A conjecture on the spectral radius of a bipartite graph

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Spectral radius

When C is a real square matrix, the spectral radius $\rho(C)$ is defined as

$$\rho(C) := \max\{ \, |\lambda| \; \mid \lambda \text{ is an eigenvalue of } C\},$$

where $|\lambda|$ is the magnitude of complex number λ.

When C is nonnegative, $\rho(C)$ is known to be an eigenvalue of C.

A snapshot of the method that we develop

The following is well known from the majorization-monotone property of spectral radii of nonnegative matrices :

$$\rho\left(\begin{array}{cc|c} 2 & 2 & 1 \\ 0 & 3 & 2 \\ \hline 1 & 2 & 1 \end{array}\right) \geq \rho\left(\begin{array}{cc|c} 2 & 1 & 1 \\ 0 & 3 & 1 \\ \hline 1 & 2 & 1 \end{array}\right) = \rho\begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} = 4.$$

Our main result implies

$$\rho\left(\begin{array}{cc|c} 2 & 2 & {\color{red}1} \\ 0 & 3 & {\color{red}2} \\ \hline 1 & 2 & {\color{red}1} \end{array}\right) \geq \rho\left(\begin{array}{cc|c} 2 & 1 & {\color{red}2} \\ 0 & 3 & {\color{red}2} \\ \hline 1 & 2 & {\color{red}1} \end{array}\right) = \rho\begin{pmatrix} 3 & 2 \\ 3 & 1 \end{pmatrix} = 2+\sqrt{7}.$$

(One column exception is allowed in majorization-monotone property if the row-sums of two matrices are the same.)

Dual result

$$\rho\left(\begin{array}{ccc|cc|cc} 2 & 1 & 3 & 3 & 3 & 12 & 0 \\ 4 & 2 & 1 & 4 & 2 & 6 & 4 \\ 2 & 3 & 1 & 4 & 1 & 8 & 3 \\ \hline 3 & 5 & 3 & 1 & 1 & 3 & 4 \\ 5 & 6 & 1 & 1 & 0 & 3 & 3 \\ \hline 0 & 2 & 1 & 2 & 2 & 6 & 0 \\ 2 & 2 & 0 & 2 & 1 & 1 & 4 \end{array}\right) \qquad \text{(same row-sums sequence)}$$

$$\leq \rho\left(\begin{array}{ccc|cc|cc} 2 & 2 & 3 & 3 & 3 & 12 & -1 \\ 4 & 2 & 1 & 4 & 2 & 6 & 5 \\ 2 & 3 & 2 & 4 & 2 & 8 & 3 \\ \hline 4 & 5 & 3 & 1 & 1 & 3 & 3 \\ 5 & 6 & 1 & 1 & 1 & 3 & 3 \\ \hline 1 & 2 & 1 & 2 & 2 & 6 & -1 \\ 2 & 2 & 0 & 2 & 2 & 1 & 4 \end{array}\right) = \rho\begin{pmatrix} 7 & 6 & 11 \\ 12 & 2 & 6 \\ 4 & 4 & 5 \end{pmatrix} \approx 18.69$$

Outline

0. Introduction

1. The non-complete bipartite graph with e edges which has the maximum spectral radius

2. The (non-complete) bipartite graph with e edges and bi-order p, q which has the maximum spectral radius

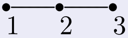
Appendix. Spectral bounds of a nonnegative matrix

Notations

Let G denote a graph with $e = e(G)$ edges without isolated vertices. Let $A = A(G)$ be the adjacency matrix of G. The spectral radius $\rho(G)$ of G be the spectral radius of A.

Example

G



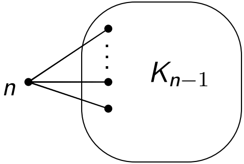
$$A = A(G) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$e = e(G) = 2, \ \rho(G) = \rho(A) = \sqrt{2}.$$

Brualdi-Hoffman Conjecture (1976)

Conjecture

If $\binom{d}{2} < e \leq \binom{d+1}{2}$, the graph with the maximum spectral radius consists of the complete graph K_d to which a new vertex of degree $e - \binom{d}{2}$ is added, together with probably some isolated vertices.



Rowlinson proved this conjecture in 1988.

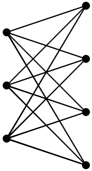
From now on, we assume G is bipartite with e edges.

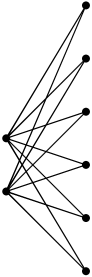
A. Bhattacharya, S. Friedland, and U.N. Peled show the following.

