SECOND INDIA-TAIWAN CONFERENCE ON DISCRETE MATHEMATICS

September 8 to 11, 2011

EXTENDED ABSTRACTS

Organized by

Department of Mathematics

Amrita School of Engineering

AMRITA VISHWA VIDYAPEETHAM

Coimbatore - 641 112, India.

SECOND INDIA-TAIWAN CONFERENCE

on

DISCRETE MATHEMATICS

September 8th - 11th, 2011

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DEPARTMENT OF MATHEMATICS
AMRITA UNIVERSITY

The Department of Mathematics is one of the vibrant departments of Amrita Vishwa Vidyapeetham. Currently, there are twenty four faculty members in the department. The department offers various courses in Mathematics for the B.Tech, M.Tech and M.C.A students and also runs a Ph.D. program in which research is going on in the areas: Graph Theory, Linear Algebra, Network on Chip (NoC), Statistical Pattern Classifications, Six Sigma, Lie Algebra and Lie Groups, Fluid Dynamics, Wavelet Transforms, Statistics, Finite Element Analysis, Fractal Theory and Cryptography. Two senior faculty members have written books on Mathematics for Engineers and also in Probability and Statistics.

The department has organized two International Conferences, viz., an International Conference on Discrete Mathematics and its Applications during December 9 -11, 2004 and an International Conference on Graph Theory and its Applications during December 11 - 13, 2008. It also organized a National conference on Quality Improvement Concepts in Higher Educational Institutions during December 10 - 11, 2009.

To provide a platform to the students to manifest their mathematical talents and to kindle their mathematical quest, the department has a student forum by name "ANANTHAM", which organizes workshops, quiz programs, puzzle corners and other activities for students. It also organizes inter-collegiate competitions. Recently, the forum conducted an International on-line quiz contest "Amrita Mathematics Olympiad" which was well received all over the world.

THIS CONFERENCE

The conduct of India-Taiwan conferences in Discrete Mathematics is an offshoot of the move originally mooted by Professor Xuding Zhu at the time of International Conference on Discrete Mathematics held at I.I.Sc., Bangalore in 2006 (ICDM 2006) and subsequently supported by Professor Gerard Jennhwa Chang and Ko-wei Lih.

The first India-Taiwan conference was held at the National Taiwan University, Taipei, Taiwan, during November 9-12, 2009. 15 Indian and 38 Taiwanese discrete mathematicians participated in this conference. The present conference is the second one in the series.

In this conference, about 120 scholars, consisting of senior professors and other researchers from both India and Taiwan are expected to participate. The main goal of this conference is to generate joint projects by groups of the two countries by identifying common areas of research interest and problems in discrete mathematics.

(With inputs from K. Somasundaram and R. Balakrishnan)

Schedule

DAY 1 - September 8, 2011

Venue: e-Lecture Hall

08.45 - 09.30	Registration
09.30 a.m.	Inauguration - Welcome address: R.Balakrishnan
	Session 1 Chair: E. Sampathkumar
09.45 - 10.30	Plenary Talk - Xuding Zhu
10.30 - 11.00	Tao-Ming Wang
11.00 - 11.30	TEA
	Session 2 Chair: G. Ravindra
11.30 - 12.00	L. Pushpa Latha
12.00 - 12.30	Hong-Gwa Yeh
12.30 - 02.00	LUNCH
	Session 3 Chair: A. Muthusamy
02.00 - 02.30	Hsin-Hao Lai
02.30 - 03.00	Tsai-Lien Wong
03.00 - 03.30	V. Swaminathan
03.30 - 04.00	S. Monikandan
04.00 - 04.30	TEA
04.30 p.m.	Group Discussions
	Co-ordinators:
	1. G. Ravindra - Graph Coloring
	2. S. Arumugam - Domination Theory
	3. Tao-Ming Wang - Graph Labelings
07.30 p.m.	DINNER

DAY 2 - September 9, 2011

Venue: e-Lecture Hall

	Session 4 Chair: B. D. Acharya
09.15 a.m. - 10.00	Plenary Talk - Sunil Chandran
10.00 - 10.30	S.M. Hedge
10.30 - 11.00	Chin-Hung Yen
11.00 - 11.30	TEA
	Session 5 Chair: H.P. Patil
11.30 - 12.00 noon	Chiuyuan Chen
12.00 - 12.30	D.J. Guan
12.30 - 12.45	GROUP PHOTO
12.45 - 02.00	LUNCH
	Session 6 Chair: R. Sampathkumar
02.00 - 02.30	A.P. Santhakumaran
02.30 - 03.00	Chin-Mei Fu
03.00 - 03.30	Mukti Acharya
03.30 - 04.00	N. S. Narayanaswamy
04.00 - 04.30	TEA
04.30 - 06.00	Group Discussions
04.30 - 06.00	Indian Dance by the Troupe of ARADHANA Dance School, Chennai Venue: Amriteswari Auditorium, Amrita University
08.00 p.m.	Conference Banquet Venue: Guest House Lawn

DAY 3 - September 10, 2011

Venue: e-Lecture Hall

	Session 7 Chair: P. Paulraja
09.15 a.m. - 10.00	Plenary Talk - Gerard Jennhwa Chang
10.00 - 10.30	G. Indulal
10.30 - 11.00	Bit-Shun Tam
11.00 - 11.30	TEA
	Session 8 Chair: T. Tamizh Chelvam
11.30 - 12.00 noon	G. R. Vijayakumar
12.00 - 12.30	Chih-Wen Weng
12.30 - 02.00	LUNCH
	Session 9 Chair: S. Ramachandran
02.00 - 02.30	H.S. Ramane
02.30 - 03.00	Li-Da Tong
03.00 - 03.30	Yaotsu Chang
03.30 - 04.00	KM. Kathiresan
04.00 - 04.30	TEA
	Session 10 Chair: Manoj Changat
04.30 - 05.00	Hung-Lin Fu
05.00 - 06.00	Group Discussions
07.30 p.m.	DINNER

DAY 4 - September 11, 2011

Venue: e-Lecture Hall

	Session 11 Chair: Sr. Germina
09.15 a.m. - 10.00	Plenary Talk - S. Arumugam
10.00 - 10.30	Roushini Leely Pushpam
10.30 - 11.00	Sen-Peng Eu
11.00 - 11.30	TEA
	Session 12 Chair: Shyam Kamath
11.30 - 12.00 noon	Shin-Shin Kao
12.00 - 12.30	Arijit Bishnu
12.30 - 02.00	LUNCH
	Session 13 Chair: T. Tarakeshwar Singh
02.00 - 02.30	Hui-Chuan Lu
02.30 - 03.00	Manu Basavaraju
03.00 - 03.30	C. R. Subramanian
03.30 - 04.30	Problem Session (Coordinator: Xuding Zhu)
04.30 - 05.00	TEA
05.30 - 05.30	Valediction: Vote of Thanks - K. Somasundaram
07.30 p.m.	DINNER

Contents

Circular colouring of graphs

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Keywords: *Circular chromatic number, circular perfect graphs, circular chromatic index*

2000 MR Subject Classification: *05C15*

A homomorphism from a graph G to a graph H is a mapping $f : V(G) \to V(H)$ which preserves edges, i.e., $f(x)f(y) \in E(H)$ whenever $xy \in E(G)$. A homomorphism from G to K_n is equivalent to an n-colouring of G. The set of complete graphs $\{K_1, K_2, K_3, \ldots, \}$ form a chain in the homomorphism order of graphs. The chromatic number of G is defined as $\chi(G) = \min\{n : G \text{ admits a homomorphism to } K_n\}$. We extend the chain of complete graphs to a larger set $\{K_{p/q} : p \geq 2q \geq 2, (p,q) = 1\}$, where the graph $K_{p/q}$ has vertices $\{0, 1, \ldots, p-1\}$ in which ij is an edge if and only if $q \leq |i-j| \leq p-q$. This set of graphs, called circular complete graphs, form a chain with $K_{p/q}$ admits a homomorphism to $K_{p'/q'}$ if and only if $p/q \leq p'/q'$. This chain is used to define a refinement of chromatic number of graphs – the circular chromatic number of graphs. The circular chromatic number of a graph G is defined as $\chi_c(G) = \min\{p/q : G \text{ admits a homomorphism to } K_{p/q}\}$. The circular chromatic number of graphs has been studied extensively in the past twenty years. It has become an important branch of chromatic graph theory with many deep results and new techniques. The study of circular colouring of graphs stimulates challenging problems and in many cases leads to better understanding of chromatic graph theory. In this talk I will survey some results in this area, especially some very recent new results.

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Nowhere-zero constant sum flows

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Keywords: nowhere-zero flows, nowhere-zero k-flows, zero-sum flow, zero-sum k-flow, constant-sum flow, constant-sum k-flow, index set, flow index.

AMS 2010 MR Subject Classification: 05C21, 05C15, 05C70, 05C78

1 Introduction to Zero-Sum Flows

Here a graph means a finite undirected graph with possibly multiple edges or loops unless otherwise stated. We use $\mathbb{Z}$ to stand for the set of all integers, and $\mathbb{Z}^* = \mathbb{Z} - \{0\}$ the set of all non-zero integers. Let G be a directed graph. A **k-flow** on G is an assignment of integers with maximum absolute value $k-1$ to each edge such that for every vertex, the sum of the values of incoming edges is equal to the sum of the values of outgoing edges. A **nowhere-zero k-flow** is a k-flow with no zero edge labels. A celebrated conjecture of Tutte [6] says that every bridgeless graph has a nowhere-zero 5-flow. Jaeger [4] showed that every bridgeless graph has a nowhere-zero 8-flow. Next Seymour [5] proved that every bridgeless graph has a nowhere-zero 6-flow.

One may consider the undirected graph analogue of nowhere-zero flows and it was shown that both concepts are related. For an undirected graph G, a **zero-sum flow** is an assignment of non-zero integers to the edges such that the sum of the values of all edges incident with each vertex is zero. Moreover a **zero-sum k-flow** is a zero-sum flow with edge labels $\pm 1, \pm 2, \ldots, \pm(k-1)$. There is a conjecture for zero-sum flows raised by S. Akbari et al. [1] in 2009, similar to the Tutte's 5-flow Conjecture for nowhere-zero flows as follows: If G is a graph with a zero-sum flow, then G admits a zero-sum 6-flow. It was proved by Akbari et al. [2] that the above zero-sum 6-flow conjecture is equivalent to the Bouchet's 6-flow conjecture[3] for bi-directed graphs. They also validated the zero-sum 6-flow conjecture for 2-edge connected bipartite graphs, and every r-regular graph with r even, $r > 2$, or $r = 3$. Moreover they have the following more detailed information for the zero-sum k-flows of regular graphs: Let G be an r-regular graph ($r \geq 3$), then G admits a zero-sum 7-flow, and if $3 \mid r$, then G admits a zero-sum 5-flow.

2 Flow Indices and Constant-Sum Flows

We define the following notion which optimizes the choice of k one may have to accomplish the zero-sum k-flow:

Definition 1 *For a graph G, we define the* **flow index** $F(G)$ *as the minimum value over all k such that G admits a zero-sum k-flow.*

Therefore one may rephrase the zero-sum 6-flow conjecture as for any graph G, $F(G) \leq 6$. Note that calculating the flow index is more general and difficult than finding the flows. On the other hand, we also extend the notion zero-sum flows(zero-sum k-flows) to a more general one, namely **constant-sum flows(constant-sum k-flows)** as follows:

Definition 2 *For an undirected graph G, if there exists $f : E(G) \to \mathbb{Z}^*$ such that the sum of values of all incident edges with the vertex v is equal to C for each $v \in V(G)$, we call f a* **constant-sum flow** *of G, or simply a C-sum flow of G. We call such constant C an* **index***, and denote the set of all possible indices for G by $I(G)$, which is the* **index set** *of G. Moreover a constant-sum flow is called a* **constant-sum k-flow** *if the absolute values of all assigned edge labels are less than k.*

Among others we give infinite families of graph examples of small flow indices, and we are able to determine completely the index sets of r-regular graphs for $r \geq 3$, also index sets of fans and wheels[7]. We find the minimum values of k for fans and wheels admitting zero-sum k-flows, also justify the zero-sum 6-flow conjecture for fans and wheels as a byproduct[8]. In fact one may extend the theory of zero-sum flows to designs and even hypergraphs by studying the null spaces of corresponding incidence matrices.

3 Summary of Recent Results

We list some of more recent results in the following :

Theorem 1 *The index sets of r-regular graphs G of order n are*

$$I(G) = \begin{cases} \mathbb{Z}^*, & r = 1. \\ \mathbb{Z}, & r = 2 \text{ and } G \text{ contains even cycles only.} \\ 2\mathbb{Z}^*, & r = 2 \text{ and } G \text{ contains an odd cycle.} \\ 2\mathbb{Z}, & r \geq 3,\ r \text{ even and } n \text{ odd .} \\ \mathbb{Z}, & r \geq 3, \text{ and } n \text{ even .} \end{cases}$$

Theorem 2 *The index sets of fans F_n are as follows:*

$$I(F_n) = \begin{cases} \emptyset, & n = 3. \\ 2\mathbb{Z}, & n = 2k,\ k \geq 2. \\ \mathbb{Z}, & n = 2k+1,\ k \geq 2. \end{cases}$$

Theorem 3 *The index sets of wheels W_n are as follows:*

$$I(W_n) = \begin{cases} 2\mathbb{Z}, & n = 2k,\ k \geq 2. \\ \mathbb{Z}, & n = 2k+1,\ k \geq 1. \end{cases}$$

Theorem 4 *The flow indices of fans and wheels are as follows:*

$$F(F_n) = \begin{cases} 3, & n = 3k+1,\ k \geq 1. \\ 4, & otherwise. \end{cases}$$

$$F(W_n) = \begin{cases} 3, & n = 3k,\ k \geq 1. \\ 5, & n = 5. \\ 4, & otherwise. \end{cases}$$

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3-Consecutive colorings of graphs

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Classical theory of graph coloring began with the goal of minimizing the number of colors. Recently there has been a spurt of research activity in the field, wherein the number of colors is to be maximized under certain conditions. The present talk is on such a topic which is currently receiving a lot of attention. Both the vertex and edge versions of certain parameters are considered and their relations to various other graph parameters are examined.

1 3-consecutive vertex colorings of graphs

Three vertices u, v and w in a graph $G = (V, E)$ are consecutive if uv and vw are edges in E. A 3-consecutive coloring of G is a coloring of the vertices of G such that vertex v receives the color of either u or v. The 3-consecutive coloring number $\psi_{3c}(G)$ of G is the maximum number of colors that can be used in such a coloring.

Some results:

1. $\psi_{3c}(G) = n$ if, and only if, each connected component of G is either K_1 or K_2.

2. If $\psi_{3c}(G) = 1$, then every edge of G lies on a cycle.

3. If every edge of G lies in a triangle, then $\psi_{3c}(G) = 1$.

4. Every maximal planar graph G has $\psi_{3c}(G) = 1$.

2 3-consecutive edge coloring of a Graph

Three edges e_1, e_2 and e_3 in a graph G are consecutive if they form a path or a cycle of length 3. The 3-consecutive edge coloring of G is an assignment of colors to the edges of G such that if e_1, e_2 and e_3 are three consecutive edges in graph G, then e_1 or e_3 receives the color of e_2. The 3-consecutive edge coloring number $\psi'_{3c}(G)$ of G is the maximum number of colors permitted in such a coloring.

An equivalent definition is that a coloring of a graph is a 3-consecutive edge coloring of G if at least one end of each edge is monochromatic, in the sense that all edges incident at that vertex have the same color.

Some results:

1. For a graph G without isolated vertices, and for every integer $k \geq 2$, $\psi'_{3c}(G) \geq k$ if and only if G has a stable k-separator.

2. If u and v are two non-adjacent vertices in a graph G of order $p \geq 4$, then $\psi'_{3c}(G) - \psi'_{3c}(G + uv) \leq p - 3$, and the bound is tight for all p.

3. If G is a graph of order p, then $\psi'_{3c}(G) \leq p - i(G)$, where $i(G)$ is the independent domination number of G.

4. If G is a connected graph of order p and maximum degree Δ, then $\psi'_{3c}(G) \leq p - \frac{(p-1)}{\Delta}$ and this inequality holds for infinitely many p for every fixed Δ.

5. If G is a graph of order p and minimum degree δ, then $\psi'_{3c}(G) \leq p - \delta$, and the bound is tight for all $\delta \leq \frac{p}{2}$.

6. If G is a bipartite graph with bipartition $V = V_1 \cup V_2$, and G has no isolated vertices, then $max\{|V_1|, |V_2|\} \leq \psi'_{3c}(G) \leq \beta_0(G)$.

7. For a graph G with q edges, $\psi'_{3c}(G) = q$ if and only if each component of G is a star or an isolated vertex.

3 1-open neighborhood edge coloring of a graph

The 1-open neighborhood edge coloring number $\psi'_{1n}(G)$ of a graph $G = (V, E)$ is the maximum number of colors permitted in a coloring of the edges of G such that for each edge e in G, at most one edge adjacent to e receives a color different from that of e.

1. For a graph G, $\psi'_{1n}(G) = q$, the number of edges in G, if and only if, each component of G is either K_2 or $K_{1,2}$.

2. If G is a connected graph such that degree of every vertex is either 1 or at least 3, then $\psi'_{1n}(G) = 1$.

3. For a connected graph G, $\psi'_{1n}(G) = 1$ if, and only if, $G - M$ is connected, where M is any maximum set of independent vertices of degree 2.

4. If M is a maximum set of independent vertices if degree 2 in a graph G and $|M| = \beta_{02}(G)$, then $\psi'_{1n}(G) \leq \beta_{02}(G) + 1$.

5. For a tree T, $\psi'_{1n}(T) = \beta_{02}(T) + 1$.

Results on the above topics and related concepts have been published in various journals. A few of them are mentioned here.

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Some notes on relations between coloring and scheduling

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Keywords: Circular coloring, edge-weighted digraph, min-max theorem

2000 MR Subject Classification: 05C15

This talk describes some connections between coloring and scheduling. A scheduling of G is a mapping $f : \{1, 2, 3, \cdots\} \to 2^{V(G)}$, where $f(i)$ consists of processes that are operating at round i. The rate of f is defined as

$$\text{rate}(f) = \limsup_{n\to\infty} \sum_{i=1}^{n} |f(i)|/n|V(G)|,$$

which is the average fraction of operating processes at each round. The operating rate of G is defined to be the maximum rate of a scheduling.

