On degrees and average 2-degrees in graphs

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Degree, average 2-degree, degree pair

Let G be a simple connected graph with vertex set $VG = \{1, 2, \ldots, n\}$ and edge set EG. Let d_i and m_i be the **degree** and **average 2-degree** of the vertex $i \in VG$ respectively, define as follows.

$$d_i := |G_1(i)|,$$
$$m_i := \frac{1}{d_i} \sum_{ji \in EG} d_j,$$

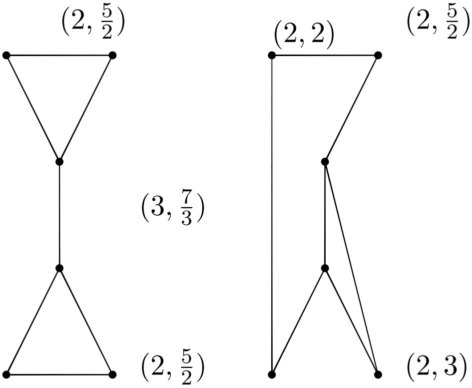
where $G_1(i)$ means the set $\{j \in VG \mid ji \in EG\}$ of neighbors of i.

The sequence of pairs

$$\{(d_i, m_i)\}_{i \in VG}$$

of G are called sequence of **degree pairs** of G.

The degree pairs (d_i, m_i)



Two graphs with their degree pairs (d_i, m_i).

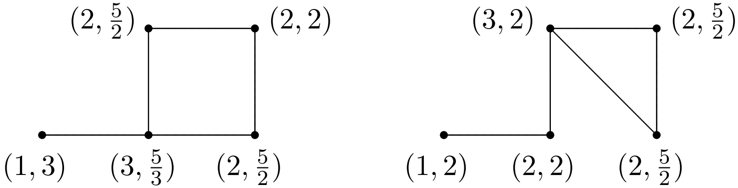
Generating the degree pair

$$\begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{pmatrix} = A \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} m_1 \\ m_2 \\ \vdots \\ m_n \end{pmatrix} = \begin{pmatrix} d_1^{-1} & & & \\ & d_2^{-1} & & \\ & & \ddots & \\ & & & d_n^{-1} \end{pmatrix} A \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{pmatrix},$$

where A is the adjacency matrix of G.

Determine a graph from degree pairs

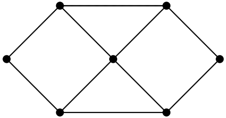


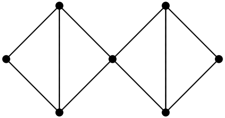
Two graphs uniquely determined by their sequence of degree pairs.

We will show that

$$\max d_i m_i = 5 \geq 5 = n \quad \Rightarrow \quad \exists C_3 \text{ or } C_4.$$

Two graphs with the same degree pairs I

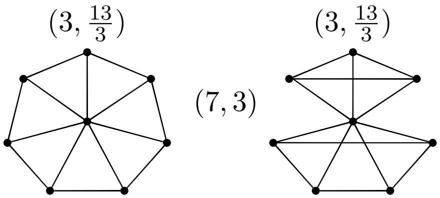




Two graphs with the same sequence of degree pairs

$$(2,3),(3,3),(3,3),(4,3),(3,3),(3,3),(2,3).$$

Two graphs with the same degree pairs II



Two graphs with the same degree pairs.

A feasible condition

$\sum_{i \in VG} d_i m_i = \sum_{i \in VG} d_i^2.$

Proof.

$$\sum_{i \in VG} d_i m_i = \sum_{i \in VG} d_i \frac{\sum_{ji \in EG} d_j}{d_i} = \sum_{j \in VG} \sum_{ij \in EG} d_j = \sum_{j \in VG} d_j^2.$$

□

Another feasible condition

There are even number of odd values $d_i m_i$ among $i \in VG$.

Proof.