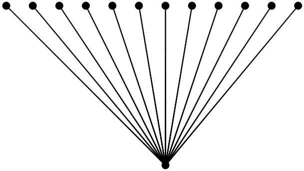
Theorem (BFP 2008)

$$\rho(G) \leq \sqrt{e(G)}$$

with equality iff G is a complete bipartite graph with possible some isolated vertices. □







$e(G) = 12$

$\rho(G) = \sqrt{12}$

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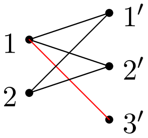
The graphs $K^{\pm}_{s,t}$

Define

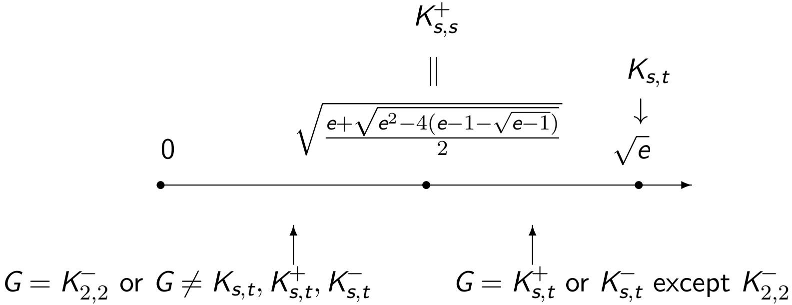
$$K^{-}_{s,t} := K_{s,t} - \{\overline{st}\},$$
$$K^{+}_{s,t} := K_{s,t} + \{\overline{s(t+1)'}\} \qquad (2 \le s \le t).$$

Example:



$$K^{-}_{2,3} = K^{+}_{2,2}$$

Value of $\rho(G)$ when e is fixed



Moreover we find

$$e = st - 1,\ s \searrow,\ t \nearrow \quad \Rightarrow \quad \rho(K^-_{s,t}) \nearrow,$$
$$e = st + 1,\ s \searrow,\ t \nearrow \quad \Rightarrow \quad \rho(K^+_{s,t}) \nearrow,$$

Extremal graphs

Theorem

If G has maximum spectral radius among bipartite non-complete graphs with e edges then

e	$(e-1, e+1)$	G
odd		$K^{-}_{2,t}$
even	*(prime,not prime)*	$K^{-}_{s,t}$ *with* $s \geq 2$ *the least*
even	*(not prime,prime)*	$K^{+}_{s,t}$ *with* $s \geq 2$ *the least*
even	*(not prime,not prime)* *neither primes case*	$K^{-}_{s,t}$ *with* $s \geq 2$ *the least or* $K^{+}_{s,t}$ *with* $s \geq 2$ *the least*
even	*(prime,prime)* *twin primes case*	*unknown (no* $K^{\pm}_{st}$ *with* $s \geq 2$*)*

□

Numerical comparisons of the neither primes case

In the case that $e \leq 100$ is even and neither $e-1$ nor $e+1$ is a prime, we determine which G of $K^{-}_{s,t}$ with $s \geq 2$ the least and $K^{+}_{s',t'}$ with $s' \geq 2$ the least has larger eigenvalue, where $e = st - 1 = s't' + 1$.

e	$\rho(K^{-}_{s,t})$	$\rho(K^{+}_{s',t'})$	winner
26	$\sqrt{13+3\sqrt{17}}$	$\sqrt{13+\sqrt{149}}$	$-$
34	$\sqrt{17+\sqrt{265}}$	$\sqrt{17+\sqrt{267}}$	$+$
50	$\sqrt{25+\sqrt{593}}$	$\sqrt{25+\sqrt{583}}$	$-$
56	$\sqrt{28+\sqrt{748}}$	$\sqrt{28+\sqrt{740}}$	$-$
64	$\sqrt{32+\sqrt{976}}$	$\sqrt{32+\sqrt{982}}$	$+$
76	$\sqrt{38+\sqrt{1384}}$	$\sqrt{38+\sqrt{1394}}$	$+$
86	$\sqrt{43+\sqrt{1813}}$	$\sqrt{43+\sqrt{1781}}$	$-$
92	$\sqrt{46+\sqrt{2096}}$	$\sqrt{46+\sqrt{2078}}$	$-$
94	$\sqrt{47+\sqrt{2137}}$	$\sqrt{47+\sqrt{2147}}$	$+$

A theorem for twin primes case

Let $\rho(e)$ denote the maximum $\rho(G)$ of a bipartite non-complete graph G with e edges.