The first part of the talk considers fair, weakly fair and strongly fair schedulings of graphs [15]. Scheduling is fair if adjacent vertices alternate their turns in operating. Fair schedulings of a graph was first studied by Barbosa and Gafni [1, 2, 3, 4, 5]. They introduced the method of "scheduling by edge reversal" which derives a fair scheduling through an acyclic orientation. Denote by $\gamma^*(G)$ the maximum rate of a fair scheduling of G. Through scheduling by edge reversal, Barbosa and Gafni related $\gamma^*(G)$ to the structure of acyclic orientations of G. We point out that this relation implies that $\gamma^*(G)$ is equal to the reciprocal of the circular chromatic number of G [15]. We also prove that the rate of an optimal strongly fair scheduling of a graph G is also equal to the reciprocal of the circular chromatic number of G, and the rate of an optimal weakly fair scheduling of G is equal to the reciprocal of the fractional chromatic number of G.

The second part of the talk considers circular chromatic number $\chi_c(\vec{G}, c)$ of a digraph $\vec{G}$ equipped with an assignment c of positive weights on each arc. The parameter $\chi_c(\vec{G}, c)$ was introduced in [9]. We show there is an unexpected connection between $\chi_c(\vec{G}, c)$ and some sorts of schedules on $(\vec{G}, c, T)$ [17], where $(\vec{G}, c, T)$ (called timed marked graph) is an edge-weighted digraph $(\vec{G}, c)$ equipped with an assignment T of nonnegative integer numbers of tokens on each arc.

The final part of this talk is going to generalize the following five theorems. Notation in the following theorems can be found in [7].

Theorem 1 [8] (Minty's Theorem) *G is k-colorable if and only if G has an orientation ω such that*

$$\max_{C\in\mathcal{M}(G)} \frac{|C|}{|C^+_\omega|} \leq k.$$

Theorem 2 [14] (Tuza's Theorem) *Suppose k is an integer ≥ 2. Then G is k-colorable if and only if G has an orientation ω such that*

$$\max_{C\in\mathcal{T}(G,k)} \frac{|C|}{|C^+_\omega|} \leq k.$$

Theorem 3 [6] (Goddyn, Tarsi and Zhang's Theorem) *G is (k,d)-colorable if and only if G has an orientation ω such that*

$$\max_{C\in\mathcal{M}(G)} \frac{|C|}{|C^+_\omega|} \leq \frac{k}{d}.$$

Theorem 4 [19] (Zhu's Theorem) *G is (k,d)-colorable if and only if G has an orientation ω such that*

$$\max_{C\in\mathcal{Z}(G,k,d)} \frac{|C|}{|C^+_\omega|} \leq \frac{k}{d}.$$

Theorem 5 [9] (Mohar's Theorem) *Let $(\vec{G},\ell)$ be an edge-weighted symmetric digraph with positive weights on the arcs. Suppose that r is a real number with $r \geq L(\vec{G},\ell)$. Then $(\vec{G},\ell)$ has a circular r-coloring if and only if $\vec{G}$ has a good initial marking T such that*

$$\max_{\vec{C}\in\mathcal{M}(\vec{G})} \frac{|\vec{C}|_\ell}{|\vec{C}|_T} \leq r.$$

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Acyclic list edge coloring of planar graphs with girth conditions

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An *edge coloring* of G is an assignment of colors to the edges of G. An edge coloring is said to be *proper* if adjacent edges receive distinct colors. The least number of colors, denoted $\chi'(G)$, needed for a proper edge coloring of G is called the *chromatic index* of G. A proper edge coloring of a graph is said to be *acyclic* if any cycle is colored with at least three colors. The *acyclic chromatic index*, denoted $a'(G)$, is the least number of colors required for an acyclic edge coloring of G.

An *edge-list* L of a graph G is a mapping that assigns a finite set of positive integers to each edge of G. An acyclic edge coloring ϕ of G such that $\phi(e) \in L(e)$ for any $e \in E(G)$ is called an *acyclic L-edge coloring* of G. A graph G is said to be *acyclically k-edge choosable* if it has an acyclic L-edge coloring for any edge-list L that satisfies $|L(e)| \geq k$ for each edge e. The *acyclic list chromatic index*, denoted $a'_{\text{list}}(G)$, is the least integer k such that G is acyclically k-edge choosable. Obviously, $\Delta(G) \leqslant \chi'(G) \leqslant a'(G) \leqslant a'_{\text{list}}(G)$.

We initiated the study of the list version of acyclic edge coloring in [5]. Let $\Delta(G)$ denote the maximum degree of a vertex in G. At the end of [5], the following conjecture was proposed.

Conjecture 1 *For any graph G, $a'_{\text{list}}(G) \leqslant \Delta(G) + 2$.*

This is the list version of the following outstanding conjecture about acyclic edge coloring independently given by Fiamčík [3] and Alon, Sudakov, and Zaks [1].

Conjecture 2 *For any graph G, $a'(G) \leqslant \Delta(G) + 2$.*

The *girth* $g(G)$ of a graph G is the length of a shortest cycle in G. In [2, 4, 6, 7], upper bounds for the acyclic chromatic indexes of several classes of planar graphs with large girth were obtained and Conjecture 2 was proved to be true for these graphs. We generalize these results to the acyclic list chromatic indexes of these graphs.

Theorem 3 *Let G be a planar graph. If $g(G) \geqslant 5$, then $a'_{\text{list}}(G) \leqslant \Delta(G) + 2$.*

Theorem 4 *Let G be a planar graph. Suppose that any of the following conditions holds.*

1. $g(G) \geqslant 6$;
2. $\Delta(G) \geqslant 11$ *and* $g(G) \geqslant 5$.

Then $a'_{\text{list}}(G) \leqslant \Delta(G) + 1$.

Theorem 5 *Let G be a planar graph. Suppose that any of the following conditions holds.*

1. $\Delta(G) \geqslant 8$ *and* $g(G) \geqslant 7$;
2. $\Delta(G) \geqslant 6$ *and* $g(G) \geqslant 8$;
3. $\Delta(G) \geqslant 5$ *and* $g(G) \geqslant 9$;
4. $\Delta(G) \geqslant 4$ *and* $g(G) \geqslant 10$;
5. $\Delta(G) \geqslant 3$ *and* $g(G) \geqslant 14$.

Then $a'_{\text{list}}(G) = \Delta(G)$.

Obviously, Conjecture 1 is true for these graphs.

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Total weight choosability of graphs

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A *k-edge weighting* of a graph G is a mapping $w : E(G) \to \{1, 2, \dots, k\}$. An edge weighting w induces a vertex coloring $f_w : V(G) \to \mathbb{N}$ defined by $f_w(v) = \sum_{e \in E(v)} w(e)$, where $E(v)$ is the set of edges of G incident to v. We say the edge weighting is proper if $f_w(u) \neq f_w(v)$ for any edge uv. Edge weighting of graphs was introduced by Karoński, Łuczak and Thomason in 2004 [5]. They posed the following conjecture which is referred as the $1, 2, 3$-conjecture:

Conjecture 1 *Every graph without isolated edges has a proper edge weighting w such that $w(e) \in \{1, 2, 3\}$ for every edge e.*

The conjecture received a lot of attention and edge weighting of graphs has been studied in [1, 2, 4, 5, 6, 7]. The best result concerning $1, 2, 3$-conjecture is obtained by M. Kalkowski, M. Karoński and F. Pfender in [6] recently. They proved that every graph without isolated edges has a proper edge weighting w such that $w(e) \in \{1, 2, 3, 4, 5\}$ for every edge e.

In 2008, T. Bartnicki, J. Grytczuk and S. Niwczyk [3] considered the list version of this problem. Suppose each edge e of G is assigned a set $L(e)$ of real weights. The graph G is *weight L-colorable* if there is a proper edge weighting $w : E \to \cup_{e \in E(G)} L(e)$ such that for each edge e, $w(e) \in L(e)$. A graph is *k-edge weight choosable* if it is weight L-colorable for any list assignment L for which $|L(e)| = k$. They posed the following conjecture:

Conjecture 2 *Every graph without isolated edges is 3-edge weight choosable.*

Conjecture 2 is verified for several classes of graphs, including complete graphs, complete bipartite graphs and some other graphs. However it is unknown if there is a constant C such that every connected graph $G \neq K_2$ is C-edge-weight-choosable.

Suppose $G = (V, E)$ is a graph. A mapping $w : V \cup E \to \mathbb{R}$ is called a proper *total weighting* of G if the *vertex-colouring* f_w *of* G *induced by* w defined as

$$f_w(x) = \sum_{e \in E(x)} w(e) + w(x)$$

is a proper colouring of G, i.e., for any two adjacent vertices x and x', $f_w(x) \neq f_w(x')$. Przybyło and Woźniak [8, 9] considered total weighting of graphs. They posed the following conjecture and named it the $1,2$-conjecture in [8]:

Conjecture 3 *Every simple graph G has a proper total weighting w such that $w(y) \in \{1, 2\}$ for all $y \in V \cup E$.*

They verified this conjecture for some special graphs, including complete graphs, 4-regular graphs and graphs G with $\chi(G) \leq 3$. They also proved that every simple graph G has a proper total-weighting w such that $w(y) \in \{1, 2, \ldots, 11\}$ for all $y \in V \cup E$. This result was improved in [6] where it was shown w can be chosen so that $w(v) \in \{1, 2\}$ for every vertex v and $w(e) \in \{1, 2, 3\}$ for every edge e.

Wong and Zhu [11] considered the list version of this problem. A *total list assignment* of G is a mapping $L : V \cup E \to \mathcal{P}(\mathbb{R})$ which assigns to each element $y \in V \cup E$ a set $L(y)$ of real numbers as *permissible weights*. Given a total list assignment L, a proper total weighting w is called a proper *L-total weighting* if for each $y \in V \cup E$, $w(y) \in L(y)$. Given a pair (k, k') of positive integers, a total list assignment L is called a (k, k')-total list assignment if $|L(x)| = k$ for each vertex $x \in V$ and $|L(e)| = k'$ for each edge $e \in E$. We say G is *(k, k')-total weight choosable* ((k, k')-choosable, for short) if for any (k, k')-total list assignment L, G has a proper L-total-weighting. It is known [11] that a graph is $(k, 1)$-choosable if and only if it is k-choosable. On the other hand, if G is $(1, k')$-choosable, then it is certainly k'-edge-weight-choosable. So the concept of (k, k')-choosable builds a bridge between the concept of conventional choosability of graphs and edge-weight-choosability of graphs, and can be viewed as a generalization of both choosability and edge-weight-choosability. The following conjectures were proposed in [11]:

Conjecture 4 *Every graph is $(2, 2)$-choosable.*

Conjecture 5 *Every graph with no isolated edges is $(1, 3)$-choosable.*

However, it is still unknown if there are constants k, k' such that every graph is (k, k')-choosable. It was shown in [11] that complete graphs, trees, cycles, generalized theta graphs are $(2, 2)$-choosable, and complete bipartite graphs $K_{2,n}$ are

$(1,2)$-choosable and $K_{3,n}$ are $(2,2)$-choosable. In [10], it is proved that complete multipartite graphs of the form $K_{n,m,1,1,\ldots,1}$ are $(2,2)$-choosable and complete bipartite graphs other than K_2 are $(1,2)$-choosable.

In this talk, we shall give some sufficient conditions for a bipartite graph to be $(1,3)$-choosable. An orientation of a bipartite graph with partite sets A and B is balanced if each vertex in A is either a source or a sink and each vertex in B has in-degree equals out-degree. It is proved that a bipartite graph which has a balanced orientation is $(1,3)$-choosable. As a consequence, hypercubes of even dimension are $(1,3)$-choosable.

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λ-Resolving sets in graphs

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Given a vertex v and a k-tuple of vertices $(v_1, v_2, \ldots, v_k)$, the code of v is determined using the distance of v from the vertices $v_1, v_2, \ldots, v_k$ [1] and in the case of neighbourhood resolvability adjacency or otherwise of a vertex v with the vertices $v_1, v_2, \ldots, v_k$ is used [10]. Eventhough the codes are used to distinguish the vertices, the elements of the codes are either the distances or 0 and 1. One may be interested in using different types of codes. An answer to this question is attempted through a new concept called λ-resolvability.

Definition. 1 *Let G be a simple graph. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r, \ldots)$ be a sequence of positive reals. Let $S = \{u_1, u_2, \ldots, u_r\}$ be a subset of $V(G)$. Let $x \in V(G)$. The λ-code of x with respect to k-tuple $(u_1, u_2, \ldots, u_r)$, denoted by $(x/S)_\lambda$, is defined as $(x/S)_\lambda = (a_1, a_2, \ldots, a_r)$ where*

$$a_j = \begin{cases} \frac{1}{\lambda_i}, & if\ d(u_j, x) = i > 0, 1 \leq j \leq r \\ 0, & if\ d(u_j, x) = 0 \end{cases}.$$

S is a λ-resolving set of G if $(x/S)_\lambda \neq (y/S)_\lambda$, whenever $x, y \in V(G)$, $x \neq y$. The minimum cardinality of a λ-resolving set of G is called λ-dimension of G and is denoted by $dim_\lambda(G)$.

Definition. 2 *Let G be a simple graph. Let $S = \{u_1, u_2, \ldots, u_r\}$ be a subset of $V(G)$. Let $x \in V(G)$. Let k be a positive integer.*
The $\lambda_{(\frac{1}{k})}$- code of x with respect to S, denoted by $(x/S)_{\lambda_{(\frac{1}{k})}}$, is defined as $(x/S)_{\lambda_{(\frac{1}{k})}} = (a_1, a_2, \ldots, a_r)$ where $a_j = \begin{cases} \frac{1}{i}, & if\ d(u_j, x) = i \leq k \\ 0, & if\ d(u_j, x) > k \quad , 1 \leq j \leq r \end{cases}$,
when $x \notin S$. If $x \in S$ and $x = u_i$, then $(u_i/S)_{\lambda_{\frac{1}{k}}} = (a_1, a_2, \ldots, a_r)$ where

$$a_j = \begin{cases} \frac{1}{l}, & if\ d(u_j, u_i) = l \leq k \\ 0, & if\ d(u_j, u_i) = l > k \end{cases}, \ if\ j \neq i \text{ and } a - i = 0.$$

S is a $\lambda_{(\frac{1}{k})}$-resolving set of G if $(x/S)_{\lambda_{(\frac{1}{k})}} \neq (y/S)_{\lambda_{(\frac{1}{k})}}$ whenever $x \neq y$, $x, y \in V(G)$.
The minimum cardinality of a $\lambda_{(\frac{1}{k})}$-dimension of G is denoted by $dim_{\lambda_{(\frac{1}{k})}}$.

Remark. 1 *The value of the code of (x/S) at the i^{th} place is denoted by $(x/S)(i)$.*

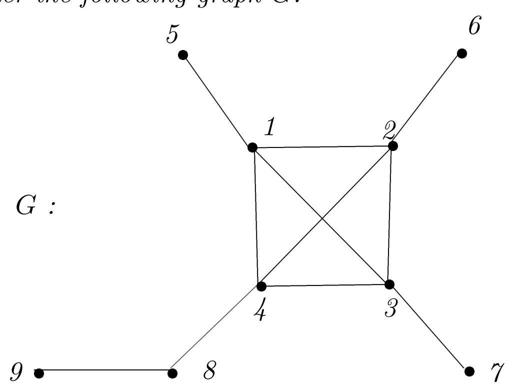
Remark. 2 *If $\lambda_i = \frac{1}{i}$ for every i, then $a_j = i = d(u_j, x)$ and resolving sets are obtained.*

Remark. 3 *If k = 1, then neighbourhood resolving sets are obtained.*

Example. 1 *Consider* P_6 *with* $V(P_6) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$.
Let $S = \{v_1, v_4\}$. *Now* $(v_1/S)_{\lambda_{(\frac{1}{3})}} = (0, \frac{1}{3})$; $(v_2/S)_{\lambda_{(\frac{1}{3})}} = (1, \frac{1}{2})$;
$(v_3/S)_{\lambda_{(\frac{1}{3})}} = (\frac{1}{2}, 1)$; $(v_4/S)_{\lambda_{(\frac{1}{3})}} = (\frac{1}{3}, 0)$; $(v_5/S)_{\lambda_{(\frac{1}{3})}} = (0, 1)$ *and* $(v_6/S)_{\lambda_{(\frac{1}{3})}} = (0, \frac{1}{2})$. *Therefore* S *is a* $\lambda_{(\frac{1}{3})}$*-resolving set of* P_6.
Since $(v_1/S)_{\lambda_{(\frac{1}{2})}} = (v_4/S)_{\lambda_{(\frac{1}{2})}} = (0, 0)$, S *is not a* $\lambda_{(\frac{1}{2})}$*-resolving set of* P_6.
Consider $S = \{v_2, v_3\}$. *Now* $(v_1/S)_{\lambda_{(\frac{1}{3})}} = (1, \frac{1}{2})$; $(v_2/S)_{\lambda_{(\frac{1}{3})}} = (0, 1)$;
$(v_3/S)_{\lambda_{(\frac{1}{3})}} = (1, 0)$; $(v_4/S)_{\lambda_{(\frac{1}{3})}} = (\frac{1}{2}, 1)$; $(v_5/S)_{\lambda_{(\frac{1}{3})}} = (\frac{1}{3}, \frac{1}{2})$ *and* $(v_6/S)_{\lambda_{(\frac{1}{3})}} = (0, \frac{1}{3})$.
Similarly, $(v_1/S)_{\lambda_{(\frac{1}{2})}} = (1, \frac{1}{2})$; $(v_2/S)_{\lambda_{(\frac{1}{2})}} = (0, 1)$;
$(v_3/S)_{\lambda_{(\frac{1}{2})}} = (1, 0)$; $(v_4/S)_{\lambda_{(\frac{1}{2})}} = (\frac{1}{2}, 1)$; $(v_5/S)_{\lambda_{(\frac{1}{2})}} = (0, \frac{1}{2})$ *and* $(v_6/S)_{\lambda_{(\frac{1}{2})}} = (0, 0)$. *Therefore* S *is both* $\lambda_{(\frac{1}{3})}$ *and* $\lambda_{(\frac{1}{2})}$*-resolving sets of* P_6.

