Spectral Radius and Degree Sequence of a Bipartite Graph

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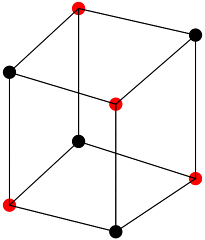
The graphs G we considered here are always bipartite.

The adjacency matrix $A = (a_{ij})$ of G is a binary square matrix with rows and columns indexed by the vertex set VG of G such that for any $i, j \in VG$, $a_{ij} = 1$ if i, j are adjacent in G.

G 

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

The eigenvalues of $A = A(G)$ give clues of the structure of G.

Can you hear the shape of a drum?

Motivation

Let $\rho(G)$ denote the largest eigenvalue of $A = A(G)$. $\rho(G)$ is called the spectral radius of G.

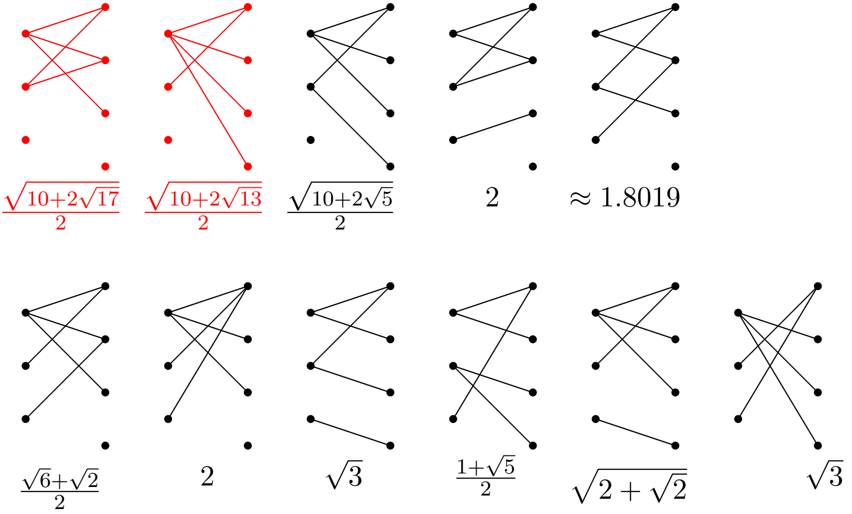
It is well-known that

$$\rho(G) \leq \sqrt{e},$$

where e is the number of edges in G. Moreover if G is connected the the above equality holds iff G is complete bipartite.

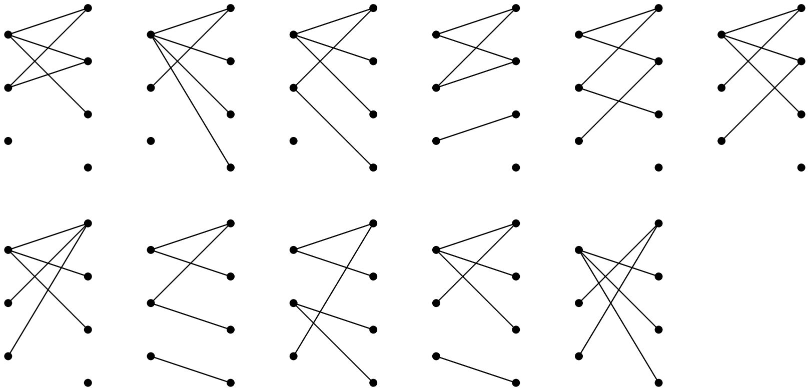
We will consider this problem on percolated bipartite graphs.

Example ($\rho(G) < \sqrt{e} = \sqrt{5}$)



Definition

Let $\mathcal{K}(p, q, e)$ denote the family of e-edge subgraphs of the complete bipartite graph $K_{p,q}$ with p and q vertices in the partite sets (See $\mathcal{K}(3, 4, 5)$ below).



Conjecture 1

In 2008, Amitava Bhattacharya, Shmuel Friedland, and Uri N. Peled proposed the following conjecture.