Theorem

If $e \geq 4$ then $(e-1, e+1)$ is a pair of twin primes if and only if

$$\rho(e) < \sqrt{\frac{e + \sqrt{e^2 - 4(e - 1 - \sqrt{e-1})}}{2}}.$$

□

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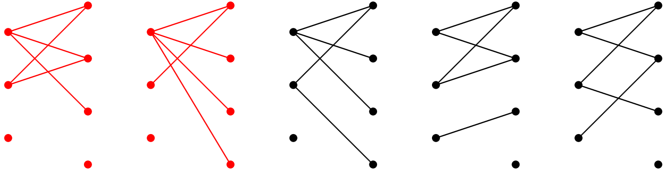
$\mathcal{K}(p, q, e)$ and $\mathcal{K}_0(p, q, e)$

Definition

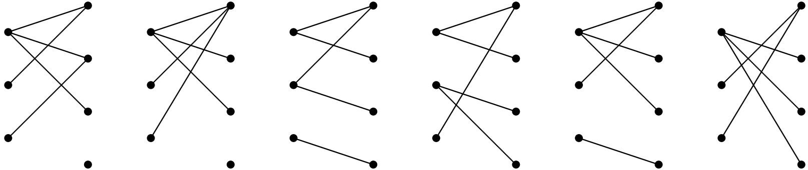
(i) $\mathcal{K}(p, q, e)$ is the family of subgraphs of $K_{p,q}$ with e edges without isolated vertices which are not complete bipartite graphs

(ii) $\mathcal{K}_0(p, q, e)$ is the subset of $\mathcal{K}(p, q, e)$ such that each graph in the subset is obtained from a complete bipartite graph by adding one vertex and a corresponding number of edges.

$\mathcal{K}(3,4,5)$, $\mathcal{K}_0(3,4,5)$ and $\rho(G)$



$\frac{\sqrt{10+2\sqrt{17}}}{2}$ $\frac{\sqrt{10+2\sqrt{13}}}{2}$ $\frac{\sqrt{10+2\sqrt{5}}}{2}$ 2 ≈ 1.8019



$\frac{\sqrt{6}+\sqrt{2}}{2}$ 2 $\sqrt{3}$ $\frac{1+\sqrt{5}}{2}$ $\sqrt{2+\sqrt{2}}$ $\sqrt{3}$

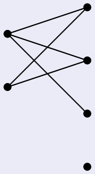
BFP Conjecture for $\mathcal{K}(p, q, e)$

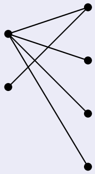
The following is a bipartite graphs analogue of Brualdi-Hoffman conjecture proposed by Bhattacharya, Friedland and Peled.

BFP Conjecture for $\mathcal{K}(p, q, e)$

If $G \in \mathcal{K}(p, q, e)$ such that $\rho(G) = \max_{H \in \mathcal{K}(p,q,e)} \rho(H)$ and $\mathcal{K}_0(p, q, e) \neq \emptyset$, then $G \in \mathcal{K}_0(p, q, e)$.

Example





$p = 2,\ q = 4,\ e = 5$

Some previous results

Theorem (Bhattacharya, Friedland and Peled 2008)

BFP Conjecture for $\mathcal{K}(p, q, e)$ *holds for* $e = st - 1$ *for* s, t *satisfying* $2 \leq s \leq p \leq t \leq t + (t-1)/(s-1)$. □

Theorem (Liu and Weng, 2015)

BFP Conjecture for $\mathcal{K}(p, q, e)$ *holds for* $e > pq - \min(p, q)$. □

Remark

The is no proper complete bipartite subgraph of $K_{p,q}$ with $e > pq - \min(p, q)$ edges.

A slight improvement

If $e \in \{st-1, st+1 \mid s \leq p, t \leq q\}$, then $K^{-}_{s,t} \in \mathcal{K}_0(p,q,e)$ or $K^{+}_{s,t} \in \mathcal{K}_0(p,q,e)$. The following theorem is an immediate consequence.

Theorem

BFP Conjecture for $\mathcal{K}(p,q,e)$ holds with

$$e \in \{st-1, st+1 \mid s \leq p, t \leq q\}.$$

□

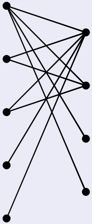
The graph G_D

For a sequence D of positive integers in nonincreasing order, one can define the bipartite graph G_D with bipartition $X = \{x_1, x_2, \ldots, x_p\}$, $Y = \{y_1, y_2, \ldots, y_{d_1}\}$ such that

$$E(G_D) = \{x_i y_j | 1 \le i \le p, 1 \le j \le d_i\}.$$

Example

For $D = (4, 2, 2, 1, 1)$ or $D = (5, 3, 1, 1)$, we have the isomorphic graph G_D.