Since $\sum_{i \in VG} d_i$ is even, there are even number of odd d_i, and so does d_i^2. Hence $\sum_{i \in VG} d_i m_i = \sum_{i \in VG} d_i^2$ is even. □

Corollary

$\sum_{i \in VG} m_i^2 \geq \sum_{i \in VG} d_i^2$ with equality iff $m_i = d_i = k$ for all i.

Proof.

$$(\sum_{i \in VG} d_i^2)(\sum_{i \in VG} m_i^2) \geq (\sum_{i \in VG} d_i m_i)^2 = (\sum_{i \in VG} d_i^2)^2$$

and equality iff $m_i = cd_i$, where $c = 1$ by the above lemma. This is also equivalent to that all neighbors of a vertex of minimum degree k also have degree k. □

Proposition

If $\max_{i \in VG} d_i m_i \geq n$ then the graph has girth at most 4.

Proof.

If the graph has girth at least 5 then

$$n - 1 = |VG| - 1 \geq |G_1(i) \cup G_2(i)| = d_i m_i.$$

for any $i \in VG$. □

In general, $d_i m_i \geq |G_1(i)| + |G_2(i)|$, and there are at least $(d_i m_i - n)/2$ triangles based on the vertex i.

Erdős-Gallai Theorem

A sequence of nonnegative integers $d_1 \geq d_2 \geq \cdots \geq d_n$ can be represented as the degree sequence of a finite simple graph on n vertices if and only if

$$\sum_{i=1}^{n} d_i$$

is even and

$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min\{d_i, k\} \qquad (1 \leq k \leq n).$$

□

An analogue of the Erdős-Gallai Theorem

If a sequence of ordered pairs of positive real numbers $(d_1, m_1) \succeq (d_2, m_2) \succeq \cdots \succeq (d_n, m_n)$ in dictionary order is a sequence of degree pairs of a simple graph G of order n, then

(i) d_i and $d_i m_i$ are both positive integers for $i = 1, 2, \ldots, n$;

(ii) $d_i m_i \le \sum_{j=1}^{d_i+1} d_j - d_{\min\{d_i+1, i\}}$ for $i = 1, 2, \ldots, n$;

(iii) $d_i m_i \ge \sum_{j=n-d_i}^{n} d_j - d_{\max\{n-d_i, i\}}$ for $i = 1, 2, \ldots, n$;

(iv) $\sum_{i=1}^{n} d_i m_i = \sum_{i=1}^{n} d_i^2$;

(v) $\sum_{i=1}^{n} d_i$ is even (and so does $\sum_{i=1}^{n} d_i m_i$);

(vi) $\sum_{i=1}^{k} d_i \le k(k-1) + \sum_{i=k+1}^{n} \min\{d_i, k\}$ for $k = 1, 2, \ldots, n$; and

(vii) $\sum_{i=1}^{k} d_i m_i \le \sum_{i=1}^{k} d_i \min\{d_i, k-1\} + \sum_{i=k+1}^{n} d_i \min\{d_i, k\}$ for $k = 1, 2, \ldots, n$.

□

However, the sufficiency is not completed.

The square graph G^2 and its independent number

Let G^2 be the **square** of G, i.e.

$$V(G^2) = V(G) \quad \text{and} \quad E(G^2) = \{ij \mid d(i,j) = 1 \text{ or } 2\},$$

where $d(i,j)$ denotes the distance between vertices i and j in G.

The **independent number** $\alpha(G)$ of a graph G is the maximum size of a vertex subset consisting of pairwise nonadjacent vertices.

Proposition

Let G be a simple graph with no isolated vertices and of degree pair sequence $(d_i, m_i)_{i=1}^n$. Then the independence number of the square G^2 of G satisfies

$$\alpha(G^2) \geq \sum_{i=1}^{n} \frac{1}{1 + d_i m_i}.$$

□

The proof is using probabilistic method.

Harmonic graphs

A simple graph G with no isolated vertices is k-**harmonic** if its average 2-degree $m_i = k$ for every $i \in V(G)$.

