An Introduction to **Hamiltonian Graph Theory**

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https://jupiter.math.nctu.edu.tw/~weng/powerpoint/2021_01_18_Weng.pdf

https://jupiter.math.nctu.edu.tw/~weng/powerpoint/Weng_Exercise_20210118_for%20print.pdf

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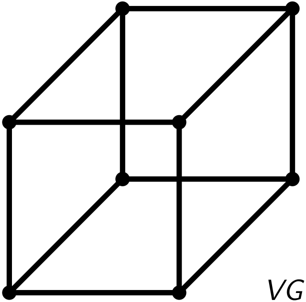
1. Background

Graphs

1. A **graph** G is a finite set V with a family $E \subseteq \binom{V}{2}$ of 2-subsets of V.
2. The set $V = VG$ is called **vertex set** and $E = EG$ is called **edge set**.
3. Two vertices x, y are **adjacent** if $xy \in EG$.



$$VG = \{0_{0_0}, 0_{0_1}, 0_{1_0}, 0_{1_1}, 1_{0_0}, 1_{0_1}, 1_{1_0}, 1_{1_1}\},$$

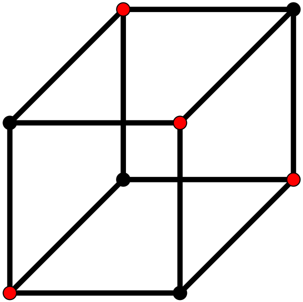
$$EG = \{\ i_{j_k} i'_{j'_{k'}} \mid \text{Exactly one of } i \neq i',\ j \neq j' \text{ or } k \neq k' \text{ holds}\}.$$

Bipartite graph

A graph G is **bipartite** if there exists a bipartition $X \cup Y$ of VG such that

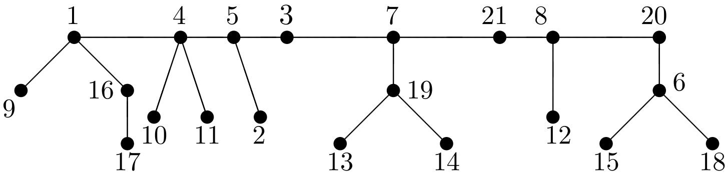
$$EG \cap \binom{X}{2} = \emptyset \quad \text{and} \quad EG \cap \binom{Y}{2} = \emptyset.$$

Moreover G is **balanced** if $|X| = |Y|$.



Trees

A graph T is a **tree** if it is **connected** and contains no **cycle**.



Degrees, maximum degree and k-regular graphs

Let G be ag graph.

1. For $x \in VG$, the set
$$\mathbf{G_1(x) := \{y \mid xy \in EG\}}$$
is called the set of **neighbors** of x in G.
2. The **degree** of x in G is $|G_1(x)|$.
3. The **maximum degree** $\Delta(G)$ of G is the maximum value of $|G_1(x)|$ for $x \in VG$.
4. G is **k-regular** if every vertex of G has degree k.

First theorem of graph theory

If G is a graph such that vertex $x \in G$ has degree $\deg(x)$ then

$$\sum_{x \in VG} \deg(x) = 2|EG|.$$

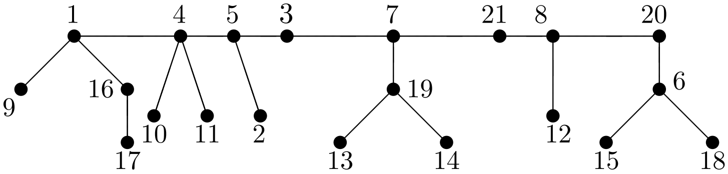
Proof.

By counting the pairs (x, e) with $x \in VG$, $e \in EG$ and $x \in e$, we have

$$\sum_{x \in VG} \deg(x) = 2|EG|.$$

$\square$

Example



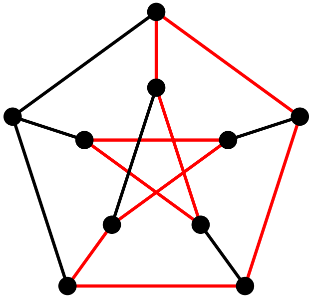
$$3+1+2+4+3+3+3+3+1+1+1+1+1+1+1+2+1$$
$$+1+3+2+2=\sum_{x\in VG}\deg(x)=2|EG|=2\cdot 20.$$

Exercise 1

If n points are placed in a plane with pairwise distances at least 1, then there are at most $3n$ unordered pairs of points at distance exactly 1.

Hamiltonian cycles and paths

1. A **Hamiltonian cycle** (resp. **Hamiltonian path**) of G is a cycle (resp. path) that contains all vertices of G.
2. A graph is **Hamiltonian** (resp. **traceable**) if it contains a Hamiltonian cycle (resp. Hamiltonian path).



Petersen graph is not Hamiltonian, containing a Hamiltonian path.

Examples

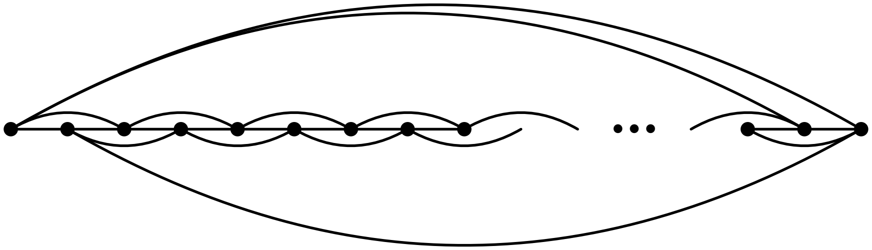
1. The **path** P_n of order n is not Hamiltonian.
2. The **cycle** C_n of order n is Hamiltonain.
3. The **complete graph** K_n is Hamiltonian if $n \geq 3$.
4. The **complete bipartite graph** $K_{s,t}$ is Hamiltonian if and only if $s = t \geq 2$.

Remark

(Karp, 1972) Determine if a graph is Hamiltonian is *NP*-complete.

Exercise 2

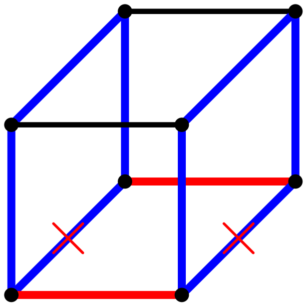
Let G be a graph with vertex set $VG = [n] := \{1, 2, \ldots, n\}$ and $EG = \{ij : |i - j| \in \{1, 2, n-2, n-1\}\}$. Show that if $n \geq 5$, then the edges of G can be partitioned into two Hamiltonian cycles.