$G_{(4,2,2,1,1)} = G_{(5,3,1,1)}$

Disproof of the BFP conjecture

Proposition

If $q > p+2 \geq 5$ then BFP Conjecture for $\mathcal{K}(p,q,p(q-1))$ is false.

Proof.

With sequences

$$D_1 =(q, q-1, \ldots, q-1, q-2),$$
$$D_2 =(q, q, \ldots, q, q-p),$$

$G_{D_1}, G_{D_2} \in \mathcal{K}(p,q,p(q-1))$ and $\mathcal{K}_0(p,q,p(q-1) = \{G_{D_2}\}$. By direct computation, $\rho(G_{D_2}) < \rho(G_{D_1})$. □

$\mathcal{C}(p, q, e)$

From now on the complete bipartite graphs will be included in our consideration.

Definition

(i) $\mathcal{C}(p, q, e)$ is the family of subgraphs of $K_{p,q}$ with e edges without isolated vertices.

(ii) $\mathcal{C}_0(p, q, e)$ is the subset of $\mathcal{C}(p, q, e)$ such that each graph in the subset is a complete bipartite graph or a graph obtained from a complete bipartite graph by adding one vertex and a corresponding number of edges.

Weak BFP Conjecture

We propose the following conjecture.

Weak BFP Conjecture for $\mathcal{C}(p, q, e)$

If $G \in \mathcal{C}(p, q, e)$ such that $\rho(G) = \max_{H \in \mathcal{C}(p,q,e)} \rho(H)$, then $G \in \mathcal{C}_0(p, q, e)$.

$e \geq pq - \max(p, q)$ or $p \leq 5$

We have the following two theorems.

Theorem

If $e \geq pq - \max(p, q)$ then the weak BFP Conjecture for $\mathcal{C}(p, q, e)$ is true. □

Theorem

If $\min(p, q) \leq 5$ then the weak BFP conjecture for $\mathcal{C}(p, q, e)$ is true. □

The proofs of the above two Theorems employ some new sharp upper bounds of the spectral radii of nonnegative matrices which will be included in the end of this file. Anyone is free to download.

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Thank you for your attention.

Appendix

Spectral bounds of a nonnegative matrix

Motivation

A bipartite graph G has adjacency matrix of the block form

$$A(G) = \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix}.$$

Then

$$A^2 = \begin{pmatrix} BB^T & O \\ O & B^TB \end{pmatrix}.$$

Since BB^T and B^TB have the same spectral radius,

$$\rho^2(G) = \rho(BB^T) = \rho(B^TB).$$

Because BB^T is no longer a binary matrix, we need spectral theory for general nonnegative matrices C.

Motivation

Let C and C' be two $n \times n$ nonnegative matrix. It is well-known as a consequence of Perron-Frobenius Theorem that

$$C \leq C' \quad \Rightarrow \quad \rho(C) \leq \rho(C').$$

Moreover if C is irreducible then $\rho(C) = \rho(C')$ if and only if $C = C'$.

We might expect to find another matrix C' such there are many C related to C' in some way and $\rho(C) \leq \rho(C')$. Moreover we expect the matrix C with $\rho(C) = \rho(C')$ is not unique.

Matrix notation

For a matrix M and a subset α of the set of row indices and a subset β of the set of column indices, we use $M[\alpha|\beta]$ to denote the submatrix of M which restricts the positions in $\alpha \times \beta$.

Example

$$M = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \quad M[[4]|[3]] = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \; M[\{4\}|[4]] = (1, 1, 0, 0),$$

where $[n] := \{1, 2, \ldots, n\}$.

Rooted matrix

An $m \times n$ matrix $C' = (c'_{ij})$ is rooted if

$$c'_{ij} \geq c'_{nj} \qquad \text{for } 1 \leq i \leq m \text{ and } 1 \leq j \leq n-1, \text{ and}$$

$$r'_i := \sum_{j=1}^{n} c'_{ij} \geq r'_n := \sum_{j=1}^{n} c'_{nj} \qquad \text{for } 1 \leq i \leq m-1.$$

Example

The matrix

$$C' = \begin{pmatrix} 6 & 6 & -1 \\ 8 & 2 & 0 \\ 5 & 2 & 0 \end{pmatrix}$$

with row-sum vector $(r'_1, r'_2, r'_3)^T = (11, 10, 7)^T$, which is a rooted vector.

