A few inequalities related to spectral excess theorem

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quasi-spectral characterization

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$$f(\text{eigenvalues}) \quad \leq \quad g\left(\begin{matrix}\text{graph parameters}\\ \sout{\text{eigenvectors}}\end{matrix}\right) \quad \leq \quad h(\text{eigenvalues}).$$

spectral characterization

(still working)

Preliminaries

1. Throughout let $G = (VG, EG)$ be a simple connected graph of order n and diameter D.

2. Assume that adjacency matrix A has $d+1$ distinct eigenvalues $\lambda_0 > \lambda_1 > \ldots > \lambda_d$ with corresponding multiplicities $1 = m_0, m_1, \cdots, m_d$. d is called the spectral diameter of G.

3. It is well-known that $D \leq d$.

4. $$Z(x) := \prod_{i=0}^{d}(x - \lambda_i)$$

   is the minimal polynomial of A.

Inner product

Consider the vector space $\mathbb{R}_d[x] \cong \mathbb{R}[x]/\langle Z(x)\rangle$ with the inner product

$$\langle p(x), q(x)\rangle := \frac{1}{n}\mathrm{tr}(p(A)q(A)) = \frac{1}{n}\sum_{i,j}(p(A)\circ q(A))_{ij},$$

for $p(x), q(x) \in \mathbb{R}_d[x]$, where $\circ$ is the entrywise product of matrices.

Predistance polynomials

Definition 1.1

(i) The orthogonal polynomials $1 = p_0(x), p_1(x), \ldots, p_d(x)$ in $\mathbb{R}_d[x]$ satisfying

$$\deg\, p_i(x) = i \qquad \text{and} \qquad \langle p_i(x), p_j(x) \rangle = \delta_{ij} p_i(\lambda_0)$$

are called the predistance polynomials of G.

(ii) The polynomial

$$H(x) := n \prod_{i=1}^{d} \frac{x - \lambda_i}{\lambda_0 - \lambda_i}$$

is called the Hoffman polynomial of G. Moreover, G is regular iff $H(A) = J$, the all 1's matrix.

The sum of all predistance polynomials gives the Hoffman polynomial

$$H(x) = p_0(x) + p_1(x) + \cdots + p_d(x)$$

and

$$H(A) = p_0(A) + p_1(A) + \cdots + p_d(A).$$

Three -term recurrence

The predistance polynomials satisfy a three-term recurrence:

$$x p_i(x) = c'_{i+1} p_{i+1}(x) + a'_i p_i(x) + b'_{i-1} p_{i-1}(x) \qquad 0 \le i \le d,$$

where c'_{i+1}, a'_i, $b'_{i-1} \in \mathbb{R}$ with $b'_{-1} = c'_{d+1} := 0$.

Three -term recurrence for bipartite graph

1. If G is bipartite, then the predistance polynomials satisfy a three-term recurrence of the form

$$x^2 p_i(x) = X_{i+2} p_{i+2}(x) + Y_i p_i(x) + Z_{i-2} p_{i-2}(x) \qquad 0 \le i \le d, \tag{1}$$

where

$$\begin{aligned} X_{i+2} &= c'_{i+1} c'_{i+2}, \\ Y_i &= b'_i c'_{i+1} + b'_{i-1} c'_i, \\ Z_{i-2} &= b'_{i-2} b'_{i-1}. \end{aligned}$$

2. Moreover, if G is bipartite, then for $0 \le j \le d$, $a'_j = 0$, and $p_j(x)$ is even or odd depending on whether j is even or odd.

Distance polynomials

1. Let α be the eigenvector of A corresponding to λ_0 such that $\alpha^t\alpha = n$ and all entries are positive. Note that $\alpha = (1,1,\dots,1)^t$ iff G is regular.

2. The matrix A_i, indexed by VG, satisfying $(A_i)_{uv} = \begin{cases} \alpha_u\alpha_v, & \text{if } \partial(u,v) = i; \\ 0, & \text{else.} \end{cases}$ is called the i-th (weighted) distance matrix of G.

3. $A_0 = p_0(A)(= I)$ iff G is regular.

4. $A_0 + A_1 + \cdots + A_D = H(A) = p_0(A) + p_1(A) + \cdots + p_d(A)$.

The spectral excess and the excess

1. $p_d(\lambda_0)$ is called the spectral excess of G.
2. $\delta_d := \frac{1}{n}\sum_{i,j}(A_d \circ A_d)_{ij}$ is called the excess of G, where $A_d := 0$ if $D < d$.

If G is regular, then δ_d is the average number of vertices to have distance d to a vertex.

