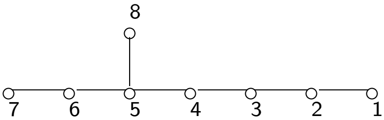
E_6



E_7



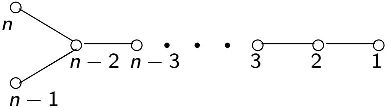
E_8



Trees with degenerate quadratic form q

$A_n (n \equiv 3 \pmod 4)$

$D_n (n \geq 4)$



By using the non-degenerate property of q, Reeder proved:

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Theorem

(Mark Reeder, 2005) Suppose G is a tree with odd number of matchings of size $\lfloor n/2 \rfloor$, but not a path. If O is a Reeder's puzzle orbit, then exactly one of the the following (i)-(iii) holds.

(i) $O = \{u\}$ *for some* $u \in F_2^n$ *with* $Au = 0$; *(O contains a single unmovable configuration)*

(ii) $O = \{u \in F_2^n \mid q(u) = 1\}$;

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Let G be a tree with a subgraph E_6. If O is a Reeder's puzzle orbit, then exactly one of the the following (i)-(iii) holds.

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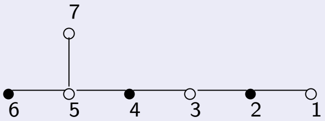
If G is a tree with a subgraph E_6 then number of black vertices in a movable configuration can be reduced to one or two by a sequence of moves; moreover, the one or two black vertices can be anywhere.

Exercise



$\{u \in F_2^7 \mid q(u) = 1\}$ is not an orbit.

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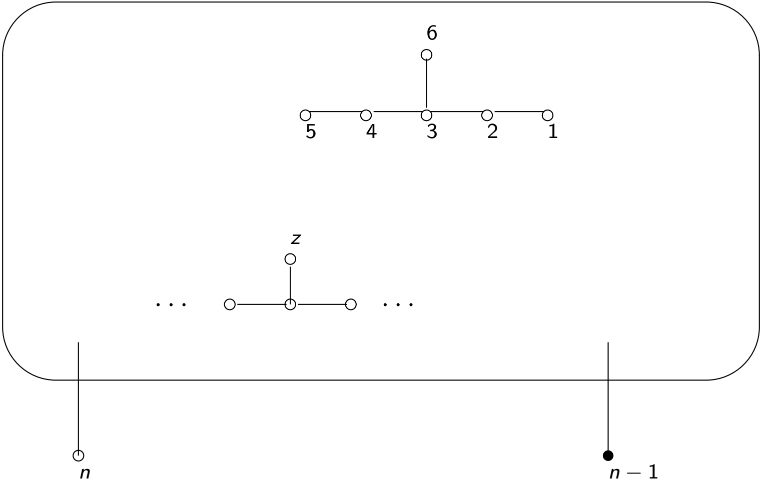
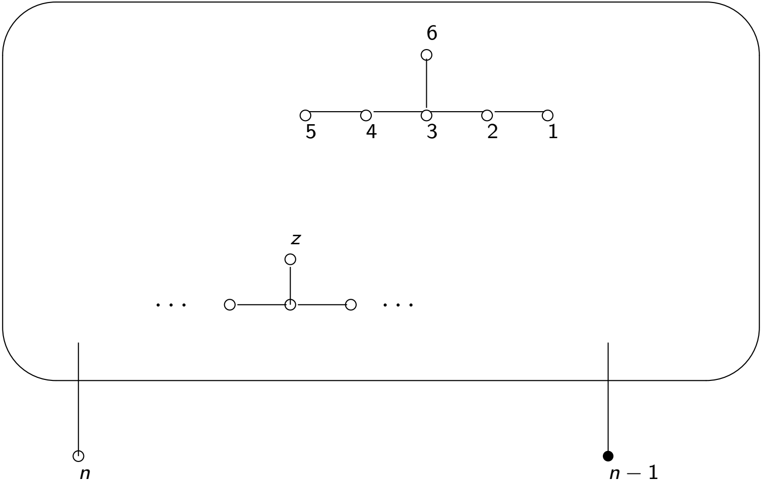
10. Can assume there is another leaf outside E_6, say $n - 1$.



n, $n-1$, are leafs not in E_6. n is white, and is the unique movable vertex.