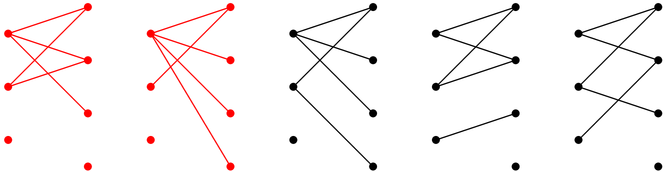
Conjecture 1

Let $2 \leq p \leq q$, $1 < e < pq$ be integers. An extremal graph that solves

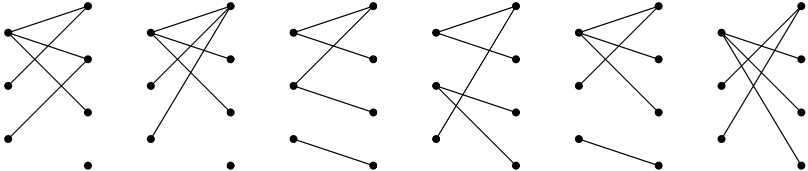
$$\max_{G \in \mathcal{K}(p,q,e)} \rho(G)$$

is obtained from a complete bipartite graph by adding one vertex and a corresponding number of edges.

Example $\mathcal{K}(3,4,5)$



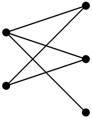
$\frac{\sqrt{10+2\sqrt{17}}}{2}$ $\quad$ $\frac{\sqrt{10+2\sqrt{13}}}{2}$ $\quad$ $\frac{\sqrt{10+2\sqrt{5}}}{2}$ $\quad$ 2 $\quad$ ≈ 1.8019



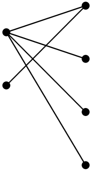
$\frac{\sqrt{6}+\sqrt{2}}{2}$ $\quad$ 2 $\quad$ $\sqrt{3}$ $\quad$ $\frac{1+\sqrt{5}}{2}$ $\quad$ $\sqrt{2+\sqrt{2}}$ $\quad$ $\sqrt{3}$

Two different classes of extremal graphs

Let $K_{p,q}^{[e]}$ (resp. $K_{p,q}^{\{e\}}$) denote the graph which is obtained from the complete graph $K_{p,q}$ by deleting $pq - e$ edges incident on a common vertex in the partite set whose order is at most (resp. at least) the order of other partite. Thus $K_{p,p}^{[e]} = K_{p,p}^{\{e\}}$.



$K_{2,3}^{\{5\}}$



$K_{2,4}^{[5]}$

Conjecture 2

In 2010, Yi-Fan Chen, Hung-Lin Fu, In-Jae Kim, Eryn Stehr and Brendon Watts proposed the following conjecture as a refinement of Conjecture 1.

Conjecture 2

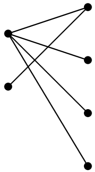
Suppose $pq - e < \min(p, q)$. Then for $G \in \mathcal{K}(p, q, e)$,

$$\rho(G) \leq \rho\left(K_{p,q}^{\{e\}}\right).$$

The condition $pq - e < \min(p, q)$ ensures that G is connected.

Notations

Assume that G has degree sequences $d_1 \geq d_2 \geq \cdots \geq d_p$ and $d'_1 \geq d'_2 \geq \cdots \geq d'_q$ according to the two parts respectively.



$K_{2,4}^{[5]}$

$$d_1 = 4, d_2 = 1$$
$$d'_1 = 2, d'_2 = 1, d'_3 = 1, d'_4 = 1.$$

The number $\phi_{s,t}$

For $1 \leq s \leq p$ and $1 \leq t \leq q$, set

$$\phi_{s,t} := \sqrt{\frac{X_{s,t} + \sqrt{X_{s,t}^2 - 4Y_{s,t}}}{2}},$$

where

$$X_{s,t} = d_s d'_t + \sum_{i=1}^{s-1}(d_i - d_s) + \sum_{j=1}^{t-1}(d'_j - d'_t),$$

$$Y_{s,t} = \sum_{i=1}^{s-1}(d_i - d_s) \cdot \sum_{j=1}^{t-1}(d'_j - d'_t).$$

Bipartite Sum

Let H, H' be two bipartite graphs with given ordered bipartitions $VH = X \cup Y$ and $VH' = X' \cup Y'$. The the bipartite sum $H + H'$ of H and H' with respect to the given ordered bipartitions $VH = X \cup Y$ and $VH' = X' \cup Y'$ is the graph obtained from H and H' and adding an edge between x and y for each pair $(x, y) \in X \times Y' \cup X' \times Y$.

$$\begin{aligned} K_{1,2} + K_{2,3} =& K_{3,5} \\ C_6 + C_6 =& K_{6,6} - \text{perfect matching.} \end{aligned}$$

Theorem 1

For $1 \leq s \leq p$ and $1 \leq t \leq q$,

$$\rho(G) \leq \phi_{s,t}.$$

Moreover, if G is connected then the above equality holds iff there exists nonnegative integers $s' < s$ and $t' < t$ and a bipartite biregular graph H of bipartition orders $p - s'$ and $q - t'$ respectively such that $G = K_{s',t'} + H$.