Theorem. 1 *For any graph* G*, every neighbourhood resolving set of* G *is a* $\lambda_{(\frac{1}{2})}$*-resolving set of* G.

Remark. 4 *The converse of theorem 1 is not true.*
Consider the following graph G.



Let $S = \{1, 2, 3\}$.
$(1/S)_{\lambda_{(\frac{1}{2})}} = (0, 1, 1)$; $(2/S)_{\lambda_{(\frac{1}{2})}} = (1, 0, 1)$;
$(3/S)_{\lambda_{(\frac{1}{2})}} = (1, 1, 0)$; $(4/S)_{\lambda_{(\frac{1}{2})}} = (1, 1, 1)$;
$(5/S)_{\lambda_{(\frac{1}{2})}} = (1, \frac{1}{2}, \frac{1}{2})$; $(6/S)_{\lambda_{(\frac{1}{2})}} = (\frac{1}{2}, 1, \frac{1}{2})$;
$(7/S)_{\lambda_{(\frac{1}{2})}} = (\frac{1}{2}, \frac{1}{2}, 1)$; $(8/S)_{\lambda_{(\frac{1}{2})}} = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$; *and* $(9/S)_{\lambda_{(\frac{1}{2})}} = (0, 0, 0)$. *Therefore* S *is a* $\lambda_{(\frac{1}{2})}$*-resolving set of* G.
Since $nc_S(8) = nc_S(9) = (0, 0, 0)$, S *is not a* $\lambda_{(\frac{1}{1})}$*-resolving set of* G.

Remark. 5 *For any connected graph* G, $dim_{\lambda_{(\frac{1}{2})}}(G) \leq nr(G)$.

Theorem. 2 *Let* G *be a simple graph. Let* $G \neq K_n$. *Then*
$dim_{\lambda_{(\frac{1}{2})}}(G) \leq n - 2$.

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The reconstruction conjecture: some reductions

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A *vertex-deleted subgraph* (or *card*) $G - v$ of a graph G is the unlabeled subgraph obtained from G by deleting v and all edges incident with v. The collection of all cards of G is called the *deck* of G. A graph H is called a *reconstruction* of G if H has the same deck as G. A graph is said to be *reconstructible* if it is isomorphic to all its reconstructions. A family $\mathscr{F}$ of graphs is *recognizable* if, for each $G \in \mathscr{F}$, every reconstruction of G is also in $\mathscr{F}$, and *weakly reconstructible* if, for each graph $G \in \mathscr{F}$, all reconstructions of G that are in $\mathscr{F}$ are isomorphic to G. A family $\mathscr{F}$ of graphs is *reconstructible* if, for any graph $G \in \mathscr{F}$, G is reconstructible (i.e. if $\mathscr{F}$ is both recognizable and weakly reconstructible).

The *Reconstruction Conjecture* [9] (RC) asserts that all graphs on at least three vertices are reconstructible. Kelly [3] first proved the RC for disconnected graphs and trees. Several other classes of graphs are already proved to be reconstructible in the hope that one day enough classes to include all graphs would come into the fold. [1],[5], [4], and [6] are some surveys of workdone on RC and related problems.

Yang Yongzhi [10] proved the following reduction for RC in 1988.

Reduction 1. RC is true if and only if all 2-connected graphs are reconstructible.

"RC for digraphs" is already disproved. So a proof for RC, if any will depend on some property for graphs which does not extend to digraphs. One such property which arises out of distance in complement is given by the following theorems.

Theorem 1. A graph G is reconstructible if and only if $\overline{G}$ is reconstructible.

Theorem 2. If $diam(G) > 3$, then $diam(\overline{G}) < 3$.

Theorem 3. If $rad(G) \geq 3$, then $rad(\overline{G}) \leq 2$.

The following reduction was proved by S. K. Gupta et al. [2] in 2003 using Theorems 1 and 2.

Reduction 2. RC is true if and only if all graphs G with $diam(G) = 2$ and all graphs G with $diam(G) = diam(\overline{G}) = 3$ are weakly reconstructible.

Using Reductions 1 and 2, Ramachandran and Monikandan [7] have proved Reductions 3 and 4.

Reduction 3. RC is true if and only if all 2-connected graphs G having a vertex v lying on more than one induced P_4 such that $diam(G) = 2$ or $diam(G) = diam(\overline{G}) = 3$ are reconstructible.

Reduction 4. RC is true if and only if all 2-connected graphs G having a vertex v lying on more than one induced P_4 such that $rad(G) = 2$ are reconstructible.

For any two vertices u and v in G, the set $I(u,v) = \{w \in V(G) : w$ lies on a shortest $u - v$ path$\}$ is the interval in G between u and v. A connected graph G is *interval-regular* if $|I(u,v) \cap N(u)| = d_G(u,v)$ for all $u, v \in V(G)$. Graph in which every pair of vertices have unique shortest path is a *geodetic graph*. We have shown [8] that geodetic graphs of diameter two and interval-regular graphs are reconstructible and proved that all graphs are reconstructible if and only if all non-geodetic and non-interval-regular blocks G with $diam(G) = 2$ or $diam(G) = diam(\overline{G}) = 3$ are reconstructible. In this talk, we discuss about the above reductions of RC.

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Rainbow colouring of graphs

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2000 MR Subject Classification: ****

Consider an edge coloring $c : E(G) \to N$, (not necessarily proper) of a graph G. A rainbow path between two vertices is a path such that no two edges in the path have the same colour. The colouring c is called a rainbow (edge) colouring of G if there is a rainbow path between every pair of vertices in G with respect to c. The rainbow connection number $rc(G)$ of a graph G, is the minimum number of colours required in a rainbow coloring of G. For example the $rc(K_n) = 1$, for the complete graph K_n on n vertices, $rc(T) = n - 1$ for a tree T on n vertices. Note that $rc(G)$ is defined only when G is connected.

The concept of rainbow colouring was introduced by Chartrand, Johns, McKeon and Zhang in 2008 [2]. Chakraborty et al. showed that computing the rainbow connection number of a graph is NP-Hard [3]. To rainbow colour a graph, it is enough to ensure that every edge of some spanning tree in the graph gets a distinct colour. Hence order of the graph minus one is an upper bound for rainbow connection number. Many authors view rainbow connectivity as one 'quantifiable' way of strengthening the connectivity property of a graph [4, 3, 5].

The following are some of the known results about rainbow colouring from literature:

1. Let G be a graph of order n. If G is 2-edge-connected (bridgeless), then $rc(G) \leq 4n/5 - 1$.
2. If G is 2-vertex-connected, then $rc(G) \leq \min\{2n/3, n/2 + O(\sqrt{n})\}$ [4].
3. For a graph G of minimum degree δ, $rc(G) \leq 20n/\delta$, [5].

In this talk, we will discuss about two recent results regarding rainbow connection number, from our research group.

1. In [7] we show that for every bridgeless graph G with radius r, $rc(G) \leq r(r+2)$. This bound is the best possible for $rc(G)$ as a function of r, not just for bridgeless graphs, but also for graphs of any stronger connectivity. It may be noted that,

for a general 1-connected graph G, $rc(G)$ can be arbitrarily larger than its radius ($K_{1,n}$ for instance). We further show that for every bridgeless graph G with radius r and chordality (size of a largest induced cycle) k, $rc(G) \leq rk$. Hitherto, the only reported upper bound on the rainbow connection number of bridgeless graphs is $4n/5 - 1$, where n is order of the graph.

2. In [6] we show that for every connected graph G, with minimum degree at least 2, the rainbow connection number is upper bounded by $\gamma_c(G) + 2$, where $\gamma_c(G)$ is the connected domination number of G. Bounds of the form $diameter(G) \leq rc(G) \leq diameter(G) + c$, $1 \leq c \leq 4$, for many special graph classes follow as easy corollaries from this result. This includes interval graphs, AT-free graphs, circular arc graphs, threshold graphs, and chain graphs all with minimum degree at least 2 and connected. In most of these cases, we also demonstrate the tightness of the bounds. An extension of this idea to two-step dominating sets is used to show that for every connected graph on n vertices with minimum degree δ, the rainbow connection number is upper bounded by $3n/(\delta + 1) + 3$. This improves the previously best known bound of $20n/\delta$ [5]. Moreover, this bound is seen to be tight up to additive factors by a construction mentioned in [4].

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Graceful directed graphs [1]

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EXTENDED ABSTRACT

1 INTRODUCTION

The concept of graceful labeling of an undirected graph [7] was extended to a digraph by Bloom and Hsu [1] as follows:

Definition 1.1 *A digraph D with p vertices and q arcs is labeled by assigning a distinct integer value $g(v)$ from $\{0, 1, ..., q\}$ to each vertex v. The vertex values, in turn, induce a value $g(u, v)$ on each arc (u, v) where $g(u, v) = (g(v) - g(u))(mod\ q+1)$. If the arc values are all distinct then the labeling is called a graceful labeling of a digraph.*

Bloom and Hsu [1] established some connections between graceful digraphs and latin squares, Abelian groups, Galois field, and neofields. Also they defined a complete mapping and showed necessary and sufficient conditions for a union of unicycles to be graceful in terms of complete mapping.
In the year 1994, Du and Sun [2] proposed a conjecture as:

Conjecture 1.2 *For any positive even n and any integer m, the digraph $n.\overrightarrow{C_m}$ is graceful.*

Here $n.\overrightarrow{C_m}$ denotes the digraph obtained from n copies of the directed cycle $\overrightarrow{C_m}$ which has one common vertex.

Definition 1.3 *$n - \overrightarrow{C_m}$ denotes the digraph having n copies of the unidirectional cycles $\overrightarrow{C_m}$ with one common arc.*

Xirong Xu, Jirimutu [8] proved that, the digraph $n - \overrightarrow{C_m}$ is graceful for n even and $m = 4, 6, 8, 10$.
Lingqi Zhao, Jirimutu [9] proved that, the digraph $n - \overrightarrow{C_m}$ is graceful for n even and $m = 5, 7, 9, 11, 13$. Also they conjectured that:

Conjecture 1.4 *For any even n and any $m \geq 14$, the digraph $n - \overrightarrow{C_m}$ is graceful.*

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Bloom and Hsu [1] mentioned that "for $n \leq 11$, all unicyclic wheels $\overrightarrow{W_n}$ are known to be graceful except for $n = 6$ and $n = 10$. Can graceful labelings for these be found? And more generally, will the results for unicyclic wheels be as straight forward as for undirected wheels, i.e., is the following conjecture true?".

Conjecture 1.5 *All unicyclic wheels are graceful.*

In this talk, we present the proof of Conjectures 1.2, 1.4 and 1.5.

Lemma 1.6 *Let D be a graceful digraph with p vertices and q arcs. Suppose that, $\overrightarrow{C_n}$ be a unidirectional cycle contained in the digraph D. Then the sum of the labels on the arcs of unidirectional cycle $\overrightarrow{C_n}$ is congruent to zero $(mod\ q+1)$.*

Lemma 1.7 *The unidirectional cycle $\overrightarrow{C_n}$ is graceful if and only if the sum of the elements $1, 2, ..., n$ is congruent to zero $(mod\ n+1)$ and there exists an arrangement of these elements in a circular way with the sum of $m(m < n)$ consecutive elements is not congruent to zero $(mod\ n+1)$.*

Theorem 1.8 *In a unidirectional cycle $\overrightarrow{C_n}$ (n is even), if the orientation of the arcs of any path is reversed, then the resulting directed cycle is not graceful.*

Definition 1.9 *The digraph D is said to be conservative, if indegree and outdegree of all vertices are the same.*

Theorem 1.10 *If a conservative digraph D is graceful then number of arcs in D is even.*

2 RESULTS

2.1 GRACEFULNESS OF THE DIGRAPH $n.\overrightarrow{C_m}$

Theorem 2.1 *[3] Suppose n is odd and $k \mid (n-1)$, where $k > 1$. Then the nonzero residues $(mod\ n+1)$ can be partitioned into $(n-1)/k$ sets of cardinality k, so that the sum of the elements of each set is $\equiv 0(mod\ n)$.*

Theorem 2.2 *The digraph $n.\overrightarrow{C_m}$ is graceful if and only if nm is even.*

2.2 GRACEFULNESS OF THE DIGRAPH $n - \overrightarrow{C_m}$ FOR n EVEN.

We use the following lemmas for proving Conjecture 1.4.

Lemma 2.3 *For any even m and even n, the elements of the set $S = \{1, 2, ..., r-1, r+1, ..., (m-1)n+1\}$, where $r = \frac{(m-1)n+2}{2}$ can be partitioned into n disjoint subsets of cardinality $(m-1)$, so that the sum of the elements of each subset is equal to $\frac{2mn(m-1)+2m+n-2}{2}$.*

Lemma 2.4 *Let $S = \{1, 2, ..., r-1, r+1, ..., (m-1)n+1\}$, where m is odd, n is even and $r = \frac{(m-1)n+2}{2}$. Then S can be partitioned into n disjoint subsets each of cardinality $(m-1)$, in such a way that:*
(a) The sum of the elements of each of $\frac{n}{2}$ subsets is equal to $\frac{m^2n-3mn+2m+2n-4}{2}$.
(b) The sum of the elements in each of the remaining $\frac{n}{2}$ subsets is equal to $\frac{m^2n-mn+2m}{2}$.

Lemma 2.5 *Corresponding to the matrix B, we can construct a new matrix $C = [c_{ij}]$, having the following properties:*
(i) The last column in C, remains the same as that of B.
(ii) For any row i in C, $(c_{i1}, c_{i2}, ..., c_{i(m-1)})$ is just a rearrangement of $(b_{i1}, b_{i2}, ..., b_{i(m-1)})$. This arrangement satisfies the following property: $\sum_{l=j}^{j'} b_{il} \not\equiv 0 (mod\ (m-1)n+2)$ for all $\{(j, j') : 1 \leq j < m,$ $j < j' \leq m$ and $(j, j') \neq (1, m)\}$.
(iii) For any two rows i and i' in C, $\sum_{l=j}^{m} b_{il} + \sum_{t=1}^{r} b_{i't} \not\equiv 0 (mod\ (m-1)n+2)$ for $1 < j \leq m$, $1 \leq r < m-1$.

2.3 GRACEFUL LABELINGS OF UNICYCLIC WHEELS

Proposition 2.6 *If an outspoken unicyclic wheel $\overrightarrow{W_n}$ is graceful, then the sum of the labels of the spokes are congruent to zero $(mod\ 2n+1)$.*

Theorem 2.7 *All unicyclic wheels $\overrightarrow{W_n}$ are graceful.*

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Tessellation of the plane using a polyomino

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Let *the Euclidean plane*, also called *the plane*, be divided into *unit squares*, that is, the four corners of a square have coordinates $(x, y), (x+1, y), (x, y+1), (x+1, y+1)$ for some integers x and y. And for each unit square, we use the coordinate of its lower left corner to label itself. Clearly, unit square (x, y) and unit square (x', y') share a side if either $|x' - x| = 1$ and $y' = y$ or $x' = x$ and $|y' - y| = 1$ holds. Moreover, a finite and nonempty set of unit squares, denoted by U, is called *connected* if, for any two unit squares (x, y) and (x', y') in U, there exists a sequence $(x, y) = (x_1, y_1), (x_2, y_2), \ldots, (x_{k-1}, y_{k-1})$, $(x_k, y_k) = (x', y')$ such that (i) unit square (x_i, y_i) belongs to U for $1 \le i \le k$ and (ii) unit square (x_i, y_i) and unit square (x_{i+1}, y_{i+1}) share a side for $1 \le i \le k-1$.

A *polyomino* is defined as a finite, nonempty, and connected set of unit squares. And a *configuration* generalizes the notion of polyomino by dropping the requirement "being connected". Hence, a polyomino is a connected configuration and a configuration is either a polyomino or a union of polyominoes. Polyominoes are the sources of many combinatorial problems and have fostered significant research in mathematics.

Let $\mathbb{Z}$ denote the set of all integers and $\mathbb{Z}_n$ denote the set $\{0, 1, 2, \ldots, n-1\}$ for some positive integer n. A polyomino P or a configuration C is said to *tessellate* the plane if the plane is consisting of the images of P or C under the *translations* of vectors in portions of $\mathbb{Z} \times \mathbb{Z}$ that do not overlap except along their sides. We also say that there exists a *tessellation* of the plane using a polyomino P or a configuration C if P or C tessellates the plane. Actually, there does exist several methods for determining whether a polyomino or a configuration tessellates the plane. Three of those methods are introduced as follows:

First, for an positive integer N, an N*-skewing scheme* is defined as a 2-dimension function S such that $S(i, j) = k$, where $(i, j) \in \mathbb{Z} \times \mathbb{Z}$ and $k \in \mathbb{Z}_N$. Moreover, a *data template* T is a set of ordered pairs of nonnegative integers in which no two components are identical. And an *instance* of a data template T through a vector $v = (v_x, v_y)$ in $\mathbb{Z} \times \mathbb{Z}$ is a set of ordered pairs of integers which formed by componentwise addition of

(v_x, v_y) to T. Then an N-skewing scheme S is *valid* for a data template T, if and only if, for any two ordered pairs (i_1, j_1) and (i_2, j_2) satisfying $S(i_1, j_1) = S(i_2, j_2)$, there exists no instance of T which contains both (i_1, j_1) and (i_2, j_2) as components. Besides, for each data template T, there exists uniquely a polyomino or a configuration which is corresponding to T. In 1978, Shapiro [3] proved that there is a valid N-skewing scheme for a data template T if and only if the polyomino or the configuration corresponding to T tessellates the plane.

Second, two polyominoes P and Q in the plane are called *simply neighboring* if they have a nonempty intersection with empty interior and such an intersection is also a connected set. Moreover, let $P_0, P_1, \ldots, P_{k-1}$ denote the images of a polyomino P under the translations of vectors $v_0, v_1, \ldots, v_{k-1}$ in $\mathbb{Z} \times \mathbb{Z}$, respectively. Then a *surrounding* of a polyomino P is a sequence $P_0, P_1, \ldots, P_{k-1}$ such that, for $i = 0, 1, \ldots, k-1$, P and P_i are simply neighboring, as also are P_i and P_{i+1} (indices are defined modulo k), and the union of $P, P_0, P_1, \ldots, P_{k-1}$ form a polyomino which does not have internal *holes*. In 1991, Beauquier and Nivat [1] showed that the plane can be tessellated by a polyomino P if and only if there exists a surrounding of P.