2. Construction

The k-cube

The ***k*-cube** $Q_k = Q_{k-1} \square P_2$ $(k \geq 2, Q_1 = P_2)$ is Hamiltonian.



Cartesian product graphs

The **Cartesian product graph** $G_1 \Box G_2$ of graphs G_1 and G_2 is a graph with vertex set and and edge set as follows

$$\begin{aligned} V(G_1 \Box G_2) =& \{u_v : u \in V(G_1), v \in V(G_2)\}, \\ E(G_1 \Box G_2) =& \{u_v u_w : u \in V(G_1), vw \in E(G_2)\} \\ & \cup \{u_v w_v : v \in V(G_2), uw \in E(G_1)\}. \end{aligned}$$

Example

$$Q_1 = K_2 = P_2, \qquad Q_k = \underbrace{(((P_2 \Box P_2) \Box P_2) \Box \cdots) \Box P_2}_{k \text{ times}}.$$

Corollary

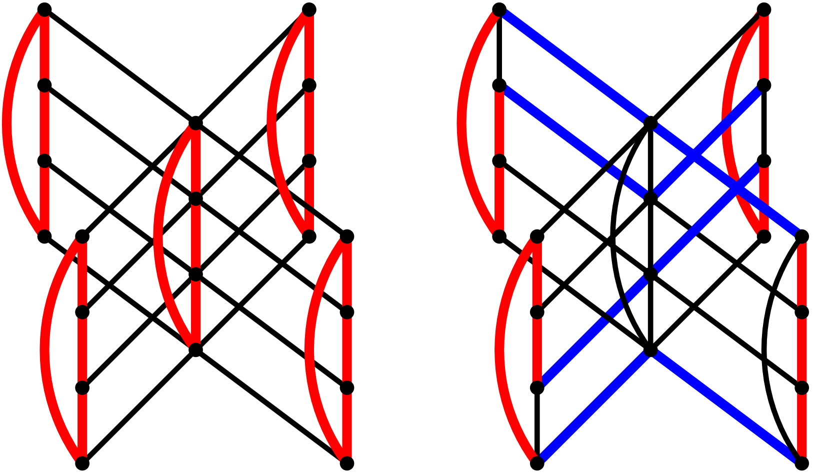
If G is Hamiltonian then $G \square P_2$ is Hamiltonian. $\square$

Question

Can we generalize the above Corollary with the replacing of P_2 by a tree T.

Theorem (Rosenfeld, Barnette, 1973)

If T is a tree and $n \geq \Delta(T)$ then $C_n \square T$ is Hamiltonian.



The converse (Exercise 3)

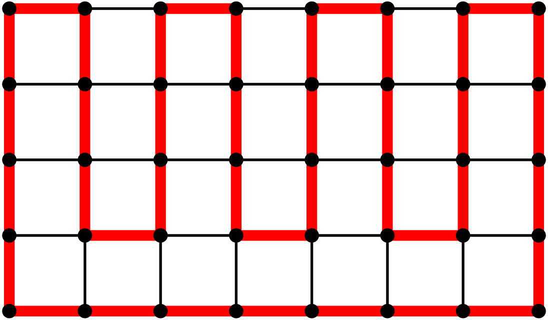
If T is a tree and G is a graph such that $G \square T$ is Hamiltonian, then $|VG| \geq \Delta(T)$.

Question

Can we modify Rosenfeld-Barnette Theorem with the replacing of $C_n \square T$ by $P_n \square T$?

Grid graph $P_m \square P_n$

The **grid graph** $P_m \square P_n$ is Hamiltonian if and only if $m, n \geq 2$ and mn is even.



1-factor

A 1-**factor** (**perfect matching**) of a graph is its 1-regular spanning subgraph.

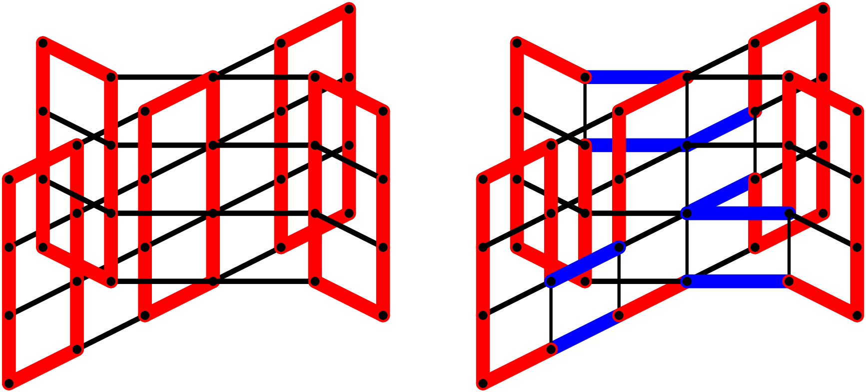
Example

1. The path P_n has a 1-factor iff n is even.
2. The grid graph $P_m \Box P_n$ is Hamiltonian iff $m, n \geq 2$ and one of P_m and P_n has a 1-factor.

Proposition (Kao)

If T is a tree with a 1-factor and $n \geq \Delta(T)$, then $P_n \Box T$ is Hamiltonian.

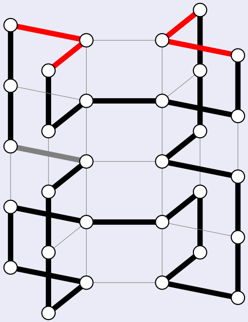
Proof.



Remark

If T is a tree and $P_n \square T$ is Hamiltonian then T has a path factor, i.e. T has a spanning subgraph such that each component is isomorphic to a path of order at least two.

Proof.



□

Theorem (J. Akiyama, D. Avis and H. Era, 1980)

Let G be a graph. Then G has a path factor if and only if

$$i(G - S) \leq 2|S|$$

for all $S \subseteq V(G)$, where $i(G)$ is the number of isolated vertices in G.

Example

$K_{1,3}$ does not have a path factor.