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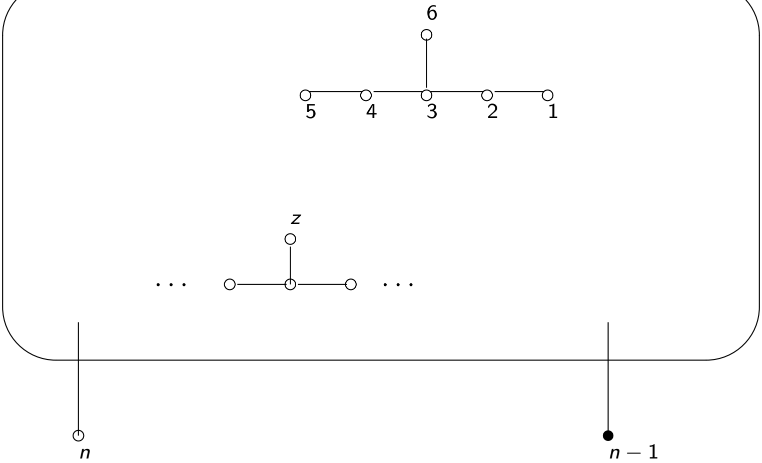

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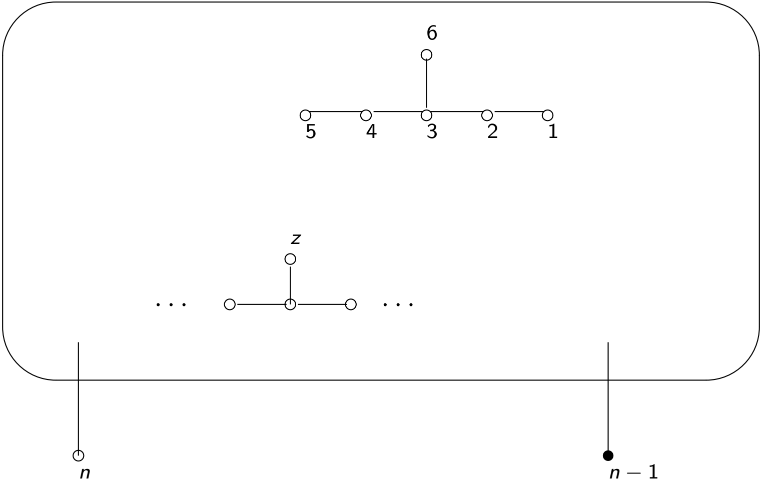
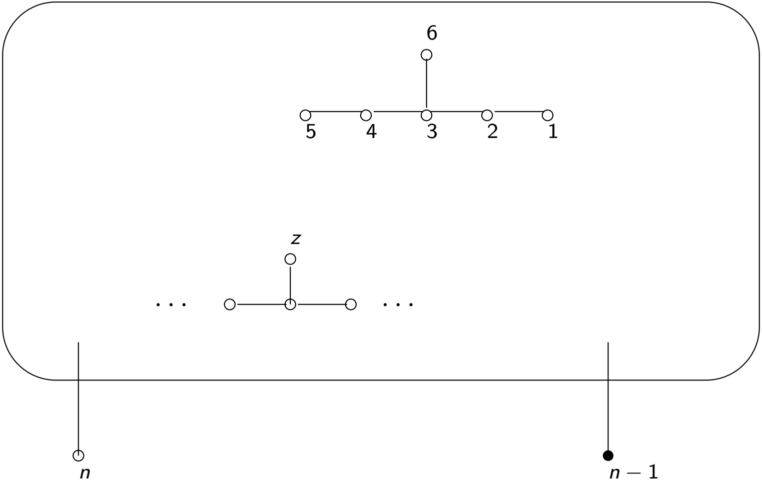


n, $n-1$, are leafs not in E_6. n is white, and is the unique movable vertex. $n-1$ must be black. Applying the moves by selecting the vertices consecutively along the path from n to $n-1$. The colors of n and $n-1$ are switched,



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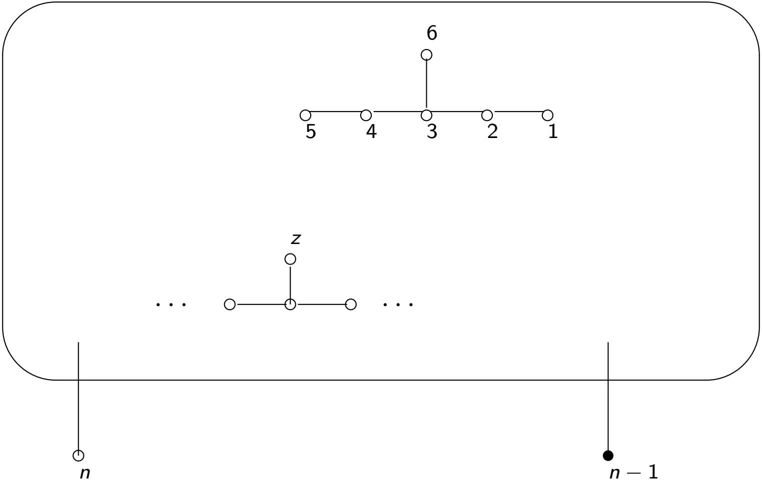