Third, a polyomino P or a configuration C of N unit squares has a 2*-linear labeling* if we can label the unit squares of P or C by using the elements of $\mathbb{Z}_f \times \mathbb{Z}_{N/f}$ exactly once for some $f \geq 1$ and $f|N$, such that the labels of unit squares in each row is an arithmetic sequence with skip parameter $A = (a_1, a_2)$ and the labels of unit squares in each column is an arithmetic sequence with skip parameter $B = (b_1, b_2)$, where $A, B \in \mathbb{Z}_f \times \mathbb{Z}_{N/f}$ and $A = B$ is allowed. In 2006, Chen, Hwang, and Yen [2] proved that a polyomino P tessellates the plane if and only if P has a 2-linear labeling.

In this talk, we will study some problems derived from these methods mentioned above for determining tessellating polyominoes.

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On the imperfection ratio of unit disc graphs

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A wireless ad hoc network (or simply a wireless network) consists of a set of nodes that communicate with each other without any physical infrastructure or centralized administration. The interference problem between nodes in a wireless network is important and difficult and it can be modeled using graph theoretic techniques, in particular, the theory associated with Unit Disc Graphs (UDGs). As we will see below, the chromatic number of a UDG model of a wireless network is directly related to the interference problem.

The chromatic number is a graph invariant. The clique number is another graph invariant and is closely related to the chromatic number. It is obvious that for any graph G, the chromatic number is always lower bounded by the clique number, i.e., $\chi(G) \geq \omega(G)$. For the special case of "perfect graphs", the chromatic number and the clique number have equal values in every induced subgraph. It is well known that the graph coloring problem is NP-complete and that even the problem of approximating the chromatic number within any constant ratio is NP-hard [5]. In [2], Clark et al. proved that the coloring problem remains NP-complete for UDGs. In fact, Clark et al. proved that the problem of determining, given a UDG G, whether G is 3-colorable is NP-complete. In [1], Breu and Kirkpatrick have provn that the problem of determining, given a graph G, whether G is a UDG is NP-hard. In [4], Graf et al. improved the result of Clark et al. by showing that the problem of determining, given a UDG G and a fixed integer k, whether G is k-colorable remains NP-complete for any fixed $k \geq 3$; they also proposed a 3-approximation algorithm for the coloring problem.

The transmission range *(TR) of a given node is defined as the maximum distance at which the nodes transmission can be successfully received, and all nodes that lie within the transmission range of a given node are called its* communicating neighbors*. The* interference range *(IR) is defined as the maximum distance at which a given node's transmission can interfere with or corrupt a simultaneous transmission or reception attempt by another node, and all nodes that lie within interference range of a given node are called its* interfering neighbors*. Clearly, all communicating neighbors are interfering neighbors and vise versa.*

Recently, in [6], Mani and Petr treated the case in which IR is the same for all nodes. The graph model is a UDG and is called an interference graph*. In such a graph, if two nodes share an edge, then they are mutually interfering and hence cannot transmit simultaneously in the same timeslot. There are two possible scenarios: balanced load scenario and unbalanced load scenario. In the balanced load scenario, each node require the identical number of transmission timeslots per second to suit their traffic requirements; as a result, the chromatic* *number gives the minimum number of timeslots per second. However, a balanced load scenario rarely occurs in the real world. In [6], Mani and Petr considered the unbalanced load scenario, wherein the traffic rates of each node need not be identical.*

For most classes of graphs, computing $\chi(G)$ and $\omega(G)$ are both NP-complete. But for UDGs, while computing $\chi(G)$ is still NP-complete, computing $\omega(G)$ can be done in polynomial time. This raises the question: How close is $\omega(G)$ to $\chi(G)$? For general graphs, $\chi(G)/\omega(G)$ can be very large. In [7], Peeters has observed that $\chi(G) \leq 3\omega(G) - 2$ if G is a UDG. In [6], Mani and Petr performed extensive simulations with UDGs of random networks and observed that in a UDG G, the clique number $\omega(G)$ and the chromatic number $\chi(G)$ were typically very close to one another. To evaluate the closeness of $\chi(G)$ and $\omega(G)$, Mani and Petr used the measure "imperfection ratio"

$$imp(G) = \sup_R \frac{\chi(G')}{\omega(G')}$$

of a transformed weighted graph G', defined as the supremum of the ratio of its chromatic number to its clique number. Here the supremum is computed over all possible weight vectors R.

It has been proven that the theoretical bound of $imp(G)$ is 2.155 and $imp(G) = 1$ if and only if G is perfect [3]. Based on the simulation results, Mani and Petr [6] concluded that a practical bound for UDGs is $imp(G) \leq 1.2079$, which is far less than the conjectured upper bound of 1.5 or the theoretic upper bound of 2.155. The purpose of this paper is to show that there exist UDGs such that $imp(G) > 1.2079$. In particular, we show that: if G is an odd cycle of length ≥ 5 or the Harary graph $H_{2m,3m+2}$ where m is odd, then $imp(G) = 1.5$; if G is the wheel W_6, then $imp(G) = 4/3$.

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An algorithm for computing two bottleneck shortest paths in weighted graphs

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We consider the problem of finding two bottleneck shortest paths in a weighted graph. The two paths need not be disjoint; they can share vertices and edges. The algorithm is useful in routing messages in a wireless sensor network, which consists of spatially distributed autonomous sensors to monitor physical or environmental conditions, such as temperature, sound, vibration, pressure, motion or pollutants. Each sensor usually has only limited resources. Therefore, it is important to minimize the resources required at each sensor in the transmission of the data collected at that sensor. In this talk, I will define the problem of transmitting message in the network. I will present algorithms for solving the case in which only one pair of sensors need to communicate. I will also list the problems which we have proved to be NP-hard.

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Geodetic number of a graph

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Keywords: geodetic set, geodetic number, minimal geodetic set, upper geodetic number.

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For two vertices u and v in a connected graph $G = (V, E)$, the distance $d(u, v)$ is the length of a shortest $u - v$ path in G. A $u - v$ path of length $d(u, v)$ is called a $u - v$ geodesic. A subset S of vertices is a geodetic set of G if every vertex of G lies on a geodesic joining a pair of vertices in S. The minimum cardinality of a geodetic set of G is the geodetic number $g(G)$ of G. A geodetic set of cardinality $g(G)$ is called a g-set of G. Some general properties of geodetic sets are studied and the geodetic numbers of certain classes of graphs are determined. Graphs of order p for which $g(G) = 2$, $p - 1$ or p are characterized. Bounds for the geodetic number of a graph G are obtained and realization results are proved. A geodetic set S of G is a minimal geodetic set if no proper subset of S is a geodetic set of G. The upper geodetic number $g^+(G)$ of G is the maximum cardinality of a minimal geodetic set of G. Results regarding the geodetic and upper geodetic numbers of a graph are investigated

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The Doyen-Wilson theorem for house designs

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A house is a graph which is obtained by attaching two edges to two vertices of a triangle. In what follows we will denote the house by $(a, b, c; d, e)$ or $(a, c, b; e, d)$ if the triangle is (a, b, c) and two edges are bd and ce. A house-design of a graph G is an ordered pair $(X, \mathcal{A})$, where X is the vertex set of G and $\mathcal{A}$ is an edge-disjoint decomposition of G into copies of houses. A house-design of order n is a house-design of K_n. A partial house-design of order n is a house-design of a subgraph of K_n. Following the design terminology, we call these copies blocks.
A house-design of order n exists precisely when $n \equiv 0, 1 \pmod 5$ [6]. In this paper, we are interested in the embedding problem for house-designs. A (partial) house-design $(X_1, \mathcal{A}_1)$ is said to be embedded in a house-design $(X_2, \mathcal{A}_2)$ if $X_1 \subset X_2$ and $\mathcal{A}_1 \subset \mathcal{A}_2$. In 1973, Doyen and Wilson [5] set the standard for embedding problems by proving the following result: Let $m, n \equiv 1$ or $3 \pmod 6$. Any STS(m) can be embedded in a STS(n) if $n \geq 2m + 1$. This lower bound on n is the best possible.
Over the years, any problem involving trying to prove a similar result for a given combinatorial structure has come to be called Doyen-Wilson problem. Recently, some papers investigated Doyen-Wilson theorems for special blocks. Huang and Yang [11] considered this problem for extended directed triple systems; Castellana and Raines [4] for extended Mendelsohn triple systems; Lo Faro and Tripodi [12] for kite systems; Wang and Shen [14] for nested Steiner triple systems; Hoffman and Kirkpatrick [10] for kite-designs; Raines [13] for extended triple systems of all indices; Fu and Lindner [7] for maximum packings of K_n with 4-cycles; Fu, Lindner and Rodger [8] for maximum packings with triples; Fu, Lindner and Rodger [9] for minimum coverings with triples; Bryant and Rodger [2] for 5-cycle systems; Bryant and Rodger [3] for m-cycle systems. The focus of this paper is to produce Doyen-Wilson theorem for house-designs.

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Labeling elements of a signed graph

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Abstract

Signed graph S is an ordered triple (V, E, σ) where $G = (V, E)$ is a graph, called its *underlying graph*, and σ is a function, called a *signing* of G, that assigns a *weight* (*sign*) +1 ('plus') or -1 ('minus') to each edge of G. An *element* of S is either an element of V, called a *vertex* of S (or of G), or an element of E, called an *edge* of S. Hence, a *labeling* of S is a function $\ell : V \cup E \to \mathcal{W}$ that assigns to each element of S an element of the chosen universe of discourse $\mathcal{W}$ such as a subset of real numbers or a family of subsets of a nonempty *ground set* X. The main aim of this paper is to creatively review the existing literature on various notions of labeling the elements of a signed graph (digraph) and identify some important research problems for further work.

1 Introduction

For all terminology and notation in graph theory (digraph theory), we refer the reader to Harary [8] (Harary *et al.* [7]). Accordingly, unless mentioned otherwise, all graphs (digraphs) treated in this article are simple, self-loop-free and finite.

In this article, we shall go one step further and indicate studies on different ways to 'label' a given signed graph (digraph) $S = (V, E, \sigma)$. Mainly we cover the following themes. (i) Marking theme (ii) Coloring theme (iii) Graceful theme.

2 Marking theme

Marking of a given signed graph (digraph) $S = (V, E, \sigma)$ is a function $\mu : V(S) \to \{-1, +1\}$; essentially, it is a labeliing of the vertices of S using the labels from the group $M = \{-1, +1\}$. Given μ, we write S_μ to indicate that S is given together with one of its markings μ, whence we call S_μ a *marked signed graph*; in particular, if $G = (V, E)$ is a graph (digraph) it may be treated as a marked graph (digraph) in which all the marks are positive and hence it may be denoted as G_μ if required, where $\mu(u) = +1,\ \forall\ u \in V(G)$, whence G_μ will be called a *marked graph*. Formally, a signed graph (digraph) S is *balanced* if every cycle (semicycle) in S contains an even number of negative edges (arcs).

Theorem 2.1 *A signed graph $S = (V, E, \sigma)$ is balanced if and only if it admits a marking μ such that*

$$\sigma(uv) = \mu(u)\mu(v)\ \forall\ uv \in E. \tag{1}$$

Theorem 2.2 *A signed graph $S = (V, E, \sigma)$ is clusterable if and only if no cycle in S has exactly one negative edge.*

Definition 2.1 *A signed graph $S = (V, E, \sigma)$ is consistent if it admits a marking μ such that every positive cycle in S contains an even number of negative vertices in μ; in other words*

$$\prod_{e \in E(Z)} \sigma(e) = + = \prod_{v \in V(Z)} \mu(v) \quad \textit{for every positive cycle } Z \textit{ in } S.$$

2.1 Canonical markings

Given any signed graph $S = (V, E, \sigma)$, its *canonical marking* μ_σ is uniquely defined as given by $\mu_\sigma(u) = \prod_{uv \in E_u} \sigma(uv)\ \forall u \in V$, where E_u denotes the set of edges that contain u and is called the *edge neighborhood* of u (see [2, 5]).

3 Coloring theme

Definition 3.1 *For any positive integer k, a proper k-coloring of a signed graph $S = (V, E, \sigma)$ is an assignment of integers from the set $\{0, \pm 1, \pm 2, \ldots, \pm k\}$ to the vertices of S such that end vertices of every positive edge of S receive different numbers and end vertices of any negative edge in S do not receive two numbers in the set that are negatives of each other; S is k-colorable if it admits a proper k-coloring.*

In general, an *r-semiclustering* of a signed graph S is a partition π of $V(S)$ into r subsets such that the induced subgraph of no subset contains a negative edge; we shall call these subsets *nonnegative sets* in S. Further, a signed graph is *r-semiclusterable* if it admits an r-semiclustering. Thus, if S has a proper k-coloring then it is $(2k+1)$-semiclusterable. A proper k-coloring of S is said to be 0-*free* if 0 (zero) does not appear as one of the colors on the vertices of S [14]. In particular, a 0-free 1-coloring of a signed graph S induces a 2-semiclustering of S; a signed graph S in general is said to be *semibalanced* if it admits a 2-semiclustering (see [3]). The following is now fairly evident.

Proposition 1 *If S admits a* 0*-free* 1*-coloring then S is semibalanced, but not conversely.*

Like semibalanced signed graphs have a simple characterization (see [3]), one would now look for a characterization of 0-free 1-colorable signed graphs; the following is still a straightforward characterization of such signed graphs, where a signed graph is called *antibalanced* whenever its negation is balanced.

Theorem 3.1 *[11] A signed graph S is* 0*-free* 1*-colorable if and only if S is antibalanced.*

Remark 3.1 *In fact, in general, the induced subgraph on the $+i$- or $-i$-colored vertices in any proper k-colored signed graph S is always an all-negative subgraph of S.*

Remark 3.2 *[11] If we allow* 0 *in a proper* 1*-coloring of a signed graph S, then the answer is: There is a stable (independent) vertex set whose deletion leaves an antibalanced signed graph. For integers $k \geq 2$ such a characterization seems to get complicated.*

The least positive integer k for which a signed graph S admits a proper (0-free) k-coloring is called the (*zero-free*) *chromatic number* of S, denoted $\chi(S)$ (respectively, $\chi^*(S)$). Clearly, for any signed graph S, $\chi(S) \leq \chi^*(S)$. In fact, as conveyed by Zaslavsky [12], we know that

$$\chi(S) = \chi^*(S) \text{ or } \chi(S) = \chi^*(S) - 1. \tag{3}$$

Nothing else is known about these parameters, aside from a few specific examples. Hence, it would be interesting to solve

Problem 1: Determine signed graphs S for which equalities in (3) hold.

Further, there is an open problem to extend the notion of proper colorings of a signed graph to the realm of signed digraphs. One can possibly take the recent approach based on a method suggested by Sampathkumar [6].

4 Graceful theme

Here, in this section, we shall deal with vertex labelings that induce edge labelings, so that one need not separately label the edges. Such labeling schemes in general have been found to be useful in optimizing the size of the sets of symbols required for encoding the graph; the purpose of this section is to creatively review such schemes developed recently for signed graphs (digraphs). It is too unwieldy to cite here references to literature as they are too numerous for graphs (e.g., see [9]); however, since they are widely known we choose to ignore such citations as far as possible.

4.1 Graceful signed digraphs

Let $S = (G, \sigma)$ be any signed graph with $E^+(S)$ as its set of positive edges and $E^-(S)$ as its set of negative edges; if, in particular, if G is a (p, q)-graph, $|E^+(S)| = m$ and $|E^-(S)| = n$, then $|E(S)| = m + n = q$. Given positive integers d and k, S is (d, k)-*graceful* (see [4, 10]) if the vertices of S can be labeled with distinct integers from the set $\{0, 1, 2, \ldots, k+(q-1)d\}$ such that when each edge uv of S is assigned the product of its sign and the absolute difference of the integers assigned to u and v the values of the edges of $E^+(S)$ and $E^-(S)$ form the sets of integers $\{k, k+d, k+2d, \ldots, k+(m-1)d\}$ and $\{-k, -(k+d), -(k+2d), \ldots, -(k+(n-1)d)\}$, respectively; such a labeling is called a (d, k)-*graceful numbering* of S and S is said to be a (d, k)-*graceful signed graph* if it admits a (d, k)-graceful numbering. (It is not difficult to see generalizations of these notions to the class of infinite signed graphs.) In particular, a $(1, 1)$-graceful signed graph is called a *graceful signed graph* and a $(1, 1)$-graceful numbering of S is called a *graceful numbering* of S. Clearly, when a graph is regarded as an all-positive signed graph, the notions of a (d, k)-graceful numbering and that of a (d, k)-graceful signed graph reduce respectively to those of a (d, k)-*graceful numbering* and that of a (d, k)-*graceful graph* [1].

For graceful signed graphs and consistency in signed graphs the reader is referred to [13] and [5] respectively.

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Tree path labeling of path hypergraphs: A generalization of consecutive ones property

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We consider the following constraint satisfaction problem. Given (i) a set system $\mathcal{F} \subseteq (powerset(U) \setminus \emptyset)$ of a finite set U of cardinality n, (ii) a tree T of size n and (iii) a bijection ℓ, defined as *tree path labeling*, mapping the sets in $\mathcal{F}$ to paths in T, does there exist at least one bijection $\phi : U \rightarrow V(T)$ such that for each $S \in \mathcal{F}$, $\{\phi(x) \mid x \in S\} = \ell(S)$? A tree path labeling of a set system is called *feasible* if there exists such a bijection ϕ. In this paper, we characterize feasible tree path labeling of a given set system to a tree. This result is a natural generalization of results on matrices with the Consecutive Ones Property. Moreover, we pose some interesting algorithmic questions which extend from this work.

Consecutive ones property (COP) of binary matrices is its property of rearrangment rows (columns) in such a way that every column (row) has its 1s occuring consecutively. The problem of COP testing is also a constraint satisfaction problem of a set system as follows. In a binary matrix, if every column is represented as a set of indices of the rows with 1s in that column, then if the matrix has the COP, a reordering of its rows will result in sets that are intervals. The COP is equivalent to the problem of finding interval assignments to a given set system [3] with a single permutation of the universe which permutes each set to its interval. Clearly COP is a special instance of tree path labeling problem described above when T is a path. The result in [3] characterize interval assignments to the sets which can be obtained from a single permutation of the rows - the cardinality of the interval assigned to it must be same as the cardinality of the set, and the intersection cardinality of any two sets must be same as the interesction cardinality of the corresponding intervals - Intersection Cardinality Preserving Interval Assignment (ICPIA). This is obviously necessary and was discovered to be sufficient.