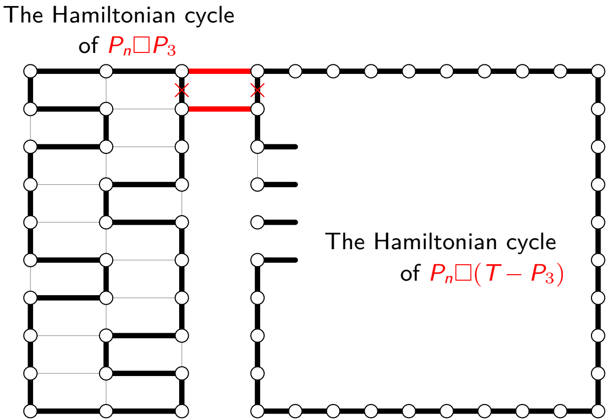
Jin Akiyama (Japanese: 秋山仁, born 1946) is a Japanese mathematician, known for his appearances on Japanese prime-time television (NHK) presenting magic tricks with mathematical explanations.

https://en.wikipedia.org/wiki/Jin_Akiyama

Theorem (Kao)

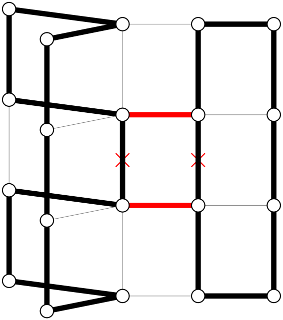
Let G be a connected graph with a path factor and n be an even integer with $n \geq 4\Delta(G) - 2$. Then $P_n \square G$ contains a Hamiltonian cycle.

Proof. (Sketch)



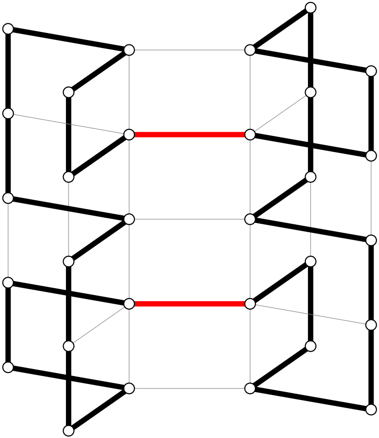
Do we really need $n \geq 4\Delta(G) - 2$?

$P_2 \square P_3$



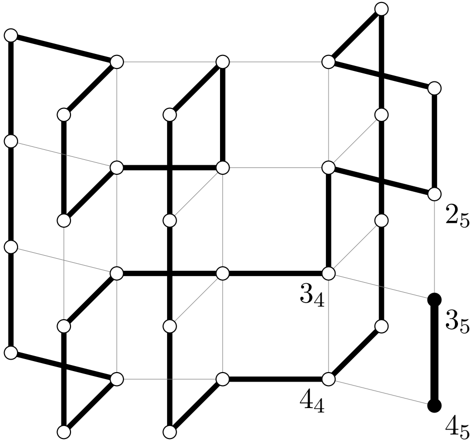
A Hamiltonian graph $P_n \square G$ with $n = \Delta(G) + 1$

Do we need n being even ?



A Hamiltonian graph $P_n \square G$ with odd n

Is $n \geq \Delta(G) + 1$ enough ?



A non-Hamiltonain graph $P_4 \Box G$ with $\Delta(G) = 3$.

Conjectures

- For $k \geq 3$, there is a connected graph G with a path factor such that $\Delta(G) = k$ and $P_{\mathbf{4k-4}} \Box G$ is not Hamiltonian.
- Let G be a connected graph with a path factor and let n be an integer such that $\mathbf{n|VG|}$ is even and $n \geq 4\Delta(G) - 2$. Then $P_n \Box G$ contains a Hamiltonian cycle.

3. Line graph

Eulerian graphs

Let G be a graph.

1. A **closed trail** in G is a **closed walk** without repeated edges.
2. G is **Eulerian** if it contains a closed trail using all edges in G.

First Paper of Graph Theory, L. Euler, 1736

Let G be a graph without isolated vertices. Then G is Eulerian if and only if G is connected and every vertex has even degree.

Proof.

($\Rightarrow$) This is clear.
($\Leftarrow$) Can assume G is loopless. Induction on $|EG|$. $|EG| = 1$ is clear.

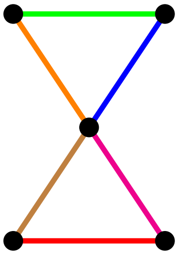
1. G contains a cycle C. (Here use $|VG| < \infty$ and $\deg(x)$ is even.)
2. Apply induction on components of $G - C$. Each component of $G - C$ has an Eulerian trail and has at least a vertex in C.
3. Glue these closed trails to an Eulerian trail of G.

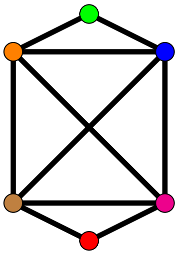
□

Line graph

The **Line graph** of a graph G, denote as $L(G)$, is a graph with vertex set $VL(G) = EG$ and edge set

$$EL(G) := \{e_1 e_2 \ : \ e_1 = uv, e_2 = vw \text{ for some } v\}.$$

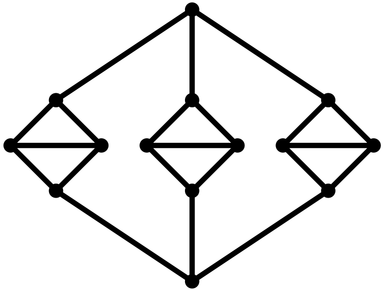


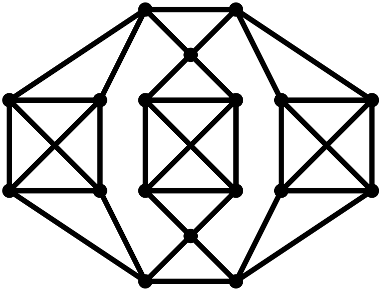


Remarks and a question

- The line graph of an Eulerian graph is Hamiltonian.
- If G has only even degrees of vertices then $L(G)$ is Hamiltonian.
- $\deg_{L(G)}(xy) = \deg_G(x) + \deg_G(y) - 2.$
- **Is $L(G)$ Hamiltonian if it has only even degrees of vertices?**

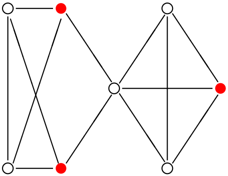
A non-Hamiltonian line graph with every vertex even degree





Independent set

An **independent set** in G is a subset I of G such that the subgraph induced on I has no edges.



Remark

If G is Hamiltonian, then $L(G)$ is Hamiltonian.

Theorem (Harary and Nash-Williams, 1965)

$L(G)$ is Hamiltonian if and only if there is a closed trail S such that $VG - VS$ is an independent set in G.

Proof.

cycle C in $L(G)$	$\to$	closed circuit $S(C)$ in G	(deleting edges)
closed circuit S in G	$\to$	cycle $C(S)$ in $L(G)$	(adding edges)

□

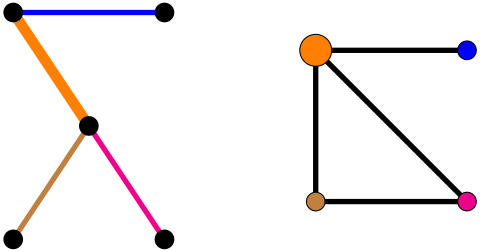
F. Harary, J. A. Nash-Williams, Every connected, locally connected nontrivial graph with no induced claw is Hamiltonian. *Canad. Math. Bull.* 8-6 (1965), 701-709.