Revisit $\phi_{s,t}$

Recall

$$\phi_{s,t} := \sqrt{\frac{X_{s,t} + \sqrt{X_{s,t}^2 - 4Y_{s,t}}}{2}},$$

where

$$X_{s,t} = d_s d'_t + \sum_{i=1}^{s-1}(d_i - d_s) + \sum_{j=1}^{t-1}(d'_j - d'_t),$$

$$Y_{s,t} = \sum_{i=1}^{s-1}(d_i - d_s) \cdot \sum_{j=1}^{t-1}(d'_j - d'_t).$$

Remark

(i) $\phi_{1,1} = \sqrt{d_1 d'_1}$ (a known result).

(ii) $\phi_{p,q} = \sqrt{\frac{2e+(q-d_p)(p-d'_q)-pq+\sqrt{(pd_p+qd'_q-d_pd_q)^2-4(pq-e)d_pd'_q}}{2}}$ (too complicate!).

The case $d_p = d'_q = 0$

$$
\begin{aligned}
&\phi_{p,q} \\
=&\sqrt{\frac{2e + (q - d_p)(p - d'_q) - pq + \sqrt{(pd_p + qd'_q - d_p d'_q)^2 - 4(pq - e)d_p d'_q}}{2}} \\
=&\sqrt{e}.
\end{aligned}
$$

It is the known result mentioned in the beginning.

It turns out that if $d'_q < p$ then

$$\frac{\partial \phi_{p,q}(d_p, d'_q)}{\partial d_p} < 0,$$

and if $d_p < q$ then

$$\frac{\partial \phi_{p,q}(d_p, d'_q)}{\partial d'_q} < 0.$$

Almost complete bipartite

A graph is almost $K_{p,q}$ if it is a connected graph obtained from $L_{p,q}$ by deleting an edge xy and some edges incidence with x or with y.

Hence an almost $K_{p,q}$ graph satisfies

$$(q - d_p) + (p - d'_q) = (pq - e) + 1.$$

Since there are only $pq - e$ almost $K_{p,q}$ graphs with a prescribed number e of edges, and their spectral radius $\phi_{p,q}(d_p, d_q)$ can be simplified by the formula $(q - d_p) + (p - d'_q) = pq - e + 1$, the following lemma is obtained by algebraic computation.

Lemma

If G is almost $K_{p,q}$. Then

$$\phi(G) \leq \phi(k_{p,q}^{\{e\}}).$$

We prove Conjecture 2 without any assumption.

Theorem 2

For $G \in \mathcal{K}(p, q, e)$,

$$\rho(G) \leq \rho\left(K_{s,t}^{\{e\}}\right)$$

for some positive integers $s \leq p$ and $t \leq q$.

Proof of Theorem 2.

Induction on $p+q$, so we can assume $d_p, d'_q > 0$.

Step 1

Indeed if for those $G \in \mathcal{K}(p, q, e)$ with $d_p = 0$, we might treat as $G \in \mathcal{K}(p-1, q, e)$, and choose $s' \leq p-1$ and $t' \leq q$ such that

$$\rho(G) \leq \rho\left(K_{s',t'}^{\{e\}}\right).$$

Similarly choose $s'' \leq p, t'' \leq q-1$ for $d'_q = 0$.

Step 2

By using the property that

$$\frac{\partial \phi_{p,q}(d_p, d'_q)}{d_p} < 0,$$

we can move the edge pq' to become a new edge iq' (so d_p is decreased) to increase the spectral radius if iq' is not an edge in the beginning for $1 \leq i \leq p-1$. We also can move an edge pj' to a new edge ik' provided that ik' is not an edge for $1 \leq i \leq p-1$ and $1 \leq k \leq q-1$. Similarly for the other part.

Hence we can assume G is almost $K_{p,q}$.

Step 3

By Lemma, we can pick $s''' \leq p$ and $t''' \leq q$ such that

$$\rho(G) \leq \rho\left(K^{\{e\}}_{s''',t''''}\right).$$

Let $(s,t) \in \{(s',t'),(s'',t''),(s''',t''')\}$ such that $\rho\left(K^{\{e\}}_{s,t}\right)$ is the maximum. $\square$

In order to prove Theorem 1, we need to quote a theorem.