From the definition of a k-harmonic graph, k is a rational number, but indeed k is an integer.

A. Dress, I. Gutman, The number of walks in a graph, *Appl. Math. Lett.* 16 (2003) 797-801.

Proposition

A k-harmonic graph on n vertices has at most $nk/2$ edges, and the maximum is obtained if and only if the graph is regular.

Proof.

Let G be a k-harmonic graph with degree pairs $\{(d_i, m_i)\}_{i=1}^{n}$, where $m_i = k$. By Cauchy's inequality,

$$2k|E(G)| = \sum_{i=1}^{n} d_i m_i = \sum_{i=1}^{n} d_i^2 \geq \frac{(\sum_{i=1}^{n} d_i)^2}{n} = \frac{4|E(G)|^2}{n},$$

we have $|E(G)| \leq nk/2$ and the equality is obtained if and only if d_i is a constant. □

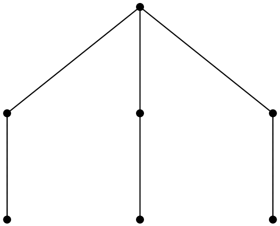
Pseudo regular graph

A graph is **pseudo k-regular** if it is k-harmonic but not k-regular.

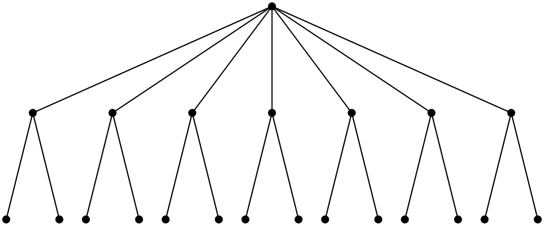
The tree T_k

For each $k \geq 2$, let T_k be the tree of order $k^3 - k^2 + k + 1$ whose root has degree $k^2 - k + 1$ and each neighbor of the root has $k - 1$ children as leafs.

T_2:



T_3:



Pseudo regular trees

For each k, a pseudo k-regular tree is the tree T_k. □

The proof is also by A. Dress and I. Gutman.

Proposition

Let G be a pseudo k-regular graph of order n with a vertex i of degree $d_i \geq k^2 - 3k + 5$. Then

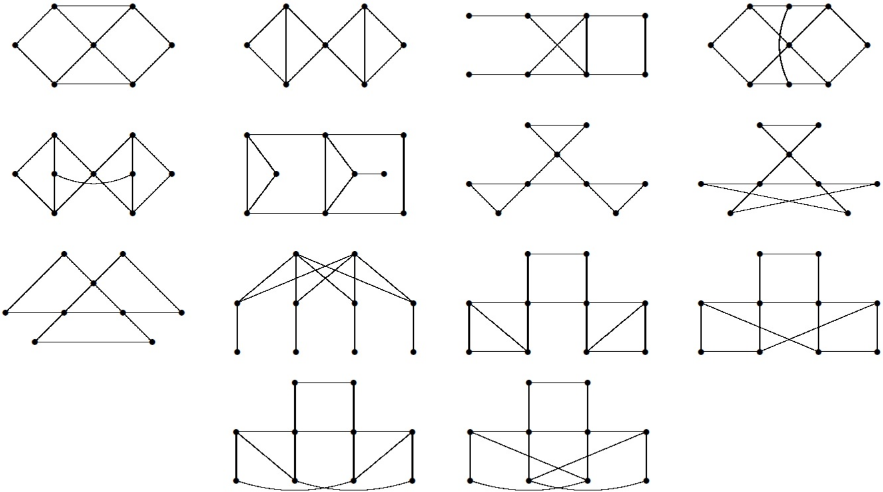
(i) every neighbor j of i has degree $d_j = k$, and

(ii) the order of G is at least

$$f(k) := \left\lceil \frac{5k^4 - 31k^3 + 94k^2 - 140k + 100}{k^2} \right\rceil .$$