$\rho_r(C')$

Remark

As a rooted matrix C' is not always nonnegative, $\rho(C')$ is not necessary to be the largest real eigenvalue of C'. Let $\rho_r(C')$ denote the largest real eigenvalue of C' (Its existence is proved).

A comment on rooted matrix

Remark

$$C' \text{ is nonnegative} \quad \Rightarrow \quad \begin{pmatrix} C' & 0 \\ u & a \end{pmatrix} \text{ is rooted and } \rho(C') = \rho \begin{pmatrix} C' & 0 \\ u & a \end{pmatrix}$$

for suitable chosen of row vector $u \geq 0$ and scalar $a \leq 0$ to have 0 row-sum in the last row.

Construct C' from C

From an $n \times n$ matrix $C = (c_{ij})$, we construct another $n \times n$ matrix $C' = (c'_{ij})$ that satisfies

(i) $C[[n]|[n-1]] \leq C'[[n]|[n-1]]$;

(ii) $r_i := \sum_{j=1}^{n} c_{ij} \leq r'_i := \sum_{j=1}^{n} c'_{ij}$ for $1 \leq i \leq n$;

(iii) $C' + kI$ is rooted for some k;

(iv) $C'[\{n\}|[n-1]] > 0$.

Example

$$C = \begin{pmatrix} 3 & 6 & 2 \\ 8 & 1 & 1 \\ 5 & 5 & 0 \end{pmatrix}, \qquad C' = \begin{pmatrix} 6 & 6 & -1 \\ 8 & 2 & 0 \\ 5 & 5 & 0 \end{pmatrix}$$

A restricted form of our main method

Theorem

With the notation from the last page, we have

$$\rho(C) \leq \rho_r(C').$$

□

The set K

To study the case of equality $\rho(C) = \rho(C')$ of the theorem in the last page, we need information of the eigenvector $v' = (v'_1, v'_2, \ldots, v'_n)^T$ (known to be positive and rooted) of C' for $\rho_r(C')$ and the set

$$K = \{j \mid v'_j > v'_n\}.$$

An easier way to find K

Let $K_1 = \{i \mid r'_i > r'_n\}$, and when K_t is defined, let

$$K_{t+1} = \{i \notin \bigcup_{s \le t} K_s \mid c'_{ij} > c'_{nj} \text{ for some } j \in \textstyle\bigcup_{s \le t} K_s\}.$$

Lemma

If $r'_i > 0$ for $1 \le i \le n-1$ then

$$\begin{cases} K = \emptyset, & \text{if } K_1 = \emptyset; \\ K = \bigcup_{s=1}^{\infty} K_s & \text{otherwise.} \end{cases}$$

□

The equality part of the theorem

Theorem

With the notation in the last few pages, if C is irreducible and $r_i' > 0$ *for* $1 \leq i \leq n-1$*, then* $\rho(C) = \rho_r(C')$ *if and only if*

$$\begin{aligned} r_i &= r_i' \quad \text{for } 1 \leq i \leq n \\ c_{ij}' &= c_{ij} \quad \text{for } 1 \leq i \leq n \text{ and } j \in K. \end{aligned}$$

□

(c_{ij} is free if $j \notin K$.)

A non-example holds

$$\rho\left(\begin{array}{ccc|cc|cc} 2 & 1 & 3 & 3 & 3 & 12 & 0 \\ 4 & 2 & 1 & 4 & 2 & 6 & 4 \\ 2 & 3 & 1 & 4 & 1 & 8 & 3 \\ \hline 3 & 5 & 3 & 1 & 1 & 3 & 4 \\ 5 & 6 & 1 & 1 & 0 & 3 & 3 \\ \hline 0 & 2 & 1 & 2 & 2 & 6 & 0 \\ 2 & 2 & 0 & 2 & 1 & 1 & 4 \end{array}\right) \leq \rho\left(\begin{array}{ccc|cc|cc} 2 & 2 & 3 & 3 & 3 & 12 & -1 \\ 4 & 2 & 1 & 4 & 2 & 6 & 5 \\ 2 & 3 & 2 & 4 & 2 & 8 & 3 \\ \hline 4 & 5 & 3 & 1 & 1 & 3 & 3 \\ 5 & 6 & 1 & 1 & 1 & 3 & 3 \\ \hline 1 & 2 & 1 & 2 & 2 & 6 & -1 \\ 2 & 2 & 0 & 2 & 2 & 1 & 4 \end{array}\right)$$

Although the matrix on the right violates some pieces of the assumptions in C', the above inequality still holds.