Example

The line graph $L(K_{m,n})$ of $K_{m,n}$ is Hamiltonian.

Characterization of line graphs

- A line graph is **claw-free**, which means it contains no **induced** $K_{1,3}$.
- Line graphs are characterized in terms of some forbidden subgraphs (Beineke, 1968).



The study of Hamiltonian line graphs can be extended to graphs with prescribed forbidden subgraphs.

Remark

(Bertossi, 1981) It is *NP*-complete to decide whether a given line graph is hamiltonian.

4. Degrees

Ore Theorem, 1960

Let G be a simple graph such that $\deg(u) + \deg(v) \geq |VG|$ for any nonadjacent vertices u, v. Then G is Hamiltonian.

Proof.

- Let $u_1 = u$, u_2, ..., $u_\ell = v$ be a path of maximum order ℓ.
- The assumption implies $u_i v, u u_{i+1} \in EG$ for some i.
- Thus G contains a cycle

$$u = u_1, u_2, \ldots, u_i, v = u_\ell, u_{\ell-1}, u_{i+1}, u.$$

- Note that $\ell = n$, since G is connected.

□

Chvátal Theorem, 1972

Let G be a simple graph with vertex degrees $d_1 \leq \cdots \leq d_n$, where $n \geq 3$. If $d_i > i$ or $d_{n-i} \geq n - i$ for any $i < n/2$, then G is Hamiltonian.

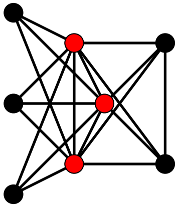
Proof.

- Find a counter example with maximum number of edges.
- Thus we can assume that $\deg(u) + \deg(v) < n$ for **any** nonadjacent vertices u, v in G.
- Pick two nonadjacent vertices u, v with maximum degree sum $\deg(u) + \deg(v)$.
- Assume $i = \deg(u) \leq \deg(v) = j$.
- From the above choice of u, v, we have $d_i \leq i$ and $d_{n-i} < n - i$, a contradiction.

□

Complement, disjoint union and joint of graphs

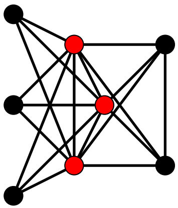
- The **complement** of a graph G is the graph $\overline{G}$ with vertex set $V\overline{G} = VG$ and edge set $E\overline{G} = \binom{VG}{2} - EG$.
- The **disjoint union** of graphs G and H is the graph $G + H$ with vertex set $V(G+H) = VG \cup VH$ and edge set $E(G+H) = EG \cup EH$.
- The **joint** of graphs G and H is the graph $G \vee H$ with vertex set $V(G \vee H) = VG \cup VH$ and edge set $E(G \vee H) = EG \cup EH \cup \{uv : u \in VG, v \in VH\}$.



The graph $(\overline{K_3} + K_2) \vee \mathbf{K_3}$.

A class of extremal graphs

The graph $(\overline{K_m} + K_{n-2m}) \vee \mathbf{K_m}$ is not Hamiltonian since the removal of K_m create $m+1$ connected components.



The graph $(\overline{K_m} + K_{n-2m}) \vee \mathbf{K_m}$ has degree sequence

$$\underbrace{m, \ldots, m}_{m}, \underbrace{n-m-1, \ldots, n-m-1}_{n-2m}, \underbrace{n-1, \ldots, n-1}_{m},$$

satisfying negation of Chvátal Conditions

$$\neg(\forall i < \frac{n}{2}(d_i > i \text{ or } d_{n-i} \geq n-i) \equiv \exists m < \frac{n}{2}(d_m \leq m \text{ and } d_{n-m} \leq n-m-1).$$

Exercise 4

Suppose $|EG| \geq \binom{n-1}{2} + 2,$ **where** $n \geq 3.$ **Show that** G **is Hamiltonian.**

Exercise 5

“ If a simple graph G of order n contains two nonadjacent vertices whose degrees sum is at least n then G is Hamiltonian.” Find a counterexample of the above statement.

Exercise 6

Find the first mistake in the following proof of the statement "If a simple graph G of order n contains two nonadjacent vertices whose degrees sum is at least n then G is Hamiltonian."

Wrong Proof.

(i) If the statement is false then there is a counter-example G with maximal number of edges.

(ii) There are two nonadjacent vertices u, v in G whose degrees sum is at least n.

(iii) Since $G + uv$ is not a counter-example, it is Hamiltonian,

(iv) so we have a Hamiltonian path $u = u_1, \ldots, u_n = v$ in G.

(v) Then uu_{i+1} and u_iv are edges for some $1 \leq i \leq n-1$.

(vi) Hence $u_1, u_{i+1}, u_{i+2}, \cdots, u_n, u_i, u_{i-1}, \cdots, u_1$ is a Hamiltonian cycle.

□

5. Connectivity

Connectivity and independent number

- The **connectivity** of G, written $\kappa(\mathbf{G})$, is the minimum size of a vertex set S such that $G - S$ is disconnected or has only one vertex.
- G is t-connected if $t \geq \kappa(G)$.
- The **independent number** of G, written $\alpha(\mathbf{G})$, is the maximum number of pairwise nonadjacent vertices.

Remark

$d_{\min}(G) \geq \kappa(G)$.

Example

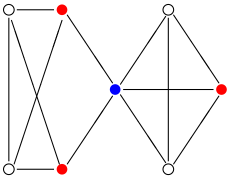


Figure. $\kappa(\mathbf{G}) = \mathbf{1}$, $\alpha(\mathbf{G}) = \mathbf{3}$.

Remark

A graph G is 2-connected if and only if any pair of vertices u, v belongs to a cycle.

Chvátal-Erdős Theorem, 1972

Suppose $G \neq K_2$ and $\kappa(G) \geq \alpha(G)$. Then G is Hamiltonian.

Proof.

- Let C be a cycle of maximum length.
- Suppose $VG \neq VC$. Let H be a connected component of $G - C$.
- Along a direction of the cycle C, consecutively pick $\kappa(G)$ vertices $u_1, u_2, \ldots, u_{\kappa(G)}$ with edges to H.
- Let a_i be a vertex immediately following u_i on C.
- Note that $a_i \neq u_{i+1}$ and $a_i a_j \notin EG$, otherwise we will have a longer cycle.
- Then $\{a_1, a_2, \ldots, a_{\kappa(G)}, x\}$ is an independent set for any $x \in VH$, a contradiction.