The spectral excess theorem (SET)

Theorem 1.2

$$\delta_d \le p_d(\lambda_0)$$

with equality iff G is a distance-regular graph.

[FG1997] M.A. Fiol and E. Garriga, From local adjacency polynomials to local pseudo-distance-regular graphs, *J. Combin. Theory Ser. B* 71 (1997), 162–183.

Related definitions

Define

$$\begin{aligned}
\delta_i &:= \frac{1}{n}\sum_{u,v}(A_i \circ A_i)_{uv}, \\
\delta_{\geq i} &:= \delta_i + \delta_{i+1} + \cdots , \\
p_{\geq i}(\lambda_0) &:= p_i(\lambda_0) + p_{i+1}(\lambda_0) + \cdots , \\
p^{even}(\lambda_0) &:= p_0(\lambda_0) + p_2(\lambda_0) + \cdots , \\
p^{odd}_{\geq i}(\lambda_0) &:= p_i(\lambda_0) + p_{i+2}(\lambda_0) + \cdots \qquad \text{for odd } i, \\
A^{odd} &:= A_1 + A_3 + \cdots \\
&\vdots
\end{aligned}$$

Modify the proof of SET

Proposition 1.3

$$\delta_{\geq i} \leq p_{\geq i}(\lambda_0)$$

with equality iff $A_{\geq i} = p_{\geq i}(A)$.

Proposition 1.4

$$\delta_{\leq i} \geq p_{\leq i}(\lambda_0)$$

with equality iff $A_{\leq i} = p_{\leq i}(A)$.

G is bipartite

Proposition 1.5

If G is bipartite and $* \in \{even, odd\}$ then

$$\delta^*_{\geq i} \leq p^*_{\geq i}(\lambda_0)$$

with equality iff $A^*_{\geq i} = p^*_{\geq i}(A)$.

Proposition 1.6

If G is bipartite and $* \in \{even, odd\}$ then

$$\delta^*_{\leq i} \geq p^*_{\leq i}(\lambda_0)$$

with equality iff $A^*_{\leq i} = p^*_{\leq i}(A)$.

Questions?

$\delta_i \leq p_i(\lambda_0)$ or $\delta_i \geq p_i(\lambda_0)$?

Which graphs have the property that $\delta_i = p_i(\lambda_0)$?

The case $i = 0$

Proposition 2.1

$$\delta_0 \geq 1 (= p_0(\lambda_0)) \quad (\text{ i.e. } \alpha_1^4 + \alpha_2^4 + \cdots + \alpha_n^4 \geq n),$$

and the following are equivalent.

1. $\delta_0 = 1$. (i.e. $\alpha_1^4 + \alpha_2^4 + \cdots + \alpha_n^4 = n$.)
2. $A_0 = I$.
3. G is regular.
4. The entries of α are all 1.

The above inequality is from $\delta_{\leq 0} \geq p_{\leq 0}(\lambda_0)$ which we mentioned earlier, but also follows from the Cauchy-Schwarz inequality

$$\alpha_1^4 + \alpha_2^4 + \cdots + \alpha_n^4 \geq (\alpha_1^2 + \alpha_2^2 + \cdots + \alpha_n^2)^2 / n \geq n.$$

The case $i = 1$

A bipartite graph with bipartition $V(G) = X \cup Y$ is biregular if there exist distinct integers $k \neq k'$ such that every $x \in X$ has degree k, and every $y \in Y$ has degree k'.

Proposition 3.1

$$\delta_1 \geq p_1(\lambda_0),$$

and the following statements are equivalent.

(i) $\delta_1 = p_1(\lambda_0)$ (or equivalently $\delta_1 \overline{k} = \lambda_0^2$),

(ii) $A_1 = p_1(A)$,

(iii) G is regular or G is bipartite biregular.

Corollary 3.2

There is no bipartite biregular graph G with exactly four distinct eigenvalues.

The idea of the proof is to modify a proof in

[DDFGG2011] C. Dalfó, E.R. van Dam, M.A. Fiol, E. Garriga and B.L. Gorissen, On almost distance-regular graphs, *J. Combin. Theory Ser. A* 118 (2011), 1094–1113.