The matrix C' after equitable quotient matters

$$\rho\left(\begin{array}{ccc|cc|cc} 2 & 1 & 3 & 3 & 3 & 12 & \color{red}{0} \\ 4 & 2 & 1 & 4 & 2 & 6 & \color{red}{4} \\ 2 & 3 & 1 & 4 & 1 & 8 & \color{red}{3} \\ \hline 3 & 5 & 3 & 1 & 1 & 3 & \color{red}{4} \\ 5 & 6 & 1 & 1 & 0 & 3 & \color{red}{3} \\ \hline 0 & 2 & 1 & 2 & 2 & 6 & \color{red}{0} \\ 2 & 2 & 0 & 2 & 1 & 1 & \color{red}{4} \end{array}\right) \qquad \text{(same row-sums sequence)}$$

$$\leq \quad \rho\left(\begin{array}{ccc|cc|cc} 2 & 2 & 3 & 3 & 3 & 12 & \color{red}{-1} \\ 4 & 2 & 1 & 4 & 2 & 6 & \color{red}{5} \\ 2 & 3 & 2 & 4 & 2 & 8 & \color{red}{3} \\ \hline 4 & 5 & 3 & 1 & 1 & 3 & \color{red}{3} \\ 5 & 6 & 1 & 1 & 1 & 3 & \color{red}{3} \\ \hline 1 & 2 & 1 & 2 & 2 & 6 & \color{red}{-1} \\ 2 & 2 & 0 & 2 & 2 & 1 & \color{red}{4} \end{array}\right) = \rho\begin{pmatrix} 7 & 6 & 11 \\ 12 & 2 & 6 \\ 4 & 4 & 5 \end{pmatrix} \approx 18.69$$

Applications

In the following few pages, we shall provide some applications of the inequality $\rho(C) \leq \rho_r(C')$.

The matrices C attaining the equality can be characterized, but for simplicity, we omit the discussion here.

Realization a result of Xing Duan and Bo Zhou

Theorem

Let $C = (c_{ij})$ be a nonnegative $n \times n$ matrix with row-sums $r_1 \geq r_2 \geq \cdots \geq r_n$ and $d := \max_i c_{ii}$, $f := \max_{i \neq j} c_{ij}$. Then for $1 \leq \ell \leq n$,

$$\rho(C) \leq \frac{r_\ell + d - f + \sqrt{(r_\ell - d + f)^2 + 4f\sum_{i=1}^{\ell-1}(r_i - r_\ell)}}{2}$$

Proof.

$$C' = \begin{pmatrix} d & f & \cdots & f & r_1 - (\ell-2)f - d \\ f & d & & f & r_2 - (\ell-2)f - d \\ \vdots & & \ddots & \vdots & \vdots \\ f & f & \cdots & d & r_{\ell-1} - (\ell-2)f - d \\ f & f & \cdots & f & r_\ell - (\ell-1)f \end{pmatrix}_{\ell \times \ell}.$$

□

A little generalization

Theorem

Let $C = (c_{ij})$ be a nonnegative $n \times n$ matrix with row-sums $r_1 \geq r_2 \geq \cdots \geq r_n$ and $d \geq \max_{1\leq i\leq \ell-1} c_{ii}$, $f \geq \max_{1\leq i\neq j\leq \ell-1} c_{ij}$. Then for $1 \leq \ell \leq n$,

$$\rho(C) \leq \frac{r_\ell + d - f + \sqrt{(r_\ell - d + f)^2 + 4f\sum_{i=1}^{\ell-1}(r_i - r_\ell)}}{2}$$

Proof.

$$C' = \begin{pmatrix} d & f & \cdots & f & r_1 - (\ell-2)f - d \\ f & d & & f & r_2 - (\ell-2)f - d \\ \vdots & & \ddots & \vdots & \vdots \\ f & f & \cdots & d & r_{\ell-1} - (\ell-2)f - d \\ f & f & \cdots & f & r_\ell - (\ell-1)f \end{pmatrix}_{\ell\times\ell}.$$

□

A theorem of Richard Stanley in 1987

Theorem

Let $C = (c_{ij})$ be an $n \times n$ symmetric (0,1) matrix with zero trace. Let the number of 1's of C be $2e$. Then

$$\rho(C) \leq \frac{-1 + \sqrt{1 + 8e}}{2}.$$

Proof.