□

Example

The complete bipartite graph $K_{m,m}$ is Hamiltonian and

$$\kappa(K_{m,m}) = m = \alpha(K_{m,m}).$$

Large connectivity does not guarantee hamiltonian

The complete bipartite graph $K_{n,n+1}$ is not Hamiltonian and $\kappa(K_{n,n+1}) = n = \alpha(K_{n,n+1}) - 1$.

Local property

We said that a graph G is **locally-$\mathcal{P}$** if for any vertex x, its neighbor set $G_1(x)$ has the property $\mathcal{P}$.

Exercise 7

A connected and locally $(k-1)$-connected graph G is k-connected.

Related articles

(Oberly, Summer, 1979) A connected, locally-connected graph with at least 3 vertices which does not contain an induced $K_{1,3}$ is Hamiltonian.

Corollaries (Oberly, Summer, 1979)

If G is connected and locally-connected on at least 3 vertices, then $L(G)$ is Hamiltonian.

A line graph cannot contain $K_{1,3}$ as an induced subgraph, so the corollary follows:
A connected, locally-connected line graph with at least 3 vertices is Hamiltonian.

If every edge of a graph G lies in some triangle, then the line graph $L(G)$ is locally-connected, so:
If every edge of G lies in some triangle, then $L(G)$ is Hamiltonian.

Conjecture (Oberly, Summer, 1979)

A connected, locally-n-connected graph with at least 3 vertices which does not contain an induced $K_{1,2+n}$ is Hamiltonian.

6. Toughness

Lemma (A necessary condition for Hamiltonian graphs)

Suppose G is Hamiltonian and $S \subseteq VG$ is nonempty. Then $c(G - S) \leq |S|$, where $G - S$ is the subgraph of G induced on $VG - S$, and $\mathbf{c(G)}$ is the number of connected components in G.

Proof.

Let C be an Hamiltonian cycle in G. Then

$$c(G - S) \leq c(C - S) \leq |S|.$$

$\square$

Exercise 8

On a chessboard, a knight can move from one square to another if they differ by 1 in one coordinate and 2 by another. A knight-tour is a path that a knight visiting every single square exactly once and return to the starting square. Show that a 4-by-n chessboard contains no knight-tour for all n.

Toughness of a graph

- A graph is ***t*-tough** if

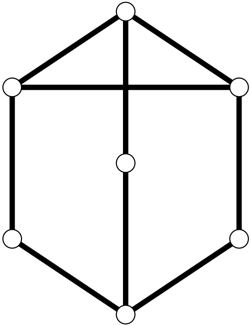
$$c(G-S)>1 \quad \Rightarrow \quad c(G-S)\le \frac{|S|}{t} \quad \forall S\subseteq VG$$

- The **toughness** a non-complete graph G is the maximum t such that G is t-tough. The toughness of a complete graph is ∞.

Remark

- A Hamiltonian graph is 1-tough.
- A t-tough graph is $2t$-connected.

A 1-tough non-Hamiltonian graph with 7 vertices



A long standing conjecture

There is a positive constant t such that every t-tough graph is Hamiltonian.

V. Chvátal, Tough graphs and hamiltonian circuits, *Discrete Math.* 5 (1973), 215-228.

Recall (J. Akiyama, D. Avis and H. Era, 1980)

Let G be a graph. Then G has a path factor if and only if

$$i(G - S) \leq 2|S|$$

for all $S \subseteq VG$, where $i(G)$ is the number of isolated vertices in G.

Remark (Kao)

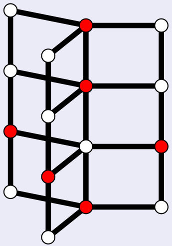
If G is bipartite without path factor then the above vertex subset S with $i(G - S) > 2|S|$ can be chosen entirely inside one part of the bipartition of VG.

Proposition (Kao)

If $P_n \square G$ is a 1-tough bipartite graph then G has a path factor.

Proof.

(Sketch) We prove the contrapositive.



$$i(G-\mathbf{x}) = 3 \not\leq 2 = 2\cdot|\{\mathbf{x}\}| \quad \Rightarrow \quad c(P_4\square G-\mathbf{S}) = 7 \not\leq 6 = |\mathbf{S}|.$$

$\square$

Related articles

(Jung, 1987) Let G be a 1-tough graph with at least 11 vertices such that $\deg(u) + \deg(v) \geq |VG| - 4$ for any nonadjacent vertices u, v. Then G is Hamiltonian.

(Moon and Moser, 1963) Let G be a 1-tough bipartite graph such that $\deg(u) + \deg(v) > |VG|/4$ for any nonadjacent vertices u, v. Then G is Hamiltonian.

(Li, Wei, Yu and Zhu, 2002) Let G be a 1-tough triangle-free graph such that $\deg(u) + \deg(v) + \deg(w) \geq |VG|$ for any independent set $\{u, v, w\}$. Then G is Hamiltonian.

(Bauer, Broersma, Heuvel, Veldman, 1995) Let G be a t-tough graph with at least 3 vertices and $(t+1)(d_{\min}(G)+1) > n$. Then G is Hamiltonian.

Related articles

(Kratsch, Lehel, and Müller, 1996) Every 3-tough $2K_2$-free graphs are Hamiltonian.

A **split** graph is a graph whose vertices can be partitioned into a clique and an independent set. A split graph is $2K_2$-free.

(Shan, 2020) Every $\frac{3}{2}$-tough split graphs are Hamiltonian.

We say a graph G to be **chordal** if every cycle of length more than 3 in G contains no chord.

(Chen, Jacobson, Kėzdy and Lehel, 1998) Every 18-tough chordal graph is Hamiltonian.

(Kabela, Kaiser, 2015) 10-tough chordal graphs are Hamiltonian.

Remark

(Bauer, Hakimi, Schmeichel, 1990) The NOT-1-tough problem is NP-complete.

7. Planar graphs

Planar Graph

A graph is **planar** if it has a drawing in $\mathbb{R}^2$ without crossings. A **plane graph** is a particular drawing of a planar graph in $\mathbb{R}^2$.

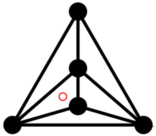
Example

1. K_4 is planar.
2. $K_{2,n}$ is planar.

Faces

Let G be a plane graph.

- Then $\mathbb{R}^2 - G$ is a finite disjoint union of "maximal connected open sets".
- Each of these maximal connected open sets is called a **face** of G.
- If the boundary of unbounded face is a cycle, it is called the **exterior cycle**.



Remark

Every face of a planar graph can be an unbounded face.