We focus on the question of generalizing the notion of an ICPIA [3] to characterize feasible path assignments. We show that for a given set system $\mathcal{F}$, a tree T, and an assignment of paths from T to the sets, there is a bijection ϕ between U and $V(T)$ if and only if all intersection cardinalities among any three sets (not necessarily distinct) is same as the intersection cardinality of the paths assigned to them and the input successfully executes our filtering algorithm (described in this paper) without prematurely exiting. Aside from finding the bijection ϕ mentioned above for a given path labeling, we also present a characterization of set systems which have a feasible tree path labeling on a given tree T and present an algorithm to find a path labeling if it exists when T is a *k-subdivided star*. A k-subdivided star is a star with all its rays subdivided exactly k times. The path from the center to a leaf is called a ray of a k-subdivided star and they are all of length $k + 2$. A *star* graph is a complete bipartite graph $K_{1,l}$ which is clearly a tree and l is the number of leaves. The vertex with maximum degree is called the *center* of the star and the edges are called *rays* of the star.

The following questions are extensions to this work.

1. The intersection graph of a set system with a feasible tree path labeling from a tree T must be a *path graph* which is a subclass of chordal graphs. This can be checked efficiently because path graph recognition is polynomial time solvable[1, 4]. However, this is only a necessary condition. It is possible to have a pair of set system and tree $(\mathcal{F}, T)$, such that the intersection graph of $\mathcal{F}$ is a path graph, but there is no feasible tree path labeling to T. Therefore, the following questions.

   (a) What is the maximal set system $\mathcal{F}' \subseteq \mathcal{F}$ such that $(\mathcal{F}', T)$ has a feasible tree path labeling?

   (b) What is the maximal subtree $T' \subseteq T$ such that $(\mathcal{F}, T)$ has a feasible tree path labeling?

   (c) Path graph isomorphism is known be isomorphism-complete[2]. An interesting area of research would be to see what this result tells us about the complexity of the tree path labeling problem.

2. A set system $\mathcal{F}$ can be alternatively represented by a *hypergraph* $\mathcal{H}_{\mathcal{F}}$ whose vertex set is *supp*$(\mathcal{F})$ and hyperedges are the sets in $\mathcal{F}$. This is a known representation for interval systems in literature [2]. We extend this definition here to path systems. Two hypergraphs $\mathcal{H}$, $\mathcal{K}$ are said to be isomorphic to each other, denoted by $\mathcal{H} \cong \mathcal{K}$, iff there exists a bijection $\phi : supp(\mathcal{H}) \rightarrow supp(\mathcal{K})$ such that for all sets $H \subseteq supp(\mathcal{H})$, H is a hyperedge in $\mathcal{H}$ iff K is a hyperedge in $\mathcal{K}$ where $K = \{\phi(x) \mid x \in H\}$. If $\mathcal{H}_{\mathcal{F}} \cong \mathcal{H}_{\mathcal{P}}$ where $\mathcal{P}$ is a path system, then $\mathcal{H}_{\mathcal{F}}$ is called a *path hypergraph* and $\mathcal{P}$ is called *path representation* of $\mathcal{H}_{\mathcal{F}}$. If isomorphism is $\phi : supp(\mathcal{H}_{\mathcal{F}}) \rightarrow supp(\mathcal{H}_{\mathcal{P}})$, then it is clear that there is an induced path

labeling $l_\phi : \mathcal{F} \rightarrow \mathcal{P}$ to the set system. So our problem of finding if a given path labeling is a feasible path labeling is a path hypergraph isomorphism problem.

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Rank and maximum nullity of graphs

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The purpose of this talk is to report recent results on rank, nullity, maximum nullity and minimum rank of graphs by Liang-Hao Huang, Hong-Gwa Yeh and myself.

The first part of this talk is on rank and nullity. The *rank* $r(G)$ (resp. *nullity* $\eta(G)$) of a graph G the rank (resp. the number of zero eigenvalues) of its adjacency matrix $A(G)$. These two concepts are equivalent in the sense that $r(G) + \eta(G) = n$ for a graph G of n vertices.

The nullity of a molecular graph G has a number of important applications in physical chemistry. In quantum mechanics, Hückel theory [16] (for the molecule corresponding to molecular graph G) says that the eigenvectors of the adjacency matrix $A(G)$ are identical to the Hückel molecular orbitals, and the eigenvalues of $A(G)$ are the energies corresponding to the Hückel molecular orbitals. The number of nonbonding molecular orbitals (NBMOs) is identical with the multiplicity of the eigenvalue zero in the spectrum of $A(G)$. If $\eta(G) > 0$, then the molecule corresponding to G have NBMOs in the Hückel spectrum, and such molecule should have open-shell ground states and be very reactive. This implies molecular instability.

There are two things concerned by people in this direction. The first one is to compute the rank using the structure of the graph. The second is to determine the structure of an n-vertex connected graph with a fixed rank k.

In 2001, Sillke [20] conjectured that the rank of a cograph (P_4-free graph) is equal to the number of distinct non-zero rows of its adjacency matrix. This conjecture is a by-product of Sillke's approach to Rank-coloring Conjecture [2]. After two years Royle [19] proved this conjecture. We [3] answered a question in [19] by giving an elementary and short proof of this rank property of cographs.

While determining the structure of an n-vertex graph G with $r(G) = k$ is well answered for $k \leq 3$, the question has not yet been fully answered for $k = 4, 5$ in the literature. Only partial results were known (see [13, 17, 10, 18, 21, 11]). We completely resolve this recently [4, 5].

The second part of this talk is on maximum nullity and minimum rank. Given a graph G on n vertices and a field F, the *maximum nullity* $M^F(G)$ (resp. *minimum rank* $\mathrm{mr}^F(G)$) of G over F is the largest possible nullity (resp. smallest possible rank) over all $n \times n$ symmetric matrices over F whose (i, j)th entry (for $i \neq j$) is nonzero whenever ij is an edge in G and is zero otherwise. Again, these two parameters determine each other as $\mathrm{mr}^F(G) + M^F(G) = n$. The maximum nullity problem of a graph G is to determine $M^F(G)$. This problem has close relation with the inverse eigenvalue problem. The problem and its variations have received considerable attention over the years (see [9, 12] and references therein).

In 2008, the AIM group [1] introduced a new concept called the zero forcing number $Z(G)$ of a graph as a useful upper bound of $M^F(G)$. For a vertex subset $S \subseteq V(G)$ of an n-vertex graph G, let $S_0 = S$ and $S_{i+1} = S_i \cup \{y\colon \{y\} = N(x) \setminus S_i$ for some $x \in S_i\}$ for $i \geq 0$. It is clear that $S_i = S_n$ for all $i \geq n$. A *zero-forcing set* of G is a subset $S \subseteq V(G)$ for which $S_n = V(G)$. The *zero-forcing number* $Z(G)$ of G is the minimum size of a zero-forcing set. The authors in [1] showed that $M^F(G) \leq Z(G)$ for any graph G and any field F. They posted an attractive question: What is the class of graphs G for which $Z(G) = M^F(G)$ for some field F? We partially answer this question [14, 15] with results improving several results in [1, 8].

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On the distance spectra of graphs

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Abstract

Let G be a connected graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. The distance matrix $D = D(G)$ of G is defined so that its (i, j)-entry is equal to $d_G(v_i, v_j)$, the distance between the vertices v_i and v_j of G. The eigenvalues of the $D(G)$ are said to be the D-eigenvalues of G and form the D-spectrum of G, denoted by $spec_D(G)$.

In this talk we present some of the recent results in the distance spectra of graphs.

1 Introduction

Let G be a connected graph with vertex set $V(G) = \{v_1, v_2, \ldots, \ldots, v_p\}$ and size (= number of edges) q . The distance matrix $D = D(G)$ of G is defined so that its $(i, j)-$ entry is equal to $d_G(v_i, v_j)$, the distance (= length of the shortest path [1]) between the vertices v_i and v_j of G. The eigenvalues of $D(G)$ are said to be the $D-$ eigenvalues of G and form the $D-$ spectrum of G , denoted by $spec_D(G)$.

The ordinary graph spectrum is formed by the eigenvalues of the adjacency matrix [2]. In what follows we denote the ordinary eigenvalues of the graph G by $\lambda_i,\ i = 1, 2, \ldots, p$, and the respective spectrum by $spec(G)$.

Since the distance matrix is symmetric, all its eigenvalues $\mu_i,\ i = 1, 2, \ldots, p$, are real and can be labelled so that $\mu_1 \geq \mu_2 \geq \ldots\ldots \geq \mu_p$.

2 Some recent results in the theory of distance spectra of graphs

Theorem 1 *For $i = 1, 2$, let G_i be an r_i-regular graph with n_i vertices and eigenvalues of the adjacency matrix A_{G_i}, $\lambda_{i,1} = r_i \geq \lambda_{i,2} \geq \lambda_{i,2} \geq \cdots \geq \lambda_{i,n_i}$. The distance spectrum of $G_1 \nabla G_2$ consists of eigenvalues $-\lambda_{i,j} - 2$ for $i = 1, 2$ and $j = 2, 3, \ldots, n_i$ and two more eigenvalues of the form*

$$n_1 + n_2 - 2 - \frac{r_1 + r_2}{2} \pm \sqrt{\left(n_1 - n_2 - \frac{r_1 - r_2}{2}\right)^2 + n_1 n_2}. \tag{1}$$

Theorem 2 *For $i = 0, 1, 2$, let G_i be an r_i-regular graph with n_i vertices and eigenvalues $\lambda_{i,1} = r_i \geq \lambda_{i,2} \geq \lambda_{i,2} \geq \cdots \geq \lambda_{i,n_i}$ of the adjacency matrix A_{G_i}. If $r_1 \neq r_2$, then the distance spectrum of $G_0\nabla(G_1 \cup G_2)$ consists of eigenvalues $-\lambda_{i,j} - 2$ for $i = 0, 1, 2$ and $j = 2, 3, \ldots, n_i$ and three more eigenvalues which are solutions of the cubic equation in ν:*

$$\begin{aligned}&(2n_0 - r_0 - 2 - \nu)(\nu + r_1 + 2)(\nu + r_2 + 2)\\&+[2(\nu + r_0 + 2) - 3n_0][n_1(\nu + r_2 + 2) + n_2(\nu + r_1 + 2)] = 0.\end{aligned} \quad (2)$$

Theorem 3 *Let G and H be two distance regular graphs on p and n vertices with distance regularity k and t respectively. Let $spec_D(G) = \{k, \mu_2, \mu_3, ..., \mu_p\}$ and $spec_D(H) = \{t, \eta_2, \eta_3, ..., \eta_n\}$. Then*

$$spec_D(G + H) = \{nk + pt, n\mu_i, p\eta_j, 0\}$$

$i = 2, ..., p$, $j = 2, ..., n$ and 0 is with multiplicity $(p - 1)(n - 1)$.

Theorem 4 *Let G be a graph with $D-$ matrix D_G and H , an $r-$ regular graph with an adjacency matrix A. Let $spec_D(G) = \{\mu_1, \mu_2, ..., \mu_p\}$ and the ordinary spectrum of H be $\{r, \lambda_2, \lambda_3, ..., \lambda_n\}$. Then*

$$spec_D G[H] = \begin{pmatrix} n\mu_i + 2n - r - 2 & -(\lambda_j + 2) \\ 1 & p \end{pmatrix}, \; i = 1 \text{ to } p \text{ and } j = 2 \text{ to } n - 1$$

Definition 1 *[3] Given two graphs G with vertex set $\{v_1, v_2, ..., v_p\}$ and H , the corona of G and H is denoted by $G \circ H$ and is defined as the graph obtained by taking p copies of H and for each i , joining the i^{th} vertex of G to all the vertices in the i^{th} copy of H , $i = 1, 2, \ldots\ldots, p$.*

Definition 2 *[3] Let H be a rooted graph rooted at u. Then given a graph G with vertex set $\{v_1, v_2, ..., v_p\}$, the cluster $G\{H\}$ is defined as the graph obtained by taking p copies of H and for each i , joining the i^{th} vertex of G to the root in the i^{th} copy of H .*

Theorem 5 *Let G be a distance regular graph on p vertices $\{v_1, v_2, ..., v_p\}$ with distance regularity k, a distance matrix D and $spec_D = \{k = \mu_1, \mu_2, ..., \mu_p\}$. Let H be an $r-$ regular graph on n vertices with an adjacency matrix A and $spec_A = \{r = \lambda_1, \lambda_2, ..., \lambda_n\}$. Then the distance spectrum of $G \circ H$ consists of the following numbers:*

(a) $\dfrac{n(2p + k) + k - r - 2 \pm \sqrt{(n(2p + k) + k - r - 2)^2 + 4(np^2 + k(r + 2))}}{2}$ *each with multiplicity 1*

(b) $\dfrac{\mu_i(n + 1) - r - 2 \pm \sqrt{(\mu_i(n + 1) - r - 2)^2 + 4\mu_i(r + 2)}}{2}$ *for each $\mu_i \in spec_D$, $i = 2, 3, ..., p$*

(c) $-\lambda_i - 2$ *with multiplicity* p *for each* $\lambda_i \in spec_A(H)\,, i = 2, 3, ..., n$.

Theorem 6 *Let* G *be a distance regular graph with distance regularity* k *,a distance matrix* D *and distance spectrum* $\{k = \mu_1, \mu_2, ..., \mu_p\}$. *Then the distance spectrum of* $G\{K_n\}$ *consists of the numbers* -1 *of multiplicity* $(n-2)p$ *,the roots of the equation*

$$\prod_{i=2}^{p} \left[x^3 - (n(\mu_i - 3) + \mu_i)\, x^2 - 2n(2\mu_i - 1)\, x - 2n\mu_i\right] = 0$$

together with the three roots of

$$x^3 - (n(k-3) + p(4n-2) + k)\, x^2 - \left(p^2(5n-4) + 2n(p + 2k - 1)\right) x - p^2(3n-2) - 2nk = 0 \tag{3}$$

3 New graphs from old and their distance spectra

In this section we obtain the distance spectrum of the following graphs.

1. Median and total graph of a cycle.
2. The complement of the subdivision graph of a regular graph G.
3. The subdivision graph of K_n.
4. The graph obtained by subdividing the edges of a Hamiltonian cycle in K_n

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Graphs whose adjacency matrices have rank equal to the number of distinct nonzero rows

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For a (simple) graph G, let rank(G) and dnzr(G) denote respectively the rank and the number of distinct nonzero rows of the adjacency matrix $A(G)$ of G. It is proved that for any graph G, with no isolated vertices, rank(G) = dnzr(G) if and only if the adjacency matrix $A(G)$ of G does not contain a nonzero null vector x with the property that $x_u = x_v$ whenever u, v are vertices of G that have the same neighbors in G. Based on the latter result (and its equivalent formulation in terms of the reduced adjacency matrix of G) and by considering the question of when two vertex-disjoint graphs G_1, G_2 satisfy $\text{rank}(G_1 \vee G_2) = \text{dnzr}(G_1 \vee G_2)$, we provide an alternative proof for the known result that every cograph G satisfies rank(G) = dnzr(G). We provide evidences which support the conjecture that for any graph G with nonsingular adjacency matrix, the sum of the entries of $A(G)^{-1}$ is greater than 1. If the conjecture is true, then it can be shown that the class of graphs G that satisfy rank(G) = dnzr(G) is closed under taking join.

It is clear that in general $\text{rank}(G) \leq \text{dnzr}(G)$. While experimenting on the rank-chromatic number question by computer, Torsten Sillke [8] observed that for all the cographs he checked, the rank is equal to the number of distinct non-zero rows of the adjacency matrix, and he conjectured that all cographs have this maximal rank property. In [7] Royle provided an inductive proof for the conjecture, which is based on the fact that every cograph can be constructed from smaller cographs by taking (disjoint) union and join, starting from trivial (single-vertex) graphs. His proof begins by examining the behavior of the characteristic polynomial when the operation of union or join is applied to two cographs. He raised the question of whether there are other natural classes of graphs for which this rank property holds. In [2] Chang, Huang and Yeh gave another proof for the conjecture, in the slightly more general setting of vertex-weighted cographs, i.e., cographs for which each vertex is assigned a weight from $[0,1)$, the adjacency matrix being the same as that of the underlying graph, except that now the diagonal entries are equal to the vertex weights of the corresponding vertices instead of equal to 0. Their proof employs elementary row and column operations and relies on the fact that every nontrivial cograph has two vertices which are (true or false) twins, i.e., distinct vertices with identical open or closed neighborhoods. In [1] Türker Biyikoğlu has provided yet another proof for the conjecture, using cotrees and threshold graphs as tools.

In this talk I provide the fourth proof for Sillke's conjecture. Like Royle, our approach is by induction and still based on the fact that every cograph can be constructed from smaller cographs by taking union and join. Instead of working with the characteristic polynomial of the adjacency matrix, we essentially use the eigenspace method (more in the spirit of the book [5]). We examine the general question of when two graphs G_1, G_2, not necessarily cographs, satisfy $\mathrm{rank}(G_1 \vee G_2) = \mathrm{dnzr}(G_1 \vee G_2)$). We need the concept of reduced adjacency matrix of a graph, which we introduce below. For more details, see [4].

For a graph G, by the *neighborhood equivalence relation* on G we mean the equivalence relation $\sim^G$ on $V(G)$ given by: $u \sim^G v$ if and only if $N_G(u) \setminus \{v\} = N_G(v) \setminus \{u\}$. The equivalence classes for $\sim^G$ are called the *neighborhood equivalence classes* of G. We use $t(G)$ to denote the number of neighborhood equivalence classes of G. For a graph G with neighborhood equivalence classes of cardinality $n_1, \ldots, n_{t(G)}$ respectively, we use $z(G)$ to denote $(n_1, n_2, \ldots, n_{t(G)})^T$, the *neighborhood class cardinalities vector* of G. By the *reduced adjacency matrix* of G we mean the $t(G) \times t(G)$ matrix $B(G) = [b_{ij}]$ given by:

$$b_{ij} = \begin{cases} (n_i - 1) & \text{if } i = j, V_i \text{ is a clique} \\ 0 & \text{if } i = j, V_i \text{ is a stable set} \\ n_j & \text{if } i \neq j, \text{ there are edges between } V_i \text{ and } V_j \\ 0 & \text{if } i \neq j, \text{ there are no edges between } V_i \text{ and } V_j \end{cases} .$$

Theorem 1. For a graph G with no isolated vertices, the following conditions are equivalent :

(a) $\mathrm{rank}(G) = \mathrm{dnzr}(G)$.