Use $2e = \sum_{i=1}^{n} r_n$ and

$$C' = \begin{pmatrix} 0 & 1 & \cdots & 1 & r_1 - (n-1) \\ 1 & 0 & \ddots & 1 & r_2 - (n-1) \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & 0 & r_n - (n-1) \\ 1 & 1 & \cdots & 1 & -n \end{pmatrix}.$$

□

A generalization to nonnegative matrices

Theorem

Let $C = (c_{ij})$ be an $n \times n$ nonnegative matrix. Let m be the sum of entries, and $d \geq \max_i c_{ii}$, $f \geq \max_{i \neq j} c_{ij}$. Then

$$\rho(C) \leq \frac{d - f + \sqrt{(d-f)^2 + 4fm}}{2}.$$

Proof.

Use $m = \sum_{i=1}^{n} r_n = n(n-1)f + nd$ and

$$C' = \begin{pmatrix} d & f & \cdots & f & r_1 - d - (n-1)f \\ f & d & \ddots & f & r_2 - d - (n-1)f \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ f & f & \cdots & d & r_n - d - (n-1)f \\ f & f & \cdots & f & -nf \end{pmatrix}.$$

Realization a result of Csikvári in 2009

Theorem

Assume that the set $\{v_1, v_2, \ldots, v_k\}$ *forms a clique in the graph* G *and* $V(G) \setminus K = \{v_{k+1}, \ldots, v_n\}$ *forms an independent set. Let* e' *be the number of edges between* K *and* $V(G) \setminus K$. *Then*

$$\rho(G) \leq \frac{k - 1 + \sqrt{(k-1)^2 + 4e'}}{2}.$$

Proof.

Use $e' = \sum_{i=1}^{k} r_i$ and

$$C' = \begin{pmatrix} 0 & 1 & \cdots & 1 & r_1 - (k-1) \\ 1 & 0 & \ddots & 1 & r_2 - (k-1) \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & 0 & r_k - (k-1) \\ 1 & 1 & \cdots & 1 & 0 \end{pmatrix}.$$

$\rho(G) \leq \rho(G_{D(G)})$

To illustrate how our method is applied to bipartite graph, we need the following theorem of A. Bhattacharya, S. Friedland, and U.N. Peled in 2008.

Theorem

If a bipartite graph G has degree sequence $D = D(G)$ of one part then $\rho(G) \leq \rho(G_D)$ with equality if and only if $G = G_D$ (up to isomorphism). □

The spectral radius of G_D

The bipartite graph G_D has adjacency matrix of the block form

$$A(G) = \begin{pmatrix} 0 & B(D) \\ B(D)^T & 0 \end{pmatrix}. \text{ Then } A^2 = \begin{pmatrix} B(D)B(D)^T & O \\ O & B(D)^T B(D) \end{pmatrix},$$

and

$$C := B(D)B(D)^T = \begin{pmatrix} d_1 & d_2 & d_3 & & d_p \\ d_2 & d_2 & d_3 & & d_p \\ d_3 & d_3 & d_3 & & d_p \\ & & & \ddots & \\ d_p & d_p & d_p & & d_p \end{pmatrix}.$$

Since $B(D)B(D)^T$ and $B(D)^T B(D)$ have the same spectral radius,

$$\rho^2(G_D) = \rho(A^2) = \rho(C).$$

A proof of the next theorem

For $D = (d_1, d_2, \ldots, d_p)$ in nonincreasing order,

$$C = \begin{pmatrix} d_1 & d_2 & d_3 & & d_p \\ d_2 & d_2 & d_3 & & d_p \\ d_3 & d_3 & d_3 & & d_p \\ & & & \ddots & \\ d_p & d_p & d_p & & d_p \end{pmatrix}$$

and fix $1 \leq \ell \leq p$, we will choose

$$C' = \begin{pmatrix} d_1 & d_1 & \cdots & d_1 & r_1 - (\ell-1)d_1 \\ d_2 & d_2 & \cdots & d_2 & r_2 - (\ell-1)d_2 \\ d_3 & d_3 & \cdots & d_3 & r_3 - (\ell-1)d_3 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ d_\ell & d_\ell & \cdots & d_\ell & r_\ell - (\ell-1)d_\ell \end{pmatrix},$$

where $r_i = (i-1)d_i + \sum_{k=i}^{p} d_k$ is the i-th row-sum of C to obtain the theorem in the next page.

A theorem for bipartite graphs

Theorem

Let G be a bipartite graph and $D = (d_1, d_2, \ldots, d_p)$ be the degree sequence of one part of G in decreasing order. Then for $1 \leq \ell \leq p$,

$$\rho(G) \leq \sqrt{\frac{r_1 + \sqrt{(2r_\ell - r_1)^2 + 4d_\ell \sum_{i=1}^{\ell}(r_i - r_\ell)}}{2}}.$$

□

The above theorem is the main tool for our proof of the weak BFP Conjecture for $C(p, q, e)$ with $e \geq pq - \max(p, q)$ or $\min(p, q) \leq 5$.