Subdivision

- In a graph G, **subdivision** of an edge xy is the operation of replacing xy with a path x, z, y through a new vertex w.
- A **subdivision** of G is a graph obtained from a sequence of subdivision processes starting from G.

Theorem (Kuratowski, 1930)

A graph is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$.

Exercise 9

Let H denote a 2-connected and non-hamiltonian graph. Then H contains a subdivision of $K_{3,2}$.

Exercise 10

If H is a graph that does not contain a subdivision of $K_{3,2}$, then the following are equivalent.

(i) H **is hamiltonian.**

(ii) H **is** 1**-tough.**

(iii) H **is** 2**-connected.**

Exercise 11

If G is a planar, then the following are equivalent.

(i) G **is locally hamiltonian.**

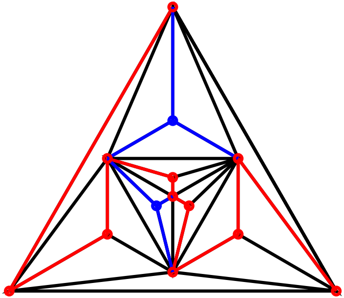
(ii) G **is locally 1-tough.**

(iii) G **is locally 2-connected.**

Tutte path in a plane graph

A path P of a plane graph G with external cycle C is called a **Tutte path** if

- each component of $G - P$ has at most three neighbors in P, or
- has at most two neighbors in P if the component contains an edge of C.



Theorem (Thomassen, 1983)

Let G be a 2-connected plane graph with external cycle C. For any $v \in V(C)$, $e \in E(C)$ and $u \in V(G - v)$, there is a Tutte path $P(G, C, v, e; u)$ from v to u using e.

C. Thomassen. A theorem on paths in planar graphs. *Journal of Graph Theory*, 7(2):169–176, 1983.

Theorem (Tutte, 1956)

A 4-connected plane graph is Hamiltonian.

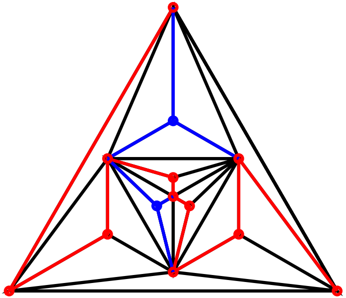
Proof.

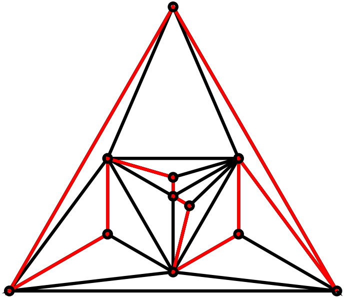
Let C be the external cycle of 4-connected plane graph G of order at least 5. Let vertex v and edge e is on the external cycle C and u be a vertex adjacent to v such that e is not incident u and v. Let $P(G, C, v, e; u)$ be a Tutte path from v to u using e. Hence P contains at least 4 vertices. Thus $V(G - P(G, C, v, e; u)) = \emptyset$ since G is 4-connected. Hence $P(G, C, v, e; u)$ is a hamiltonian path. Adding the edge vu to P, we have a hamiltonian cycle. □

W.T. Tutte, A theorem on planar graphs. *Trans. Amer. Math. Soc.* 82 (1956), 99-116.

Maximal planar graph

- A graph is **maximal planar** if it is planar and no edge can be added without losing planarity.
- A triangle T in a planar graph G is a **separating triangle** if $G - T$ is not connected.

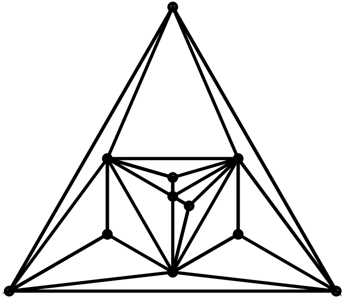




How many separating triangles? (7?5?)

Remark

If a maximal planar graph is disconnected by 3 vertices (not 4-connected) then the three vertices form a separating triangle.

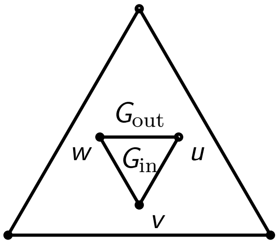


(Whitney, 1931) Every maximal planar graph without separating triangles is Hamiltonian.

Theorem (Chen, 2003 (accepted in 2000))

Any maximal planar graph with only one separating triangles is Hamiltonian.

Proof.



With separating triangle $C = uvw$ as a common exterior cycle of subgraphs $G_{\rm in}$ and $G_{\rm out}$ of G, we obtain their corresponding Hamiltonian Tutte paths $P(G_{\rm in}, C, u, \overline{wv}; v)$ and $P(G_{\rm out}, C, v, \overline{uw}; w)$. Combine these two paths consecutively and a backward step in the end to have Hamiltonian cycle

$$P(G_{\rm in}, C, u, \overline{wv}; v)(P(G_{\rm out}, C, v, \overline{uw}; w) - \overline{uw}).$$

Related articles

(Jackson, Yu, 2002) Every maximal planar graph with at most three separating triangles is Hamiltonian.

(Böhme, Harant, Tkáč, 1996) There exists a non-Hamiltonian maximal planar graph with exactly six separating triangles such that no two of them have an edge in common.

(Fujisawa, Zamfirescu, 2020) There exist infinitely many non-hamiltonian 1-tough maximal planar graphs with pairwise disjoint separating triangles.

Conjecture

Every maximal planar graph with at most five separating triangles is Hamiltonian.

Conjecture (Conflicting to the previous one)

There is a non-Hamiltonian maximal planar graph with at with four separating triangles.

Remark

(Garey, Johnson, Tarjan, 1976) The 3-connected and 3-regular planar Hamiltonian circuit problem is *NP*-complete

8. Probability

Random graphs

Different **random graph models** produce different probability distributions on graphs.

- $G_{n,p} = \{G : \ VG = [n]\}$: Every G occurs with probability

$$P(G) = p^{|EG|}(1-p)^{\binom{n}{2}-|EG|}.$$

  (Every possible edge occurs independently with probability $0 < p < 1$).