(b) $B(G)$ is nonsingular.

(c) $A(G)$ does not contain a nonzero null vector x with the property that $x_u = x_v$ whenever u, v are vertices of G that have the same neighbors in G.

We use $e^{(n)}$ to denote the vector of all 1's in $\mathbb{R}^n$.

Theorem 2. For a graph G, the following conditions are equivalent :

(a) $\mathcal{N}(B(G)) \cap (\text{span}\{z(G)\})^{\perp} = \{0\}$.

(b) $\mathcal{N}(B(G)^T) \cap (\text{span}\{e^{(t(G)}\})^{\perp} = \{0\}$.

(c) Either $B(G)$ is nonsingular, or $\text{rank}(B(G)) = t(G) - 1$ and $e^{(t)} \notin \mathcal{R}(B(G))$.

(d) Either $\text{rank}(G) = \text{dnzr}(G)$ or $\text{rank}(G) = \text{dnzr}(G) - 1$ and $e^{(n)} \notin \mathcal{R}(A(G))$.

Lemma 1. Let G_1, G_2 be vertex-disjoint graphs. If $\text{rank}(G_1 \vee G_2) = \text{dnzr}(G_1 \vee G_2)$ then G_1 and G_2 each satisfy the equivalent conditions of Theorem 2.

Theorem 3. Let G_1, G_2 be vertex-disjoint graphs. Suppose that G_1, G_2 each satisfy the equivalent conditions of Theorem 2. Assume, in addition, that we have $z(G_i)^T B(G_i)^{-1} e^{(t_i)} > 1$ whenever $B(G_i)$ is nonsingular. Then

$$\text{rank}(G_1 \vee G_2) = \text{dnzr}(G_1 \vee G_2) \text{ and } z(G_1 \vee G_2)^T B(G_1 \vee G_2)^{-1} e^{(t(G_1 \vee G_2))} > 1.$$

Conjecture. Let G be a graph with no isolated vertices. If $\text{rank}(G) = \text{dnzr}(G)$, then $z(G)^T B(G)^{-1} e^{(t(G))} > 1$.

The preceding conjecture is equivalent to the conjecture mentioned at the introductory paragraph.

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Spectral numbers and weakly spectral numbers

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We rely on [8] for terminology and notation; every graph considered in this abstract is finite and simple. Let the least eigenvalue of a graph G be denoted by $\lambda(G)$. Let the class of all graphs G with $\lambda(G) \geqslant -2$ be denoted by $\mathcal{L}_2$. It has been found that the family of all line graphs is a subclass of $\mathcal{L}_2$. (For a proof and for more information in this regard, see [2].) This fact has prompted many authors to devote considerable attention to the study of $\mathcal{L}_2$. A. J. Hoffman [5] has found an important subfamily of $\mathcal{L}_2$, whose members are called *generalized line graphs*:

For any positive integer n, the *cocktail party graph* $\mathrm{CP}(n)$ is the graph obtained from K_{2n} by removing a perfect matching; $\mathrm{CP}(0)$ is the graph without vertices. If G is a graph with vertex set $\{1, 2, \ldots, n\}$ and $\sigma_1, \sigma_2, \ldots, \sigma_n$ are nonnegative integers, then the *generalized line graph* $L(G; \sigma_1, \sigma_2, \ldots, \sigma_n)$ is obtained from the (disjoint) union of $L(G)$ and $\mathrm{CP}(\sigma_i)$, $i = 1, 2, \ldots, n$ by forming additional edges: a vertex e in $L(G)$ is adjacent to all vertices in $\mathrm{CP}(\sigma_i)$ whenever i is an endpoint of e in G.

P. J. Cameron, J. M. Goethals, J. J. Seidel and E. E. Shult have found [2] an algebraic description of the family of all generalized line graphs and a classification of $\mathcal{L}_2$. Using the results of [2] obtained in this regard, D. Cvetković, M. Doob and S. Simić have found [3] all minimal nongeneralized line graphs. (If a graph G is not a generalized line graph whereas every proper induced subgraph of G is a generalized line graph, then G is called a *minimal nongeneralized line graph.*) Doob has observed that *if G is a minimal nongeneralized line graph with $\lambda(G) < -2$, then it has an induced subgraph H with $\lambda(H) = -2$.* This fact has motivated him to try to prove that the conclusion of this observation holds for every graph G with $\lambda(G) < -2$ and to introduce the notion defined as follows: a real number ℓ is called *spectral* if every graph G with $\lambda(G) < \ell$ has an induced subgraph H with $\lambda(H) = \ell$. (Doob has not used the term 'spectral number'; he has referred to such numbers as 'numbers having induced subgraph property'.)

Relying on a result of B. McKay obtained by computer search—*every graph G of order less than 10 with $\lambda(G) < -2$ has an induced subgraph H with $\lambda(H) = -2$*—and using the algebraic properties of the class of all minimal nongeneralized line graphs found by [3], Doob has concluded [4] that -2 is a spectral number; he has easily shown that $-\sqrt{2}, -1, 0$ are the other spectral numbers.

Subsequently, F. C. Bussemaker and A. Neumaier have found [1] the entire family $\mathcal{M}$ of all minimal graphs G with $\lambda(G) < -2$ by using computer search—the cardinality of $\mathcal{M}$ is 1812; by verifying that every graph in $\mathcal{M}$ has an induced subgraph whose least eigenvalue is -2, they have confirmed that -2 is a spectral number.

A variant of the notion 'spectral number' has been introduced in [7]: A real number ℓ is called *weakly spectral*, if every tree T with $\lambda(T) < \ell$ has a subtree U

with $\lambda(U) = \ell$. Let the set of all spectral numbers and the set of all weakly spectral numbers be denoted by $\mathcal{S}$ and $\mathcal{W}$, respectively. It is easy to see that every spectral number is also a weakly spectral number; i.e., $\mathcal{S} \subset \mathcal{W}$. In this talk, we describe a process of showing that $\mathcal{S} = \{-2, -\sqrt{2}, -1, 0\}$ and $\mathcal{W} = \{-2, -\sqrt{3}, -\sqrt{2}, -1, 0\}$; this process is similar to the one found in [7] to compute $\mathcal{W}$; it relies on the result of [6] and does not depend upon any computer oriented result.

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The spectral excess theorem for non-regular graphs

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Let Γ be a connected graph on n vertices, with diameter D and adjacency matrix A. Assume that A has $d+1$ distinct eigenvalues $\lambda_0 > \lambda_1 > \ldots > \lambda_d$ with corresponding multiplies $m_0 = 1$, m_1, ..., m_d. Consider the $(d+1)$-dimensional vector spaces $\mathbb{R}_d[x]$ of polynomials of degrees at most d with inner product defined by

$$< p(x), q(x) >:= \sum_{i=0}^{d} \frac{m_i}{n} p(\lambda_i)q(\lambda_i) = tr(p(A)q(A))/n$$

for $p(x), q(x) \in \mathbb{R}_d[x]$. It is known that $\mathbb{R}_d[x]$ has a unique orthogonal basis $p_0(x) = 1, p_1(x), \ldots, p_d(x)$ such that $p_i(x)$ has degree i and $< p_i(x), p_i(x) >= p_i(\lambda_0)$. The polynomial $q_d(x) := p_0(x) + p_1(x) + \cdots + p_d(x)$ is called the *Hoffman polynomial* of Γ, and if Γ is regular then $q_d(x) = J$, the all 1's $n \times n$ matrix.

Let A_i be the i-th *distance matrix*, i.e. an $n \times n$ matrix with rows and columns indexed by the vertex set of Γ such that

$$(A_i)_{xy} = \begin{cases} 1, & \text{if } \partial(x, y) = i; \\ 0, & \text{else.} \end{cases}$$

Note that $A_0 + A_1 + \cdots + A_D = J$.

Define

$$\begin{aligned} f(x) &:= p_D(x) + p_{D+1}(x) + \cdots + p_d(x), \\ \epsilon_D &:= \text{trace } (A_D q_d(A))/n, \\ \delta_D &:= \text{trace } (A_D J)/n. \end{aligned}$$

Note that $\delta_D = \epsilon_D$ if Γ is regular, and $f(x) = p_D(x)$ if $d = D$. The positive value $f(\lambda_0) =< f(x), f(x) >$ is called the *spectral excess* of Γ, and ϵ_D^2/δ_D is called the *excess* of Γ. We shall prove the following theorem, which was first proved under the assumption that Γ is regular and $d = D$ [1] and reproved in [3, 2].

Spectral Excess Theorem. $f(\lambda_0) \geq \epsilon_D^2/\delta_D$ with equality iff $A_D = \sqrt{\frac{\delta_D}{f(\lambda_0)}} f(A)$.

It is well known that if Γ is regular and $D = d$, then the above equality holds iff Γ is distance-regular [1].

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Spectra and energy of graphs

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Let G be the graph with n vertices and m edges. Eigenvalues of the adjacency matrix $A(G)$ of a graph G are called the eigenvalues of G and their collection is called the spectrum of G [1]. The eigenvalues of the distance matrix $D(G)$ of a graph G are caled the D-eigenvalues of G and their collection is called the D-spectra of G [1].

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of G then the energy $E(G)$ of a graph G is defined as [2]

$$E(G) = \sum_{i=1}^{n} |\lambda_i|$$

Let $\gamma_1, \gamma_2, \ldots, \gamma_n$ be the D-eigenvalues of G then D-energy $E_D(G)$ of a graph G is defined as [3]

$$E_D(G) = \sum_{i=1}^{n} |\gamma_i|$$

Theorem [4]: If G is a graph with n vertices, m edges and adjacency matrix $A(G)$ then

$$\sqrt{2m + (n-1)|det(A(G))|^{2/n}} \leq E(G) \leq \sqrt{2mn}$$

Two graphs G_1 and G_2 are said to be equienergtic if $E(G_1) = E(G_2)$. Following result leads to the construction of infinetly many pairs of noncospectral equienergtic graphs having equal order and equal size.

Theorem [6]: Let G be a regular graph on n vertices and of degree $r \geq 3$, then

$$E(L^2(G)) = 2nr(r-2)$$

where $L^2(G)$ is the second line graph of G.

Also in this paper, we discuss other results on spectra and energy of various class of graphs [5, 7].

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Graphs with minimum size and minimum identifying code

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The identifying codes were first introduced by Karpovsky, Chakrabarty, and Levitin in [6]. Identifying codes of graphs are constructed for fault diagnosis in multiprocessor systems [4, 6, 7]. Charon, Hudry, and Lobstein [2] proved that the determining an identifying code with minimum cardinality in a graph is NP-hard. Recently, researchers focus on the study of identifying code and extended problems [1, 3, 5].

Let G be a graph, u be a vertex of G, and $B(u)$(or $B_G(u)$) be the set of u with all its neighbors in G. A set S of vertices is called an *identifying code* of G if, for every pair of distinct vertices u and v, $B(u) \cap S$ and $B(v) \cap S$ are nonempty and distinct. A *minimum identifying code* of a graph G is an identifying code of G with minimum cardinality and $M(G)$ is the cardinality of a minimum identifying code in G. A *minimum identifying code graph* G of order n is a graph with $M(G) = \lceil \log_2(n+1) \rceil$ having the minimum number of edges. For finding the number of edges on a minimum identifying code graph is one of important problems for reducing the cost of constructing a minimum identifying code graph or network. Moncel [8] constructed minimum identifying code graph of order $2^m - 1$ for each integer $m \geq 2$ and addressed the question for finding minimum identifying code graphs of order n for arbitrary positive integer n. In the paper, we construct minimum identifying code graphs of order n for each positive integer $n \geq 4$ and investigate related properties. Hence, this question is well-studies by us.

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Algebraic decodings of binary quadratic residue codes

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In 1958, Prange [1] introduced quadratic residue (QR) codes, which are cyclic codes with code rates greater than or equal to 1/2. There are twelve binary QR codes of length not exceed 103, namely, 7, 17, 23, 31, 41, 47, 71, 73, 79, 89, 97, 103, and except for those of lengths 71, 73, 97, the other nine are the best known codes because of their large minimum distances.

Elia [2] proposed, in 1987, the first algebraic decoding algorithm for the binary Golay code, or the QR code of length 23. Later, Reed et al. presented algebraic decoders for QR codes of lengths 31, 41, and 73 in 1990 [3], 1992 [4], and 1994 [5], respectively. The key idea is to use the Sylvester resultant or Gröbner bases methods to solve the nonlinear multivariate equations provided by the Newton identities. When the code length is large, this method encounters the difficulty of insufficient syndromes. For example, the QR code of length 47 is the smallest one cannot be decoded by this decoding scheme.

In 2001, He et al. [6] modified a matrix method proposed by Feng to express the needed unknown syndrome as a function in known syndromes. With enough "known" syndromes, the binary QR code of length 47 was decoded successively by the above decoding algorithm.

The authors et al. used the modified Feng-He matrix method to express the unknown syndromes as functions of known syndromes to obtain enough consecutive known syndromes. With the unknown syndrome representations, we can apply the famous Berlekamp-Massey algorithm to determine the error-locator polynomial and then apply the Chien search to find the error positions. Based on this decoding scheme, we developed the decoders for the binary QR codes of lengths 71, 79, and 97 in 2003 [7], of lengths 103 and 113 in 2005 [8], and of length 89 in 2008 [9]. Therefore, the decoders of all the binary QR codes of lengths not exceed 103 are developed. Furthermore, this method can also be applied to decode those QR codes with larger lengths.

Although the previous decoding algorithm can decode any binary QR code, it is not a one-step decoder. There are four steps in this algorithm: First, compute the known syndromes. Next, use the unknown syndrome representations to calculate the needed unknown syndromes. In the third and fourth steps, apply Berlekamp-Massey algorithm and Chien search to determine the error-locator polynomial and error positions, respectively. When using this scheme to correct a v-error pattern, the second and third steps are executed v rounds, and in each round different representations are used.

In 2010, the authors [10] proposed a one-step decoder for a class of binary QR codes, namely, QR codes with irreducible generator polynomials. We show that any such binary QR code possesses a unified unknown syndrome representation, which can be provided by the Lagrange interpolation formula. And, by utilizing this unified representation, each correctable error pattern can be determined by executing each of the four steps exactly once.

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Change in the total vertex irregularity strength by an edge

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A vertex irregular total $k-labeling$ $\lambda : V(G) \cup E(G) \to \{1, 2, ..., k\}$ *of a graph* G *is a labeling of vertices and edges of* G *done in such a way that for any two different vertices* x *and* y*, their weights* $wt(x)$ *and* $wt(y)$ *are distinct. The weight* $wt(x)$ *of a vertex* x *is the sum of the label of* x *and the labels of all edges incident with* x*. The minimum* k *for which a graph* G *has a vertex irregular total* $k-labeling$ *is called the total vertex irregularity strength of* G*, denoted by* $tvs(G)$*. In this talk, we disscuss how the addition of a new edge affects the total vertex irregularity strength of a graph.*

The decycling number of outerplanar graphs

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The problem of destroying all cycles in a graph by deleting a set of vertices originated from applications in combinatorial circuit design [2]; also, it has found applications in many fields. A set of vertices of a graph whose removal leaves an acyclic graph is referred to as a *decycling set* [1]. The minimum cardinality of a decycling set of G, denoted by $\tau(G)$, is referred to as the *decycling number* of G.

Besides searching for the value of the decycling number in the order of a graph, another parameter that is closely related to the decycling number is the *cycle packing number*, which is the maximum number of vertex-disjoint cycles. We denote this parameter by $c(G)$. A trivial relation between the decycling number and the cycle packing number is $c(G) \leq \tau(G)$.

A graph is said to be *outerplanar* provided that all its vertices lie on the boundary of a face (after embedding the graph in a sphere). Even for an outerplanar graph G, not much is known about $\tau(G)$. Kloks et al. [3] proved that $\tau(G) \leq 2c(G)$.

An outerplanar graph G is called *lower-extremal* if $\tau(G) = c(G)$ and *upper-extremal* if $\tau(G) = 2c(G)$. We study these two extremal cases. For simplicity, we use ij to denote an edge $\{i, j\}$.

Definition 1. *S_k is a graph with parameters (V, E) where $V = \{0, 1, \cdots, 2k-1\}$ and $E = \{i(i+1) : 0 \leq i \leq 2k-1\} \cup \{i(i+2) : i$ is even$\}$ (the indices are under modulo $2k$).*

Then $\tau(S_k) = \lceil \frac{k}{2} \rceil$ and $c(S_k) = \lfloor \frac{k}{2} \rfloor$. We define the *simplified graph* of a graph G to be the graph obtained from G by continuously deleting vertices of degree one until there is no more degree one vertex and denote it by $\lfloor G \rfloor$. Then, we have

Lemma 1. *For an outerplanar graph G with $c(G) = 1$, G is upper-extremal if and only if $\lfloor G \rfloor$ is a subdivision of S_3.*

A graph is an *S_3-tree of order t* if it has exactly t vertex-disjoint S_3-subdivisions and every edge not on these S_3-subdivisions belongs to no cycle.

For $X, Y \subseteq V(G)$, an *X, Y-path* is a path having one endpoint in X, the other one in Y, and no other vertex in $X \cup Y$, and a $\{v\}, Y$-path can be simply written as a v, Y-path. Then,

Lemma 2. *An outerplanar graph G comprised of a connected S_3-tree H of order t and two internally disjoint $v, V(H)$-paths has $t+1$ vertex-disjoint cycles for $v \notin V(H)$.*

Based on the observation in Lemma 2, we prove the necessary condition of the following result by induction on $c(G)$.