n, $n-1$, are leafs not in E_6. n is white, and is the unique movable vertex. $n-1$ must be black. Applying the moves by selecting the vertices consecutively along the path from n to $n-1$. The colors of n and $n-1$ are switched, and a branch vertex z becomes movable, a contradiction.

Binary star $P_{t,a,b}$, $t \geq 3$

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A tree $T = P_{t,a,b}$ is a *binary star* if T has a path $1, 2, \ldots, t$ from 1 to t with a more vertices $t+1, t+2, \ldots, t+a$ adjacent to 2, and b remaining vertices $t+a+1, t+a+2, \ldots, t+a+b$ adjacent to $t-1$. Hence $K_{1,a+b+2} := P_{3,a,b}$ is a *star* of $3+a+b$ vertices and $P_t := P_{t,0,0}$ is a path of t vertices.

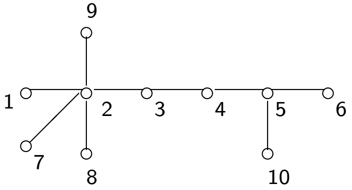

Figure. The binary star $P_{6,3,1}$.

Theorem

(Huang, Lin, W-) Let G be a tree with at least three vertices, but not a binary star. If O is a Reeder's puzzle orbit, then exactly one of the the following (i)-(iii) holds.

(i) $O = \{u\}$ *for some* $u \in F_2^n$ *with* $Au = 0$;

(ii) $O = \{u \in F_2^n \mid q(u) = 1\}$;

(iii) $O = \{u \in F_2^n \mid Au \neq 0, q(u) = 0\}$.

In particular there are $2^{\mathrm{null}\ A} + 2$ *orbits, where* $\mathrm{null}\ A$ *is the nullity of* A *over* F_2.

The standard projection in a binary star $P_{t,a,b}$.

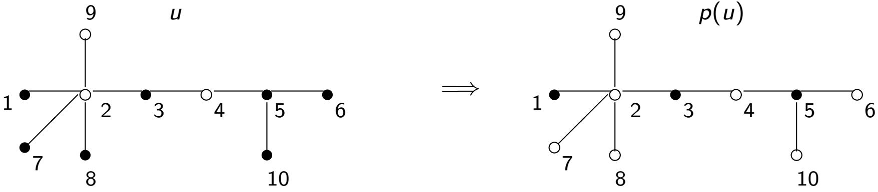

Figure. The standard projection $p(u)$.

For a configuration $u \in F_2^n$ of a binary star $P_{t,a,b}$, let $c(u)$ denote the number of connected components in the subgraph induced on the black vertices of u.

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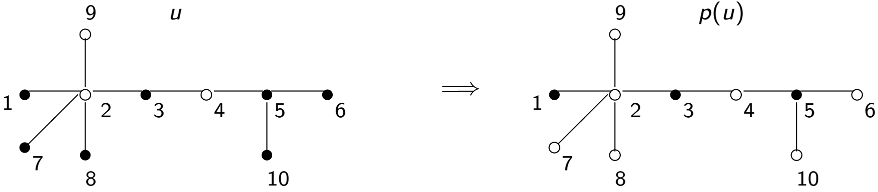

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For a configuration $u \in F_2^n$ of a binary star $P_{t,a,b}$, let $c(u)$ denote the number of connected components in the subgraph induced on the black vertices of u.

Note that $c(p(u)) \leq \lfloor t/2 \rfloor$.

Theorem

(Huang, Lin, W-) Let G be a binary star $P_{t,a,b}$. If O is a Reeder's puzzle orbit, then exactly one of the following holds.

1. $|O| = 1$, *i.e.* O *contains a unique unmovable configuration.*
2. $O = \{u \in F_2^n \text{ is movable} \mid c(p(u)) = i\}$ *for some integer* i *with* $1 \leq i \leq \lfloor t/2 \rfloor$,

where p is the standard projection of configurations in $P_{t,a,b}$. In particular there are $2^{a+b} + t/2$ orbits if t is even, and $2^{a+b+1} + (t-1)/2$ orbits if t is odd.

Thank you for your attention.