- $G_{n,m} = \{G : \ VG = [n], EG \in \binom{\binom{[n]}{2}}{m}\}$: Every G occurs with the same probability

$$P(G) = \frac{1}{\binom{\binom{n}{2}}{m}}.$$

Events H and $L(x)$ in $G_{n,p}$

For $u \in [n]$, define

$$\begin{aligned} H &:= \{G \in G_{n,p} \ : \ G \text{ has a Hamiltonian path}\}, \\ L(u) &:= \{G \in G_{n,p} \ : \ \text{Every path of maximum length in } G \text{ contains } u\}. \end{aligned}$$

Remark

$$P(H) = P\left(\bigcap_{u \in [n]} L(u)\right) = 1 - P\left(\bigcup_{u \in [n]} \overline{L(u)}\right).$$

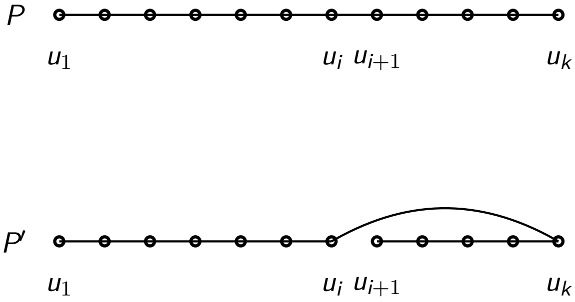
We want to choose suitable $p = p(n)$ such that

$$\lim_{n \to \infty} P(\bigcup_{u \in [n]} \overline{L(u)}) = 0.$$

(**With high probability** $G \in G_{n,p}$ is Hamiltonian.)

Pósa Rotations (Pósa, 1976)

Fix $G \in \overline{L(u)}$ and a path $P = u_1, u_2, \ldots, u_k$ of maximum length in G and that does not contain u. If $u_k u_i \in EG$ for some $1 < i < k-1$ then $P' = u_1, u_2, \ldots, u_i, u_k, u_{k-1}, \ldots u_{i+1}$ is a path of maximum length with a new endpoint u_{i+1}. We called P' a **rotation** of P.

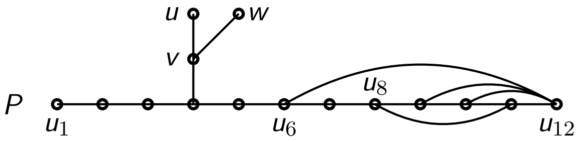


Endpoint set $X = X(G, u, P, u_1)$

Set

$$X = \{x \in VG : \ x \text{ is an endpoint of } P' \text{ obtained by a sequence of rotations starting from } P\},$$

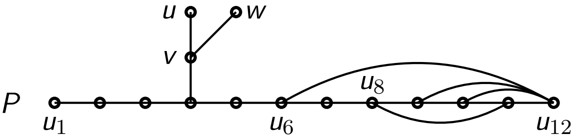
$$Y = VG - X - P_1(X) - \{u, u_1\}.$$



$$X = \{u_{12}, u_7, u_{10}, u_{11}, u_9\},$$
$$Y = \{v, w, u_2, u_3, u_4, u_5\}.$$

Lemma (Separating pair (X, Y))

$$
\begin{aligned}
&|Y| \geq n - 1 - 3|X|, \\
&G_1(u) \cap X = \emptyset, \\
&EG \cap XY = \emptyset.
\end{aligned}
$$



$$
\begin{aligned}
X &= \{u_{12}, u_7, u_{10}, u_{11}, u_9\}, \\
Y &= \{v, w, u_2, u_3, u_4, u_5\}.
\end{aligned}
$$

The probability $P(\bigcup_{u\in[n]} \overline{L(u)})$

Considering whether or not $k = |X| \le \frac{n}{4}$, and using $|Y| \ge n-1-3k$, $X \cap Y = \emptyset$, $EG \cap XY = \emptyset$ and $G_1(u) \cap X = \emptyset$, we have

$$P(\bigcup_{u\in[n]} \overline{L(u)}) \le \left[\sum_{k=1}^{n/4} \binom{n}{k} \binom{n-k}{n-1-3k} (1-p)^{k(n-1-3k)}\right] + n(1-p)^{n/4}$$

$$\to 0 \qquad \text{if } p = c\frac{\ln n}{n} \text{ with constant large } c$$

$$\text{(indeed if } p \ge \frac{\ln n + \ln\ln n + \omega(n)}{n}, \text{ where } \omega(n) \ll n).$$

Theorem (L. Pósa, 1974)

If $p = \frac{c\ln n}{n}$ for some large c then **with high probability** *a graph $G \in G_{n,p}$ contains a Hamiltonian path.*

Question

How far a traceable graph is Hamiltonian?

Theorem (L. Pósa, 1974)

If $p = \frac{c \ln n}{n}$ for some large c then with high probability a graph $G \in G_{n,p}$ is Hamiltonian.

Proof. (Sketch)

- With high probability G contains a Hamiltonian path $u_1, u_2, \ldots, u_n$.
- Then $P = u_2, u_3, \ldots, u_n$ is a Hamiltonian path in $G - u_1$ and we have its corresponding separating pair (X, Y).
- The probability of $|X| \leq n/4$ is still small, so with high probability $|X| \geq n/4$.
- Increasing probability $\ln n/n$ probability to p will make $G_1(u_1) \cap X \neq \emptyset$ with probability at least $1 - (1 - \frac{\ln n}{n})^{n/4} \to 1$.
- By the definition of X, for $w \in G_1(u_1) \cap X$, we have a Hamiltonian path P' in $G - u_1$ from u_1 to w.
- By adding edges wu_1 and u_1u_2 to P', we have a Hamiltonian cycle in G with high probability.

□

A similarity to $\delta(G)$

- **If** $p = \frac{\ln n + \ln\ln n + \omega(n)}{n}$ **then with high probability** $G \in G_{n,p}$ **has** $\delta(G) \geq 2$ **and is Hamiltonian.**
- **If** $p = \frac{\ln n + \ln\ln n - \omega(n)}{n}$ **then with high probability** $G \in G_{n,p}$ **has** $\delta(G) \leq 1$ **and is not Hamiltonian.**

Theorem (Komlós, Szemerédi, 1983)

If $m = n(\log n + \log\log n + c_n)/2$ **then**

$$\lim_{n\to\infty} Pr(G \in G_{n,m} \textbf{ is Hamiltonian}) = \begin{cases} 0 & c_n \to -\infty. \\ e^{-e^{-c}} & c_n \to c. \\ 1 & c_n \to \infty. \end{cases}$$

Random bipartite graphs

In the random balanced bipartite graph $G_{n,n,p}$ with order $2n$, each of the n^2 possible edges occurs independently with probability p.

Theorem (Frieze, 1985)

If $p = n(\log n + \log\log n + c_n)/n$ **then**

$$\lim_{n\to\infty} Pr(G_{n,n,p} \textbf{ is Hamiltonian}) = \begin{cases} 0 & c_n \to -\infty. \\ e^{-2e^{-c}} & c_n \to c. \\ 1 & c_n \to \infty. \end{cases}$$

9. Eigenvalues

Eigenvalues of graphs

Let G be a graph with $VG = [n]$.