The case $i = 2$

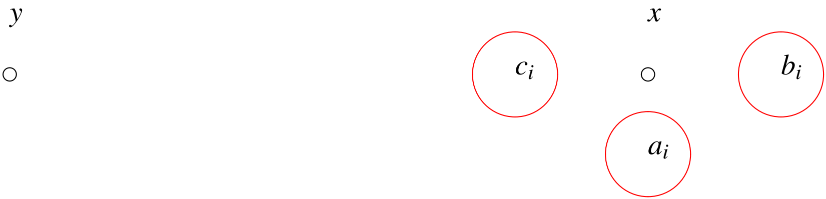
1. If $d = 2$ then $\delta_2 \leq p_2(\lambda_0)$.
2. If G is regular bipartite, then $\cancel{\delta_0 +} \delta_2 \geq \cancel{p_0(\lambda_0) +} p_2(\lambda_0)$.
3. There is no hope to determine the order of δ_2 and $p_2(\lambda_0)$ uniformly.

Definition 4.1

Let G be a graph and i is a nonnegative integer. We say the numbers c_i, a_i, b_i respectively are well-defined in G if for any $x, y \in V(G)$ with $\partial(x,y) = i$, the numbers

$$c_i := |G_1(x) \cap G_{i-1}(y)|,$$
$$a_i := |G_1(x) \cap G_i(y)|,$$
$$b_i := |G_1(x) \cap G_{i+1}(y)|,$$

respectively are independent of the choice of x, y.



Definition 4.2

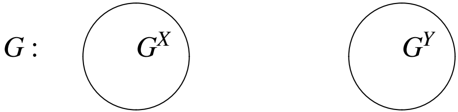
A graph G is *t-partially distance-regular* if for $2 \leq i \leq t$ and the numbers c_i, a_{i-1}, b_{i-2} are well-defined.

Lemma 4.3

$\delta_0 = p_0(\lambda_0)$ and $\delta_2 = p_2(\lambda_0)$ iff G is 2-partially distance-regular.

Definition 4.4

For a connected bipartite graph G with bipartition $X \cup Y$, the halved graphs G^X and G^Y are the two connected components of the distance-2 graph of G.



Theorem 4.5 (BCN, Prop 4.2.2, p.141)

The halved graphs of a bipartite distance-regular graph are again distance-regular and, in the case where G is vertex transitive, the two halved graphs are isomorphic.

Lemma 4.6

If $G = (X, Y)$ is a connected regular bipartite graph with $\delta_2 = p_2(\lambda_0)$ (so 2–partially distance-regular by previous lemma), then the halved graphs G^X and G^Y have the same spectrum.

Bipartite graphs with $\delta_{d-1} = p_{d-1}(\lambda_0)$

Lemma 5.1

Let G be bipartite and $\delta_{d-1} = p_{d-1}(\lambda_0)$. Then

$$A_{d-1} = p_{d-1}(A), A_{d-3} = p_{d-3}(A), A_{d-5} = p_{d-5}(A), \ldots.$$

In particular G is regular (if $A_0 = p_0(A)$) or bipartite biregular (if $A_1 = p_1(A)$).

Theorem 5.2

Let G be a connected bipartite graph with bipartition $X \cup Y$, odd d. Then the following are equivalent.

(i) $\delta_{d-1} = p_{d-1}(\lambda_0)$;

(ii) G is 2-partially distance-regular and both halved graphs G^X and G^Y are distance-regular with the same intersection numbers.

The following example shows that a bipartite graph satisfying Theorem 5.2(i)-(ii) and $D = d$ needs not to be distance-regular graph.

Example 5.3

Consider the regular bipartite graphs G on 20 vertices obtained from the Desargues graph (the bipartite double of the Petersen graph) by the GM-switching (a way to produce cospectral nonisomorphic graphs). One can check (by Maple) that $D = d = 5$, sp $G = \{3^1, 2^4, 1^5, (-1)^5, (-2)^4, (-3)^1\}$, $p_0(x) = 1$, $p_1(x) = x$, $p_2(x) = x^2 - 3$, $p_3(x) = (x^3 - 5x)/2$, $p_4(x) = (x^4 - 9x^2 + 12)/4$, $p_5(x) = (x^5 - 11x^3 + 22x)/12$, $A_i = p_i(A)$ for $i \in \{0, 1, 2, 4\}$. Hence $\delta_0 = p_0(\lambda_0) = 1$, $\delta_1 = p_1(\lambda_0 = 3$, $\delta_2 = p_2(\lambda_0) = 6$, $\delta_4 = p_4(\lambda_0) = 3)$, $\delta_3 = 32/5$, $\delta_5 = 3/5$, $p_3(\lambda_0) = 6$, $p_5(\lambda_0) = 1 \neq 3/5 = \delta_5$. Then G is not distance-regular.

The following example provides a graph G satisfying Theorem 5.2(i)-(ii) with $D = d - 1$.