A new lower bound

Our method also has dual version.

Theorem

Let $C = (c_{ij})$ be an $n \times n$ nonnegative matrix with row-sums $r_1 \geq r_2 \geq \cdots \geq r_n$. For $1 \leq t < n$, let $d = \max_{t<i\leq n} c_{ii}$ and $f = \max_{1\leq i\leq n, t<j\leq n, i\neq j} c_{ij}$. Assume that $0 < r_n - (n-t-1)f - d$. Then

$$\frac{r_t - f + d + \sqrt{(r_t - (2n-2t-1)f - d)^2 + 4(n-t)(fr_n - (n-t-1)f - d)}}{2}$$

is a lower bound of $\rho(C)$.

Proof.

$$C' = \begin{pmatrix} r_t - (n-t)f & (n-t)f \\ r_n - (n-t-1)f - d & (n-t-1)f + d \end{pmatrix}.$$

□

A general form of our main method

Theorem

Let $C = (c_{ij})$, $C' = (c'_{ij})$, P *and* Q *be* $n \times n$ *matrices. Assume that*

(i) $PCQ \leq PC'Q$;

(ii) C' *has an eigenvector* Qu *for* λ' *for some nonnegative column vector* $u = (u_1, u_2, \ldots, u_n)^T \geq 0$ *and* $\lambda' \in \mathbb{R}$;

(iii) C *has a left eigenvector* $v^T P$ *for* λ *for some nonnegative row vector* $v^T = (v_1, v_2, \ldots, v_n) \geq 0$ *and* $\lambda \in \mathbb{R}$; *and*

(iv) $v^T PQu > 0$.

Then $\lambda \leq \lambda'$. *Moreover,* $\lambda = \lambda'$ *if and only if*

$$(PC'Q)_{ij} = (PCQ)_{ij} \qquad \text{for } 1 \leq i, j \leq n \text{ with } v_i \neq 0 \text{ and } u_j \neq 0.$$

□

Quick realization

To realize the theorem in the last page, we might investigate its special case $P = Q = I$.

Theorem

Let $C = (c_{ij})$, $C' = (c'_{ij})$ be $n \times n$ matrices. Assume that

(i) $C \leq C'$;

(ii) C' *has an eigenvector* u *for* λ' *for some nonnegative column vector* $u = (u_1, u_2, \dots, u_n)^T \geq 0$ *and* $\lambda' \in \mathbb{R}$;

(iii) C *has a left eigenvector* v^T *for* λ *for some nonnegative row vector* $v^T = (v_1, v_2, \dots, v_n) \geq 0$ *and* $\lambda \in \mathbb{R}$; *and*

(iv) $v^T u > 0$.

Then $\lambda \leq \lambda'$. *Moreover,* $\lambda = \lambda'$ *if and only if*

$$(C')_{ij} = (C)_{ij} \qquad \text{for } 1 \leq i, j \leq n \text{ with } v_i \neq 0 \text{ and } u_j \neq 0.$$

□

As simple as majorization-monotone property

Theorem

Let $C = (c_{ij})$, $C' = (c'_{ij})$ *be* $n \times n$ *matrices. Assume that*

(i) $0 \leq C \leq C'$;

(ii) ~~C' *has an eigenvector* u *for* λ' *for some nonnegative column vector* $u = (u_1, u_2, \ldots, u_n)^T \geq 0$ *and* $\lambda' \in \mathbb{R}$;~~

(iii) ~~C *has a left eigenvector* v^T *for* λ *for some nonnegative row vector* $v^T = (v_1, v_2, \ldots, v_n) \geq 0$ *and* $\lambda \in \mathbb{R}$; *and*~~

(iv) $v^T u > 0$.

Then $\lambda \leq \lambda'$. *Moreover,* $\lambda = \lambda'$ *if and only if*

$$(C')_{ij} = (C)_{ij} \qquad \text{for } 1 \leq i, j \leq n \ \underbrace{\text{with } v_i \neq 0 \text{ and } u_j \neq 0}_{\text{if assume } C \text{ irreducible}}.$$

□

Remark

The restricted form is the special case of the general form by applying

$$P = I, \qquad Q = \begin{pmatrix} 1 & 0 & \cdots & 0 & 1 \\ 0 & 1 & \ddots & 0 & 1 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Many other cases should be continuously investigated.