- The **adjacency matrix** $A(G)$ is a matrix with entries $A_{ij} = 1$ if $ij \in EG$ and 0 elsewhere.
- The **eigenvalues of** G is defined to be the eigenvalues of $A(G)$.
- The largest eigenvalue of G, denoted by $\rho(G)$, is called the **spectral radius** of G.

Remark

It is a well-known result that

$$\delta(G) \leq d_{\mathrm{average}}(G) \leq \rho(G) \leq \Delta(G).$$

Hence one will expect the study of Hamiltonian graphs by assumptions on $\rho(G)$ instead on $\delta(G)$ or $\Delta(G)$.

Eigenvalues of graphs

The following two step method is widely use in Hamiltonian theory.

1. Large spectral radius (or small spectral radius of its complement) implies large number of edges.
2. Large number of edges ensures a Hamiltonian cycle.

Combining known results

- If $|EG| > \binom{n-1}{2}$, then G is Hamiltonian unless $G = K_{n-1} + e$.
- Stanley's inequality: $\rho(G) \le -\frac{1}{2} + \sqrt{2|EG| + \frac{1}{4}}$.

Theorem (Fiedler, Nikiforov, 2010)

If $\rho(G) \ge n-2$, then G is Hamiltonian unless $G = K_{n-1} + e$.

Proof.

By the Stanley's inequality, if $\rho(G) \ge n-2$, we have $2|EG| > \left(n - \frac{3}{2}\right)^2 - \frac{1}{4} = n^2 - 3n + 2 > 2\binom{n-1}{2}$, which implies that G is Hamiltonian unless $G = K_{n-1} + e$. □

Theorem (Fiedler, Nikiforov, 2010)

Let G be a graph of order n and $\overline{G}$ be the complement of G.

- If $\rho(\overline{G}) \leq \sqrt{n-1}$, then G is traceable unless $G = K_{n-1} \cup K_1$.
- If $\rho(\overline{G}) \leq \sqrt{n-2}$, then G is Hamiltonian unless $G = K_{n-1} + e$.

Theorem (Lu, Liu, Tian, 2012)

Let G be a balanced bipartite graph with $2n$ vertices, and G^* be the quasi-complement of G, i.e. $V(G^*) = V(G)$ and $E(G^*) = \{xy : x \in X, y \in Y, xy \notin E(G)\}$. If $\rho(G^*) \leq \sqrt{n-1}$, then G is Hamiltonian unless $G = K_{n,n-1} + e$.

Theorem (Nikiforov, 2016)

Let G be a graph of order n with $d_{\min}(G) \geq k$.
If $k \geq 2$, $n \geq k^3 + k + 4$ and $\rho(G) \geq n - k - 1$, then G is Hamiltonian unless $G = K_1 \vee (K_{n-k-1} + K_k)$ or $G = K_k \vee (K_{n-2k} + kK_1)$.

Theorem (Ge, Ning, 2020)

Denote by $K_{n,n} - K_{k,n-k}$ the graph obtained from $K_{n,n}$ by deleting $K_{k,n-k}$, where $n > 2k$.

Let $k \geq 1$. Let G be a balanced bipartite graph of order $2n$ with $d_{\min}(G) \geq k$, where $n \geq k^3 + 2k + 4$.

- **If G is a proper subgraph of $K_{n,n} - K_{k,n-k}$, then $\rho(G) < \sqrt{n(n-k)}$.**
- **If $\rho(G) \geq \sqrt{n(n-k)}$, then G is Hamiltonian unless $K_{n,n} - K_{k,n-k}$.**

Theorem (Brouwer and Haemers, 2020)

Let G be a k-regular k-connected graph with n vertices and smallest eigenvalue s. If G is not the Petersen graph and $k > 1$ and $-ns \leq (k+1)(k-s)$ then G is Hamiltonian.

Proof. (Sketch).

- A k-regular graph G with smallest eigenvalue s satisfies $\alpha(G) \leq -ns/(k-s) \leq k+1$.
- If $\alpha(G) \leq k$ then G is Hamiltonian by Chvátal-Erdős Theorem.
- If $\alpha(G) = k+1$ then $-ns/(k-s) = k+1$ and Petersen graph is the only such graph.

□

Signless Laplacian eigenvalues of graphs

Let G be a graph with $VG = [n]$ and adjacency matrix $A = A(G)$.

- The **signless Laplacian matrix** $Q(G)$ of G is the matrix $A + D$, where D is a diagonal matrix with degrees on the diagonals.
- The **signless Laplacian eigenvalues of** G is defined to be the eigenvalues of the signless Laplacian matrix $Q(G)$.
- The largest signless Laplacian eigenvalue of G, denoted by $q(G)$, is called the **signless Laplacian spectral radius** of G.

Related eigenvalues

Let G be a graph of order $n \geq 3$. If $q(G) > 2n-4$, then G is Hamiltonian unless $G = K_2 \vee 3K_1$ or $G = K_1 \vee (K_{n-2} + K_1)$.

G.D. Yu, Y.Z. Fan, Spectral conditions for a graph to be Hamilton-connected, *Appl. Mech. Mater.* 336–338 (2013) 2329-2334.

Let G be a graph of order $n \geq 4$ with $\delta(G) \geq 2$. If $q(G) > 2n-5+\frac{3}{n-1}$, then G is Hamiltonian unless $G = K_3 \vee 4K_1$ or $G = K_2 \vee 3K_1$.

R.F. Liu, W.C. Shiu, J. Xue, Sufficient spectral conditions on Hamiltonian and traceable graphs, *Linear Algebra Appl.* 467 (2015) 254-266.

Let k be an integer and G be a graph of order n with $\delta(G) \geq k \geq 1$. If $q(G) > q(K_k \vee (K_{n-2k} + kK_1))$, where $n > \max\{6k+5, (3k^2+5k+4)/2\}$, then G is Hamiltonian unless $G = K_k \vee (K_{n-2k} + kK_1)$.

B.L. Li, B. Ning, Spectral analogues of Erdós′ and Moon–Moser′ s theorems on Hamilton cycles, *Linear Multilinear Algebra* 64 (2016) 2252-2269.

10. Others

Hamiltonian-like properties

- A graph is **Hamiltonian-connected** if every two vertices of it are connected by a Hamiltonian path.
- A graph is **pancyclic** if for all $\ell \in \{3, 4, \ldots, |VG|\}$ it contains a cycle of length ℓ.
- A graph is **even-pancyclic** if for all even $\ell \in \{4, \ldots, |VG|\}$ it contains a cycle of length ℓ.