Perron-Frobenius Theorem

If M is a nonnegative $n \times n$ matrix with largest eigenvalue $\rho(M)$ and row-sums $r_1, r_2, \ldots, r_n$, then

$$\rho(M) \leq \max_{1 \leq i \leq n} r_i.$$

Furthermore, if M is irreducible then the equality holds if and only if the row-sums of M are all equal.

Idea of the proof of Theorem 1

Let

$$A = \begin{pmatrix} 0 & B_{p\times q} \\ B^T_{q\times p} & 0 \end{pmatrix}$$

be the adjacency matrix of G. Then

$$A^2 = \begin{pmatrix} BB^T & 0 \\ 0 & B^TB \end{pmatrix}$$

has maximal eigenvalue $\rho(G)^2$.

Applying Perron-Frobenius Theorem to

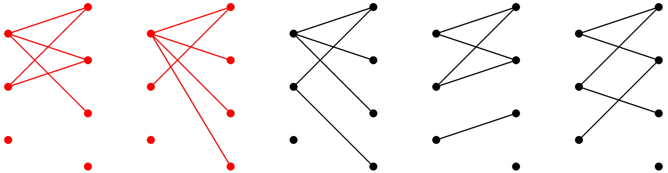
$$U^{-1}A^2U,$$

where

$$U = diag(\underbrace{x_1, \cdots, x_{s-1}, 1, \cdots, 1}_{p}, \underbrace{x'_1, \cdots, x'_{t-1}, 1, \cdots, 1}_{q})$$

with carefully chosen of variables x_i and x'_j. □

Example ($\rho(G) < \sqrt{e} = \sqrt{5}$)



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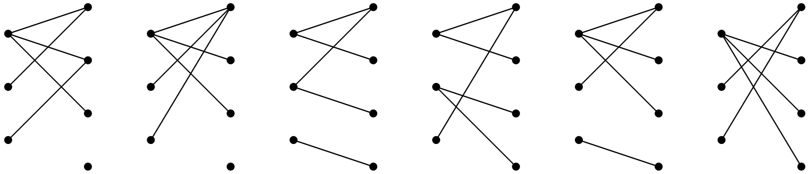

$\frac{\sqrt{6}+\sqrt{2}}{2}$ 2 $\sqrt{3}$ $\frac{1+\sqrt{5}}{2}$ $\sqrt{2+\sqrt{2}}$ $\sqrt{3}$

Example for $e = 11$

$$11 = 2 \times 5 + 1 = 3 \times 3 + 2,$$

$$\rho(K_{2,6}^{\{11\}}) = \rho(K_{2,6}^{[11]}) = \sqrt{\frac{11 + \sqrt{101}}{2}},$$

$$\rho(K_{3,4}^{\{11\}}) = \rho(K_{3,4}^{[11]}) = \sqrt{\frac{11 + \sqrt{97}}{2}}.$$

Example for $e = 13$

$$13 = 2 \times 6 + 1 = 3 \times 4 + 1.$$

$$\rho(K_{2,7}^{\{13\}}) = \frac{\sqrt{26 + 2\sqrt{145}}}{2},$$
$$\rho(K_{3,5}^{\{13\}}) = \frac{\sqrt{13} + \sqrt{137}}{2}.$$

Example for $e = 19$

$$19 = 2 \times 9 + 1 = 3 \times 6 + 1 = 4 \times 4 + 3.$$

$$\rho(K_{2,10}^{\{19\}}) = \sqrt{\frac{19 + \sqrt{361}}{2}},$$
$$\rho(K_{3,7}^{\{19\}}) = \sqrt{\frac{19 + \sqrt{313}}{2}},$$
$$\rho(K_{4,5}^{\{19\}}) = \sqrt{\frac{19 + \sqrt{313}}{2}}.$$

Conjecture

The bipartite graph whose spectral radius attains

$$\max_{G \in \mathcal{K}(p,q,e)} \rho(G)$$

is $K_{s,t}^{\{e\}}$, where $s + t$ attains the maximal value

$$\max\{s' + t' \mid s't' - e = \min\{s''t'' - e\}\},$$

among $1 \leq s', s'' \leq p$, $1 \leq t', t'' \leq q$, and $e \leq s't', s''t''$.

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Thanks for your attention.