□

Pseudo 3-regular graph of order at most 10



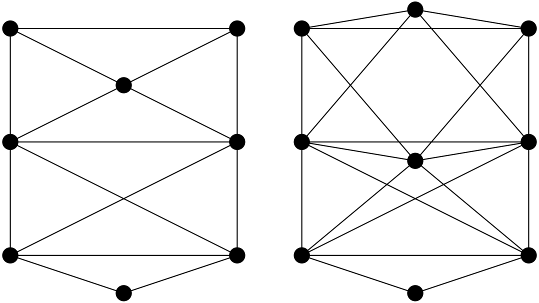
The number $N(k)$

Let $N(k)$ denote the minimum number of vertices in a pseudo k-regular graph.

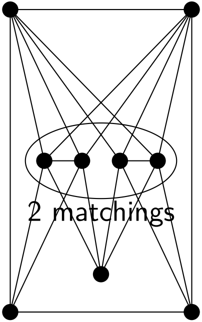
$N(k)$ for $k \leq 7$

k	$N(k)$	Possible degree sequences
2	7	3, 2, 2, 2, 1, 1, 1
3	7	4, 3, 3, 3, 3, 2, 2
4	8	5, 5, 4, 4, 4, 3, 3, 2
5	9	6, 6, 6, 5, 5, 4, 4, 4, 2
		6, 6, 5, 5, 5, 5, 4, 4, 4
6	11	8, 6, 6, 6, 6, 6, 6, 6, 6, 4, 4
7	11	8, 8, 8, 7, 7, 7, 7, 6, 6, 6, 6

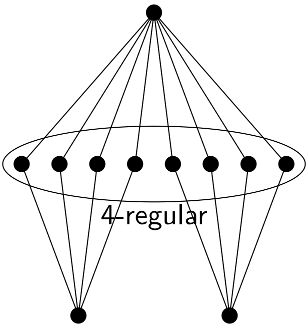
Minimal pseudo 4-regular graphs



Minimal pseudo 5-regular graphs



Minimal pseudo 6-regular graphs



Proposition

For $k = 3, 4$ there exists a pseudo k-regular graph on n vertices for every $n \geq N(k)$. □

The proof is by inductive constructions.

A lower bound of $N(k)$

For each positive integer $k \geq 2$, we have

$$N(k) \geq k + 3.$$

□

The proof uses counting arguments to disagree the existence of a pseudo k-regular graph of order $k + 2$.

An upper bound of $N(k)$

For each positive integer $k \geq 3$, we have

$$N(k) \leq \begin{cases} k+4 & \text{if } k \text{ is odd;} \\ k+6 & \text{if } k \text{ is even.} \end{cases}$$

□

The proof is by direct construction.

Open problems

1. Give a necessary and sufficient condition for a sequence of positive integers that can be the degree sequence of a finite pseudo k-regular graph with no isolated vertices for every positive integer k.
2. Give a necessary and sufficient condition for a sequence of pairs of positive real numbers that is graphic on a finite simple graph with no isolated vertices.
3. Is $N(k)$ non-decreasing? It is true for $k \leq 7$.
4. For each positive integer $k \geq 8$, determine $N(k)$, and find all pseudo k-regular graphs of order $N(k)$.
5. Does there always exist a pseudo k-regular graph on n vertices for any positive integers $k \geq 5$ and $n \geq N(k)$?
6. Give a function $g(n, k)$ for positive integers n, k that maps to the number of pseudo k-regular graphs of order n up to isomorphism. Currently we have that $g(n, 3) = 0$ for $n \leq 6$ and $g(7, 3) = 2$; $g(n, 4) = 0$ for $n \leq 7$ and $g(8, 4) = 1$; $g(n, 5) = 0$ for $n \leq 8$ and $g(9, 5) = 3$; $g(n, 6) = 0$ for $n \leq 10$; $g(n, 7) = 0$ for $n \leq 10$ and $g(11, 7) = 5$.

Thank you for your attention.