Example 5.4

Consider the Möbius-Kantor graph G, i.e., the generalized Petersen graph $GP(8,3)$ with vertex set $\{u_0, u_1, \ldots, u_7, v_0, v_1, \ldots, v_7\}$ and edge set $\{u_i v_i, u_i u_{i+1}, v_i v_{i+3} | 0 \le i \le 7\}$ with arithmetic modulo 8. One can check (by Maple) that $D = 4 < 5 = d$, sp $G = \{3^1, \sqrt{3}^4, 1^3, (-1)^3, (-\sqrt{3})^4, (-3)^1\}$, $p_0(x) = 1$, $p_1(x) = x$, $p_2(x) = x^2 - 3$, $p_3(x) = 2(x^3 - 5x)/5$, $p_4(x) = (x^4 - 10x^2 + 15)/6$, $p_5(x) = (x^5 - 56x^3/5 + 21x)/18$, $A_i = p_i(A)$ for $i \in \{0, 1, 2, 4\}$ ($\delta_0 = p_0(\lambda_0) = 1$, $\delta_1 = p_1(\lambda_0) = 3$, $\delta_2 = p_2(\lambda_0) = 6$, $\delta_4 = p_4(\lambda_0) = 1$), $\delta_3 = 5$, $p_3(\lambda_0) = 24/5$, $p_5(\lambda_0) = 1/5$. Note that $G^2 = 2X$, where X is the 16-cell graph (http://mathworld.wolfram.com/16-Cell.html), which is distance-regular with sp $X = \{6^1, 0^4, (-2)^3\}$.

The parity of d makes things different

Theorem 5.5

Let G be a connected bipartite graph with bipartition $X \cup Y$ and even d. Then the following are equivalent.

(i) G is distance-regular;

(ii) G is 2-partially distance-regular and both of the halved graphs G^X and G^Y are distance-regular of diameter $d/2$.

In the next two pages, we provide two examples of non-distance-regular graphs that satisfy $p_{d-1}(\lambda_0) = \delta_{d-1}$ when d is even. The first one is bipartite biregular and the second one is regular.

Example 5.6

Consider the bipartite graphs G on 25 vertices obtained from the Petersen graph by subdividing each edge once. One can check (by Maple) that $D = d = 6$,
sp $G = \{\sqrt{6}^1, 2^5, 1^4, 0^5, (-1)^4, (-2)^5, (-\sqrt{6})^1\}$, the Perron-Frobenius vector $\alpha = (\underbrace{\sqrt{5/4}, \cdots, \sqrt{5/4}}_{10}, \underbrace{\sqrt{5/6}, \cdots, \sqrt{5/6}}_{15})^t$,
$p_0(x) = 1$, $p_1(x) = 5\sqrt{6}x/12$, $p_2(x) = 15(x^2 - 12/5)/16$,
$p_3(x) = 5\sqrt{6}(x^3 - 4x)/12$, $p_4(x) = 25(x^4 - 21x^2/4 + 3)/28$,
$p_5(x) = 5\sqrt{6}(x^5 - 7x^3 + 10x)/24$,
$p_6(x) = 5(x^6 - 65x^4/7 + 22x^2 - 48/7)/24$, $\widetilde{A}_i = p_i(A)$ for $i \in \{1, 3, 5\}$ $(\delta_1 = p_1(\lambda_0) = 5/2,\ \delta_3 = p_3(\lambda_0) = 5,\ \delta_5 = p_5(\lambda_0) = 5)$, $\delta_0 = 25/24$, $\delta_2 = 85/24$, $\delta_4 = 85/12$, $\delta_6 = 5/6$, $p_0(\lambda_0) = 1$, $p_2(\lambda_0) = 27/8$, $p_4(\lambda_0) = 375/56$, $p_6(\lambda_0) = 10/7$. Note that G^2 is the disjoint union of the Petersen graph X and the line graph Y of X. We have sp $X = \{3^1, 1^5, (-2)^4\}$, and sp $Y = \{4^1, 2^5, (-1)^4, (-2)^5\}$.

Example 5.7

Let G be the Hoffman graph (A graph nonisomorphic but cospectral to 4-cube). Then sp $G = \{4^1, 2^4, 0^6, (-2)^4, (-4)^1\}$, $D = d = 4$, $p_0(x) = 1$, $p_1(x) = x$, $p_2(x) = (x^2 - 4)/2$, $p_3(x) = (x^3 - 10x)/6$, $p_4(x) = (x^4 - 16x^2 + 24)/24$, and $A_3 = p_3(A)$. Note that G^2 is the disjoint union of K_8 and $K_{2,2,2,2}(= K_8 - 4K_2)$, which are both distance-regular (sp $K_{2,2,2,2} = \{6^1, 0^4, (-2)^3\}$).