Theorem 3. *An outerplanar graph G is upper-extremal if and only if G is an S_3-tree.*

To prove that a property is sufficient for a graph being lower-extremal, we will use induction. A graph property is called *monotone* if it is closed under removal of vertices. We provide the following general result that is applicable to all graphs.

Lemma 4. *Suppose that a 2-connected graph is lower-extremal provided that it satisfies a monotone property $\mathcal{P}$. Then G is lower-extremal if G satisfies $\mathcal{P}$.*

Then we prove the following result by induction on $|E(G)|$.

Lemma 5. *If G is a 2-connected outerplanar graph with no S_k-subdivision for all odd number k, then G is lower-extremal.*

The property of being without S_k-subdivision is monotone. Therefore, by Lemma 4 and Lemma 5, we have

Theorem 6. *For an outerplanar graph G, if G has no S_k-subdivision for all odd number k, then G is lower-extremal.*

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Equality of domination and transversal numbers in graphs and hypergraphs

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A subset S of the vertex set of a hypergraph $\mathcal{H}$ is called a dominating set of $\mathcal{H}$ if for every vertex v not in S there exists $u \in S$ such that u and v are contained in an edge in $\mathcal{H}$. The minimum cardinality of a dominating set in $\mathcal{H}$ is called the domination number of $\mathcal{H}$ and is denoted by $\gamma(\mathcal{H})$. A transversal of a hypergraph $\mathcal{H}$ is defined to be a subset T of the vertex set such that $T \cap E \neq \emptyset$ for every edge E of $\mathcal{H}$. The transversal number of $\mathcal{H}$, denoted by $\tau(\mathcal{H})$, is the minimum number of vertices in a transversal. In the case of graphs a transversal is also called a vertex cover. A hypergraph is of rank k if each of its edges contains at most k vertices.

The inequality $\tau(\mathcal{H}) \geq \gamma(\mathcal{H})$ is valid for every hypergraph $\mathcal{H}$ without isolated vertices. In this paper we investigate the structure of graphs and hypergraphs satisfying $\tau(\mathcal{H}) = \gamma(\mathcal{H})$.

First we consider graphs with $\tau = \gamma$. The problem of characterizing graph with $\tau = \gamma$ first appeared in [4]. Later in 1981, Laskar and Walikar [2] also mentioned this problem. First, Hartnell and Rall answered the question in [1], but their characterization was quite complicated. Randerath and Volkmann established another characterization in [3], but it was precise only for graphs of minimum degree at least 2; this characterization was recently completed by Wu and Yu [5] for the case of $\delta = 1$.

Here we give the characterization in a unified simpler way, avoiding redundances. This is a similar but different formulation from those in [3] and [5]. We need the following notions.

- A vertex v of a graph $G = (V, E)$ is called *stem* if v is adjacent to a vertex of degree 1. The set of all stems of G is denoted by Stem(G) [5].
- Let $\mathcal{S}(G)$ denote the graph obtained from G by deleting all edges contained entirely in Stem(G). Note that transformation $\mathcal{S}$ does not create isolated vertices, unless G contains some component isomorphic to K_2.

Theorem. *For a connected graph G of order at least 3, $\tau(G) = \gamma(G)$ holds if and only if there exists a bipartition (A, B) of $\mathcal{S}(G)$ such that, Stem$(G) \subseteq A$, moreover for every pair $u, v \in A \setminus$ Stem(G), if u and v have some common neighbor then they have at least two common neighbors of degree two.*

By this characterization, graphs with $\tau = \gamma$ and without isolates can be recognized in polynomial time. We design such an algorithm with running time $O(\sum_{v \in V} d^2(v))$. Moreover, we note that the condition on degree-2 neighbors implies $|A| \leq |B|$.

We now consider hypergraphs $\mathcal{H}$ for which $\tau(\mathcal{H}) = \gamma(\mathcal{H})$. We prove that the corresponding recognition problem is NP-hard already on the class of 3-uniform linear hypergraphs. Structurally we focus our attention on hypergraphs in which each subhypergraph $\mathcal{H}'$ without isolated vertices fulfills the equality $\tau(\mathcal{H}') = \gamma(\mathcal{H}')$. It is shown that if each induced subhypergraph satisfies the equality then it holds for the non-induced ones as well. Moreover, we prove that for every positive integer k, there are only a finite number of forbidden subhypergraphs of rank k, and each of them has domination number at most k.

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Roman domination in graphs

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In this talk we present some results related to the Weak Roman domination number of a graph. We also discuss two more parameters of a graph namely, the Efficient Roman Domination number and the Clique star cover number. By a graph $G = (V.E)$, we mean a finite, undirected and connected graph with neither loops nor multiple edges. For graph theoretic terminology we refer to F. Harary [2] . Cockayne et al. [1] defined a Roman dominating function (RDF) on a graph $G = (V, E)$ to be a function $f : V \to \{0, 1, 2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. For a real-valued function $f : V \to R$ the weight of f is $w(f) = \sum_{v \in V} f(v)$ and for $S \subseteq V$ we define $f(S) = \sum_{v \in S} f(v)$, so $w(f) = f(V)$. The Roman domination number, denoted by $\gamma_R(G)$, is the minimum weight of an RDF in G; that is, $\gamma_R(G) = min\{w(f)|f$ is an RDF in $G\}$. An RDF of weight $\gamma_R(G)$ is called a $\gamma_R(G)$-function. If V_0, V_1 and V_2 are the sets of vertices assigned the values 0, 1 and 2 respectively under f, then there is a 1-1 correspondence between the functions $f : V \to \{0, 1, 2\}$ and the ordered partitions (V_0, V_1, V_2) of V. Thus $f = (V_0, V_1, V_2)$.

Weak Roman Domination Number

Henning and Hedetniemi [3] defined the weak roman domination number of a graph G as follows. A vertex $u \in V_0$ is undefended with respect to f, *or simply* undefended *if the function f is clear from the context, if it is not adjacent to a vertex in V_1 or V_2. A function $f : V \to \{0, 1, 2\}$ is said to be a* weak Roman dominating function (WRDF) *if each vertex $u \in V_0$ is adjacent to a vertex $v \in V_1 \cup V_2$ such that the function $f' : V \to \{0, 1, 2\}$, defined by $f'(u) = 1, f'(v) = f(v) - 1$ and $f'(w) = f(w)$ if $w \in V - \{u, v\}$, has no undefended vertex. They defined the weight $w(f)$ to be $|V_1| + 2|V_2|$.* The weak Roman domination number, *denoted by $\gamma_r(G)$, is the minimum weight of a WRDF in G, that is, a WRDF of weight $\gamma_r(G)$ is called a $\gamma_r(G)$-function. They also observed that for any graph G, $\gamma(G) \leq \gamma_r(G) \leq \gamma_R(G) \leq 2\gamma(G)$.*

We characterize trees T for which $\gamma_r(T) = \gamma(T)$. For this purpose we introduce a family $\Im$ of trees as follows. A tree $T \in \Im$ if the following conditions holds.

(i) *No vertex of T is a strong support.*

(ii) *If $u \in V(T)$ is a non support which is adjacent to a support, then $N(u)$ contains exactly one vertex which is neither a support nor adjacent to a support and all other members of $N(u)$ are either supports or adjacent to supports.*

(iii) *For any vertex u of degree at least two, there exists at least one leaf vertex v such that $d(u, v) \leq 3$.*

(iv) *Two vertices which are neither supports nor adjacent to supports are not adjacent.*

Theorem 1 [6]: For any tree $T, \gamma_r(T) = \gamma(T)$ *if and only if* $T \in \Im$.

Clique Star Cover Number

Wayne Gaddard et.al [8] defined the clique star cover number $\theta_s(G)$ as follows. A Colonization of G is defined as a partition of the vertex set into sub graphs each with a dominator (a vertex adjacent to all other nodes in the sub graph). The weight of a colonization counts 1 for each clique and 2 for each non-clique. Then $\theta_s(G)$ is the minimum weight of a colonization.

We prove the following theorem on the equality of γ_R and θ_s of a graph G.

Theorem 2 [7]: *Let G be a graph. Then $\gamma_R(G) = \theta_s(G)$ if and only if there exists a colonization $\hat{C}$ of G of minimum weight such that no member of $\hat{C}$ is a $K_t, t > 1$.*

In order to characterize the class of trees for which $\gamma_R(T) = \theta_s(T)$, we introduce a family $\Im_1$ of trees as follows. Let $\hat{A} = \{A|A$ is a star with at least 3 end vertices$\}$, $\widetilde{B} = \{B|B \cong P_3\}$ and $\hat{C} = \{C|C \cong K_1\}$. Let $H = \hat{A} \cup \widetilde{B} \cup \hat{C}$. A tree $T \in \Im_1$ if T is the union of H and a collection ε of edges subject to the following conditions.

(i) *Between any two vertices in H, there exists a unique path.*

(ii) *$\hat{C}$ is independent.*

(iii) *An end vertex of a member in $\hat{A}$ is adjacent to at most two vertices in $\hat{C}$.*

(iv) *No two end vertices of members in $\widetilde{B}$ are adjacent.*

(v) *A vertex of a member in $\widetilde{B}$ is neither adjacent to a vertex in $\hat{C}$ nor adjacent to a head vertex of a member in $\hat{A}$.*

(vi) *Corresponding to each $\hat{A}$ in A, there exist at least two end vertices x_1, x_2 in A such that each $x_i, i = 1, 2$ is either of degree one or adjacent to an end vertex in $\hat{A}$.*

Theorem 3 [7]: *Let T be a tree of order n. Then $\gamma_R(T) = \theta_s(T)$ if and only if $T \in \Im_1$.*

We prove the following theorem on the equality of γ_r and θ_s of a graph G.

Theorem 4 [7]: *Let G be a graph with a colonization $\hat{C}$ of minimum weight. Then $\gamma_r(G) = \theta_s(G)$ if and only if for every non clique colony C in $\hat{C}$ with dominator x, there exists at least one vertex in $V(C) - x$, which is neither adjacent to a vertex of a clique nor adjacent to a dominator of a non clique of $\hat{C}$.*

We define a family $\Im^*$ of trees as follows.

Let $A_1 = \{A|A$ is a star$\}$, $B_1 = \{B|B \cong P_2\}$ and $C_1 = \{C|C \cong K_1\}$.

Let $H_1 = A_1 \cup B_1 \cup C_1$. A tree $T \in \Im^*$ if T is the union of H and a collection ε of edges subject to the following conditions.

(i) *Between any two vertices in T, there is a unique path.*

(ii) *C_1 is independent.*

(iii) *Corresponding to each A in A_1, at most $n-1$ end vertices in A are adjacent to vertices of members of B_1.*

(iv) No two vertices in C_1 are adjacent to a vertex of B in B_1 and no two vertices in C_1 are adjacent to the two end vertices of a B in B_1.

Theorem 5 [7]: *Let T be a tree of order n. Then $\gamma_r(T) = \theta_s(T)$ if and only if $T \in \Im^*$.*
***Theorem 6** [7]: For any $2 \times n$ grid graph G, $\gamma_R(G) = n + 1$.*
Theorem 7 [7]: *For any $2 \times n$ grid graph G, $\theta_s(G) = n$ and $\gamma_R(G) = \theta_s(G) + 1$.*

Efficient Roman Domination

The idea of efficiency was extended to Roman domination by Rubalcaba and Slater [4]. We recall the definitions of an efficient Roman dominating function and some related terms. A (j,k)-packing is a function $f : V(G) \to \{0, 1, 2, \ldots, j\}$ with $f(N[v]) \leq k$ for all $v \in V(G)$. Thus a 2-packing is a (1,1)-packing, and in particular, a (2,2)-packing is a function $f : V(G) \to \{0, 1, 2\}$ with $f(N[v]) \leq 2$ for all $v \in V(G)$. For a function $f : V(G) \to \{0, 1, 2\}$ the Roman influence of f, denoted by $I_R(f)$ is defined to be $I_R(f) = (|V_1| + |V_2| + \sum_{v \in V_2}$ deg(v)). The efficient Roman domination number of G, denoted by $F_R(G)$ is defined to be the maximum of $I_R(f)$ such that f is a (2,2)-packing. That is $F_R(G) = max\{I_R(f) : f$ is a (2,2)-packing$\}$. A (2,2)-packing f with $F_R(G) = I_R(f)$ is called an $F_R(G)$ function. Graph G is said to be efficiently Roman dominatable, if $F_R(G) = n$ where n is the order of G and when $F_R(G) = n$, the $F_R(G)$-function is called an efficient Roman dominating function.

Definition: *We define a graph T^* to be the union of stars $K_{1,ni}, 1 \leq i \leq k$ and a collection ε of edges subject to the following conditions.*

(i) If $e = vw \in \varepsilon$, then e is an edge joining $v \in V(K_{1,ni})$ and $w \in V(K_{1,ni})$, $i \neq j$ where v and w are end vertices.

(ii) For any pair of vertices $v \in V(K_{1,ni})$ and $w \in V(K_{1,nj}), i \neq j$, there exists a unique path joining v and w.

Theorem 8 [5]: *Let T be a tree of order n. Then $F_R(T) = n$ if and only if $T \cong T^*$.*
Theorem 9 [5]: *Let G be a unicyclic graph of order n. Let C be the cycle in G. Then $F_R(G) = n$ if and only if one of the following holds.*

(i) There exists an edge $e = vw$ in C such that $G - e \cong T^*$ where either both v and w are end vertices of $K_{1,ni}$ in T^* for some i or v is an end vertex of $K_{1,ni}$ in T^* and w is an end vertex of $K_{1,nj}$ in T^* for some i and $j, i \neq j$.

(ii) There exists a vertex w in C such that the components $T_1, T_2, \ldots, T_s$ of $G - w$ are isomorphic to T^* where G is obtained by joining w to a head vertex of one component T_i and to end vertices of stars in the other components $T_j (j \neq i)$.

Open problems

- *Graphs with* $\gamma(G) = \gamma_r(G) = \gamma_R(G)$ *can be characterized. Also graphs with* $\gamma_r(G) = \gamma_R(G)$ *can be characterized.*

- *Much work has not been done related to the parameter weak Roman domination number of a graph. (a) Bounds can be obtained in terms of girth and diameter of the graph, (b) Criticality in graphs can be studied in this direction, (c) The idea of efficiency can be extended to weak Roman domination.*

- *Regarding the clique star cover number, graphs with* $\gamma_r(G) = \theta_s(G) = \gamma_R(G)$ *can be characterized.*

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Standard tableaux of bounded height

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The enumeration of standard Young tableaux (SYTs) is a fundamental problem in combinatorics and representation theory. For example, it is known that the number of SYTs of a given shape $\lambda \vdash n$ is counted by the hook-length formula [4]. However, the problem of counting SYTs of bounded height is a hard one. Let $\mathcal{T}_k(n) := \{\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \vdash n : \lambda_1 \geq \ldots \lambda_k \geq 0\}$ be the set of SYTs with n entries and at most k rows, and let $\mathcal{T}_k = \bigcup_{n=1}^{\infty} \mathcal{T}_k(n)$ be the k-rowed strip. In 1981, Regev proved that

$$|\mathcal{T}_2(n)| = \binom{n}{\lfloor \frac{n}{2} \rfloor} \qquad \text{and} \qquad |\mathcal{T}_3(n)| = \sum_{i \geq 0} \frac{1}{i+1} \binom{n}{2i} \binom{2i}{i}$$

in terms of symmetric functions [8]. Note that $|\mathcal{T}_3(n)|$ is exactly the Motzkin number m_n. In 1989, together with $|\mathcal{T}_2(n)|$ and $|\mathcal{T}_3(n)|$, Gouyou-Beauchamps derived that

$$|\mathcal{T}_4(n)| = c_{\lfloor \frac{n+1}{2} \rfloor} c_{\lceil \frac{n+1}{2} \rceil} \qquad \text{and} \qquad |\mathcal{T}_5(n)| = 6 \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2i} c_i \frac{(2i+2)!}{(i+2)!(i+3)!}$$

combinatorially, where $c_n = \frac{1}{n+1}\binom{2n}{n}$ is the Catalan number [6]. These are in fact all the simple formulae we have for $|\mathcal{T}_k(n)|$ so far [10].

Recently Regev considered the following variation among others. Given $\mu = (\mu_1, \mu_2, \mu_3)$ a partition of at most three parts, let $|\mu| := \mu_1 + \mu_2 + \mu_3$ and $\mathcal{T}_3(\mu; n - |\mu|)$ be the set of SYTs with $n - |\mu|$ entries in the "skew strip" $\mathcal{T}_3/\mu$. Regev conjectured that for $\mu = (2, 1, 0)$,

$$|\mathcal{T}_3((2, 1, 0); n - 3)| = m_{n-1} - m_{n-3},$$

a difference of two Motzkin numbers [9]. This conjecture is confirmed by Zeilberger by using the WZ method [1]. What's more, with his powerful Maple package `AMITAI`, Zeilberger could generate and rigorously prove many similar identities, among them are a list of formulae of $|\mathcal{T}_3(\mu; n - |\mu|)|$ for $\mu_1 \leq 20$, and the number of SYTs in $\mathcal{T}_3$ with the restriction that the (i, j) entry is m for $1 \leq m \leq 15$. Amazingly, each formula is a linear combination of negative shifts of the Motzkin numbers with constant coefficients. Zeilberger then asked that, besides Regev's question of finding a combinatorial proof of the $m_{n-1} - m_{n-3}$ conjecture (now a theorem, after Zeilberger), is there a uniform way to construct combinatorial proofs to all of these results, or prove that there is no natural bijection because the identities are true 'just because'.

In the first part of this work we answer Regev and Zeilberger's questions affirmatively. We shall present a simple bijection between $\mathcal{T}_3(n)$ and the set of Motzkin paths of length n, which gives another proof for $|\mathcal{T}_3(n)| = m_n$. With this bijection we can prove Regev's conjecture and consequently all of Zeilberger's identities for three-rowed SYTs combinatorially. This part of work has been published [2].

The second part of this work is to extend the bijection and reveal an (unexpected) relation between $|\mathcal{T}_{2\ell}(n)|$ and $|\mathcal{T}_{2\ell+1}(n)|$. Let $\{\mathbf{e}_1, \dots \mathbf{e}_{\ell+1}\}$ denote the standard basis of $\mathbb{R}^{\ell+1}$ and let $\mathcal{L}_{2\ell+1}(n)$ be the set of n-step lattice paths in $\mathbb{R}^{\ell+1}_{\geq 0}$ from the origin to the axis along $\mathbf{e}_1$, using $2\ell + 1$ kinds of steps $\mathbf{e}_1, \mathbf{e}_1 \pm \mathbf{e}_2, \mathbf{e}_1 \pm (\mathbf{e}_2 - \mathbf{e}_3), \mathbf{e}_1 \pm (\mathbf{e}_3 - \mathbf{e}_4), \dots, \mathbf{e}_1 \pm (\mathbf{e}_\ell - \mathbf{e}_{\ell+1})$. By combining works of Grabiner and Magyar [7] and Gessel [5], Zeilberger proved [13], equivalently, that

$$|\mathcal{T}_{2\ell+1}(n)| = |\mathcal{L}_{2\ell+1}(n)|.$$

The main result is the following, which completes the description of the SYTs with bounded height in terms of lattice paths: Let $\mathcal{L}_{2\ell}(n)$ be the set of lattice paths in $\mathcal{L}_{2\ell+1}(n)$ with the restriction that the $\mathbf{e}_1$ steps appear only on the hyperplane spanned by $\{\mathbf{e}_1, \dots, \mathbf{e}_\ell\}$. Then we have

$$|\mathcal{T}_{2\ell}(n)| = |\mathcal{L}_{2\ell}(n)|.$$

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On the unimodality of independence polynomials of very well-covered graphs

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Given a graph G, let s_k be the number of stable sets (or independence sets) of cardinality k of G, and $\alpha(G)$ the size of a maximum stable set. The *independence polynomial* of G is defined by $I(G;x) = \Sigma_{k=0}^{\alpha} s_k x^k$ [1]. A graph G is *well-covered* if all its maximal stable sets are of the same size [2]. A graph G is *very well-covered* if it has no isolated vertices, it is well-covered, and $|G| = 2\alpha(G)$ [3]. For example, appending a single pendant edge to each vertex of G yields a very well-covered graph, which is denoted by G^*. Under certain conditions, any well-covered graph equals G^* for some G [4].

A finite sequence of real numbers $\{a_0, a_1, a_2, \ldots, a_n\}$ is said to be: (1)*unimodal* if there is some k with $0 \leq k \leq n$, called the *mode* of the sequence, such that $a_0 \leq \ldots \leq a_{k-1} \leq a_k \geq a_{k+1} \geq \ldots a_n$; (2)*log-concave* if $a_i^2 \geq a_{1+1}a_{i-1}$ holds for $1 \leq i \leq n-1$. It is known that any log-concave sequence of positive numbers is also unimodal, but the converse is not true. Any polynomial is *unimodal* if the sequence of its coefficients is unimodal.

The following results will be presented in this talk:
(i) $I(G^*;x)$ is unimodal for any G^* whose skeleton G has $\alpha(G) \leq 8$.
(ii) With (i) and [5], we conjecture that $I(G^*;x)$ is unimodal and $\lceil \frac{n+1}{2} \rceil \leq mode(G^*) \leq min\{\lceil \frac{n+1}{2} \rceil + \frac{\alpha-1}{2}, \lceil \frac{2n-1}{3} \rceil\}$ for any G with $\alpha(G) \geq 3$. Concrete graphs are given to show that the bounds in our conjecture are tight.
(iii) In [6], it is shown that the independence polynomial distinguishes well-covered spiders ($K_{1,n}^*, n \geq 1$) among well-covered trees. We derive a formula for $I(K_{t,n}^*;x)$ for $t \geq 2$, and prove that $I(K_{t,n}^*;x)$ is log-concave for $2 \leq t \leq 5$. A conjecture about the log-concavity of $I(K_{t,n}^*;x)$ for $t \geq 6$ is given based on the formula and many supporting evidences by computer experiments.

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A geometric approach to computing spatial skyline points [1]

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Consider a visit to a new city for a conference! A hotel recommendation system for the city gives us the position of n hotels, located at point sites $P = \{p_1, \dots, p_n\}$. We propose to visit a set of m locations of interest, $S = \{s_1, \dots, s_m\}$, e.g. museum, garden, beach, restaurant, etc. and would like a short list of hotels near these locations. The system need not list any hotel $p \in P$ that is farther from all locations than some other hotel $q \in P$. Consider another problem of similar nature. There is a set of n buildings and multiple fires have broken out at m locations. A fire fighting team wants to identify a subset of buildings that are to be evacuated ahead of other buildings.

It turns out that both the above problems are special cases of a problem considered in the database community [2, 3]. Consider a database whose entries are objects with d attributes of interest. Given two objects p and q, we write $p \succcurlyeq q$ if every attribute of p is larger or equal to the corresponding attribute of q. If $p \succcurlyeq q$ but not $q \succcurlyeq p$, then we say that p *dominates* q. An object is called *non-dominated* or a *skyline* object if it is not dominated by any object in the database. A *skyline query* is the problem of determining the skyline objects in a database with respect to a given set of attributes. Börzsönyi et al. [2] proposed to add a *skyline operator* to solve skyline queries in an existing (relational, object-oriented or object-relational) database system.

Our hotel recommendation and fire fighting problems, which are of geometric nature, fit this framework exactly if we choose the attributes of each point in P to be

[1]This work appeared in [1]

the negative distances to the points in S. Sharifzadeh and Shahabi [3] use the term *spatial skyline query* for this special version of the problem. Son et al. [4] have also looked at this problem.

For a formal definition, let P be a set of n point sites and let S be a set of m points (locations) in $\mathbb{R}^d$. Let $d(x, y)$ be the usual L_2 distance in $\mathbb{R}^d$. For two sites $p, q \in P$ we write $p \succcurlyeq q$ if $d(p, s) \leq d(q, s)$ for every $s \in S$. $p \in P$ *dominates* $q \in P \setminus \{p\}$ simply if $p \succcurlyeq q$. Conversely, p is *not dominated* by q if and only if there is a site $s \in S$ such that $d(p, s) < d(q, s)$. It follows that a site $p \in P$ is a *skyline point* if and only if for every site $q \in P \setminus \{p\}$ there is a site $s_q \in S$ with $d(p, s_q) < d(q, s_q)$.

The distances to the points of S can be considered *attributes* describing the sites of P. A site $p \in P$ dominates $q \in P$ if and only if it is *strictly better* in at least *one* attribute and is at least as good in *all* attributes. In our scenario for a hotel recommendation system, if $p \in P$ is dominated by $q \in P$, then p need not be on the short list of hotels for a tourist visiting S.

Our problem is to extract the skyline points of P with respect to S. Let $h(p, q)$ denote the half-plane containing p that is bounded by the perpendicular bisector of p and q. A brute force approach to identify whether $p \in P$ is a skyline point is to determine, for all $q \neq p$, if at least one site $s \in S$ lies in $h(p, q)$. This takes $\Theta(mn)$ time for each p, giving a total time of $\Theta(mn^2)$.

We use lifting techniques [5] to give several geometric views of dominance and non-dominance problems, involving balls, lower envelopes of cones, and Voronoi diagrams with a convex polygonal distance function (determined by S) and additive weights (determined by P) to come up with the following characterization of *skyline points*.

Result 1 *The skyline or non-dominated points of a set P with respect to locations S are those with non-empty Voronoi cells under a convex distance function determined by S with additive weights determined by P.*

When the sites P and points S are given in the plane, then Result 1 supports especially efficient computation. The disk and cone views imply structures of size $O(nm)$. We are able to show that a Voronoi diagram for convex distance functions with additive weights is an instance of an abstract Voronoi diagram as in Klein et al. [6]. Klein et al. [6] gave a generic randomized incremental algorithm to compute the abstract Voronoi diagram of n sites in expected time $O(n \log n)$. The single primitive operation required by Klein et al. is the computation of the Voronoi diagram of five sites which is a constant time operation. But in our case this constant time operation has to be replaced by the convex distance function computation that takes $O(\log m)$ time. Recall that $S(= O(m))$ determines the convex distance function. Thus, we have an $O(m \log m + n \log n \log m)$-time randomized incremental algorithm to find the non-dominated points.

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New bounds on the average information rate of secret sharing schemes for graph-based weighted threshold access structures

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In a secret sharing scheme, a dealer distributes shares of a secret key among a set of n participants in such a way that only qualified subsets of participants can reconstruct the secret key from the shares they receive from the dealer. The set of qualified subsets is called the access structure of this scheme. The information rate (resp. average information rate) of a secret sharing scheme is the ratio between the size of the secret key and the maximum size (resp. average size) of the shares. In a weighted threshold scheme, each participant has his or her own weight. A subset is qualified if and only if the sum of the weights of participants in the subset is not less than the given threshold. Morillo et al.[11] discussed the schemes for weighted threshold access structure that can be represented by graphs, called k-weighted graphs. They characterized this kind of access structures and derived a result on the information rate. In this talk, we deal with the average information rates of the schemes for these structures. Comparing with the known results, two more sophisticated constructions are presented, each of which has its own advantages and both of them perform very well when n/k is large.

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Acyclic edge coloring of graphs

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1 Introduction

Definition 1 *A proper* edge coloring *of* $G = (V, E)$ *is a map* $c : E \to C$ *(where* C *is the set of available colors) with* $c(e) \neq c(f)$ *for any adjacent edges* e,f*. The minimum number of colors needed to properly color the edges of* G*, is called the chromatic index of* G *and is denoted by* $\chi'(G)$*.*

Definition 2 *A proper edge coloring* c *is called acyclic if there are no bichromatic cycles in the graph. In other words an edge coloring is acyclic if the union of any two color classes induces a set of paths (i.e., linear forest) in* G*. The* acyclic edge chromatic number *(also called* acyclic chromatic index*), denoted by* $a'(G)$*, is the minimum number of colors required to acyclically edge color* G*.*

The primary motivation for our work is the following conjecture by Alon, Sudakov and Zaks [2] (and independently by Fiamcik [6]):

Acyclic Edge Coloring Conjecture: For any graph G, $a'(G) \leq \Delta(G) + 2$.

2 Previous Works

Acyclic Edge Coloring was first studied by Fiamcik [5]. He solved the conjecture for subcubic graphs. His papers were not available in English till recently and hence was unknown. Alon, McDiarmid and Reed [1] introduced it independently and using probabilistic methods proved that $a'(G) \leq 64\Delta$. They also mentioned that the constant 64 could be improved with more careful application of the Lovasz Local Lemma. Later Molloy and Reed showed that $a'(G) \leq 16\Delta$. This is the best known bound currently for arbitrary graphs. The best known constructive bound for general graphs is by Subramanian [12] who showed that $a'(G) \leq 5\Delta(\log \Delta + 2)$.

Though the best known upper bound for general case is far from the conjectured $\Delta + 2$, the conjecture has been shown to be true for some special classes of graphs. Alon, Sudakov and Zaks [2] proved that there exists a constant k such that $a'(G) \leq \Delta + 2$ for any graph G whose girth is at least $k\Delta \log \Delta$. They also proved that $a'(G) \leq \Delta + 2$ for almost all Δ-regular graphs. This result was improved by Nešetřil and Wormald [10] who showed that for a random Δ-regular graph $a'(G) \leq \Delta + 1$.

Determining $a'(G)$ is a hard problem both from a theoretical and from an algorithmic point of view. Even for the simple and highly structured class of complete graphs, the value of $a'(G)$ is still not determined exactly. The difficulty in determining $a'(G)$ for complete graphs could be observed by its equivalence to the *Perfect 1-factorization Conjecture* (This long standing conjecture by Kotzig and others states that, *For any $n \geq 2$, K_{2n} can be decomposed into $2n - 1$ perfect matchings such that the union of any two matchings forms a hamiltonian cycle of K_{2n}.*). It has also been shown by Alon and Zaks [3] that determining whether $a'(G) \leq 3$ is NP-complete for an arbitrary graph G.

3 Our Work

The following are some of our main results:

1. From a result of Burnstein [4], it follows that any subcubic graph can be acyclically edge colored using at most 5 colors. We proved that any non-regular subcubic graph can be acyclically colored using only 4 colors.

2. Muthu,Narayanan and Subramanian [9] proved that $a'(G) \leq \Delta+1$ for outerplanar graphs which is a subclass of 2-degenerate graphs and posed the problem of proving the conjecture for 2-degenerate graphs as an open problem. We proved that 2-degenerate graphs are $\Delta + 1$ colorable.

3. Fiedorowicz, Hauszczak and Narayanan [7] gave an upper bound of $2\Delta+29$ for planar graphs. Independently Hou, Wu, GuiZhen Liu and Bin Liu [8] gave an upper bound of $max(2\Delta - 2, \Delta + 22)$. We improve this upper bound to $\Delta + 12$. In [7], they also gave an upper bound of $\Delta + 6$ for triangle free planar graphs. We improve the bound to $\Delta + 3$.

4. We have also worked on lower bounds. Alon et. al. [2], along with the acyclic edge coloring conjecture, also made an auxiliary conjecture stating that Complete graphs of $2n$ vertices are the only class of regular graphs which require

$\Delta + 2$ colors. We disproved this conjecture by showing infinite classes of regular graphs other than Complete Graphs which require $\Delta + 2$ colors.

Apart from the above mentioned results, our work also contributes to the acyclic edge coloring literature by introducing new techniques like Recoloring, Color Exchange (exchanging colors of adjacent edges), circular shifting of colors on adjacent edges (derangement of colors). These techniques turn out to be very useful in proving upper bounds on the acyclic edge chromatic number.

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The complexity of forbidden subgraph colorings

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The standard version of graph coloring is the problem of partitioning the vertices of a graph G into parts so that each part induces an independent set. The minimum number of parts in any such partition is known as the chromatic number of G and is denoted by $\chi(G)$. Determining $\chi(G)$ has been a very important and well-studied problem in graph theory and has motivated lots of research in structural graph theory. See [6], [10] and [11] for a comprehensive introduction to graph coloring. In particular, the book authored by Jensen and Toft [10] is a very good reference for a detailed introduction to coloring and its variants. Also, tight and exact upper bounds have been obtained for various special classes like planar graphs, partial k-trees, etc.

The algorithmic version of this problem is a notoriously hard problem and has also played an important role in the development of efficient algorithms and also in the field of computational complexity. It is one of the first few problems which were shown to be NP-complete and recently it has also been shown [12] to be unlikely to approximate within a multiplicative factor of $n^{1-\epsilon}$ (for every $\epsilon > 0$).

Grünbaum [9] proposed a variant of vertex coloring where we also require, for every pair (V_i, V_j) of parts, that the induced subgraph $G[V_i \cup V_j]$ is a forest. The minimum number of colors used in any such coloring is known as the acyclic chromatic number of G and is denoted by $a(G)$. However, while $\chi(G) \leq \Delta(G) + 1$ always, $a(G)$ can be as large as $\Delta(G)^{4/3}/\log \Delta(G)$ even for bipartite graphs for which $\chi(G) = 2$ (see [2]).

A more restrictive variant is one in which we require that the union of any two color classes induces a star forest. The associated invariant star chromatic number, denoted by $\chi_s(G)$, is bounded by $O(\Delta^{3/2})$ and can become as large as $\Delta^{3/2}/\log \Delta$, again even if $\chi(G) = 2$ ([2]). Another variant is the distance-2 coloring in which any two vertices sharing a common neighbor are colored differently.

Each of these three variants helps to model practical problems arising in several applications. For example, acyclic coloring models a partitioning problem arising in the computation of Hessians and Jacobians [8]. The distance-2 coloring is closely related to the span of a radio-coloring of a graph [7]. Radiocoloring of a graph has applications in mobile communication. One can also look at the edge analogues of these colorings. For example, an acyclic edge coloring of G is the edge analogue of the acyclic vertex coloring.

Recently, Aravind and Subramanian generalized these (vertex and edge) coloring notions to a generic coloring notion in [2] and [4]. Also results on upper bounds (and their tightness) on the associated chromatic numbers and chromatic indices were obtained. As a result, we not only obtain the above given bounds as special cases of a more general result, we also obtain bounds (several of them tight also) for many new types of constrained colorings.

Given j and a family $\mathcal{F}$ of connected j-colorable graphs, a $(j, \mathcal{F})$-coloring is a proper vertex coloring of G in which the union of *any* j color classes induces a subgraph which is free of any isomorphic copy of any $F \in \mathcal{F}$. The minimum number of colors used in any such coloring is known as the $(j, \mathcal{F})$-chromatic number of G and is denoted by $\chi_{j,\mathcal{F}}(G)$. Upper bounds on this number in terms of maximum degree $\Delta(G)$ were obtained in [2]. By specializing on j and $\mathcal{F}$, we obtain bounds on various constrained colorings with restrictions such as union of any 3 color classes induces a partial k-tree or union of any 2 color classes induces a planar subgraph. For example, using our general bound, it can be shown that a first type of coloring exists using $O(\Delta^{\frac{3k+1}{3k-3}})$ colors and a second type of coloring exists using $O(\Delta^{\frac{8}{7}})$ colors.

In fact, when $j = 2$, one can improve (as shown in [3]) the previous upper bound (of [2]) to $O(\Delta^{\frac{m}{m-1}})$ colors, where m is the minimum number of edges in any member of $\mathcal{F}$. Also, in [2], the tightness (upto a polylogarithmic factor) of this bound was also established. The proofs of the upper and lower bounds were based on probabilistic arguments. In particular, it was shown in [2] that $O(\Delta^{8/7})$ colors suffice to obtain a proper coloring in which the union of any two color classes induces a partial 2-tree.

In [4], tight upper bounds were obtained for constrained edge colorings. The corresponding invariant is known as $(j, \mathcal{F})$-chromatic index and is denoted by $\chi'_{j,\mathcal{F}}(G)$. Using our bounds, it follows that $O(d)$ colors suffice for proper edge colorings with each of the following restrictions such as (i) the union of any 3 color classes should be an outerplanar graph, (ii) the union of any 4 color classes should have treewidth at most 2, (iii) the union of any 5 color classes should be planar, (iv) the union of any 16 color classes should be 5-degenerate.

These coloring notions can be used to obtain upper bounds on other invariants like oriented chromatic numbers [3] and intersection dimensions [5].

In this talk, we will provide a brief outline of some recent results obtained by us on the computational complexity issues of computing these restricted chromatic numbers and associated colorings. We present both positive (that is, efficient algorithms) and also conditional negative results (that is, non-existence of efficient algorithms modulo some complexity assumptions).

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