D-bounded distance-regular graphs

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Notations

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$$\Gamma_i(x) := \{y \in X \mid \partial(x, y) = i\}.$$

Distance-Regular Graphs

$\Gamma = (X, R)$ is distance-regular if and only if for $i \leq D$,

$$\begin{aligned} c_i &:= |C(x, y)| \\ a_i &:= |A(x, y)|, \\ b_i &:= |B(x, y)| \end{aligned}$$

are constants subject to all vertices x, y with $\partial(x, y) = i$, where $C(x, y) = \Gamma_1(x) \cap \Gamma_{i-1}(y)$, $A(x, y) = \Gamma_1(x) \cap \Gamma_i(y)$ and $B(x, y) = \Gamma_1(x) \cap \Gamma_{i+1}(y)$.

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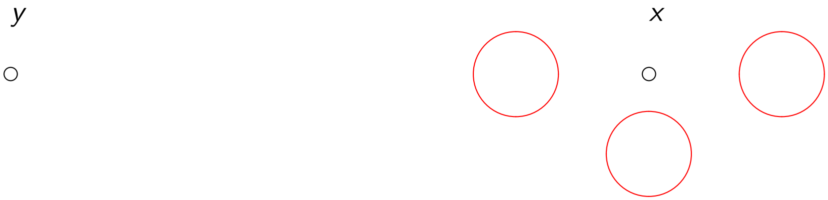

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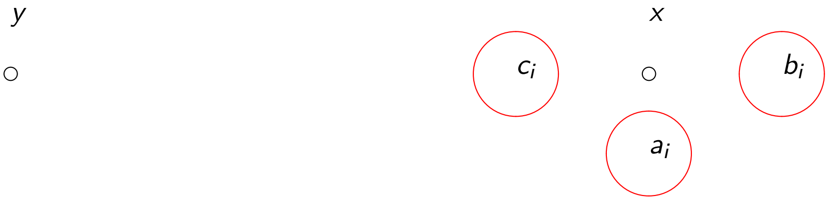

Note that $a_i + b_i + c_i = b_0$ and $k := b_0$ is the valency of Γ.

A distance-regular graph is also called a P-polynomial scheme which is an important and interesting mathematical object, and also plays the role as an underlying combinatorial structure of Coding Theory, Design Theory and Group Theory.

Examples: Hamming Graphs $H(D, 2)$

Set $F_2 = \{0, 1\}$, $X = F_2^D$, and

$$R = \{uv \mid u, v \in X \text{ differ in exact one cordinate}\}.$$

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Examples: Johnson Graphs $J(n, D)$, $2D \leq n$

Set $[n] = \{1, 2, \ldots, n\}$, $X = \binom{[n]}{D}$ (the set of D-subsets of $[n]$) and

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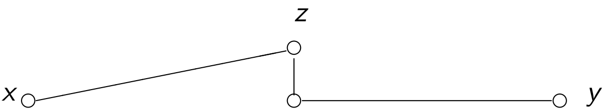
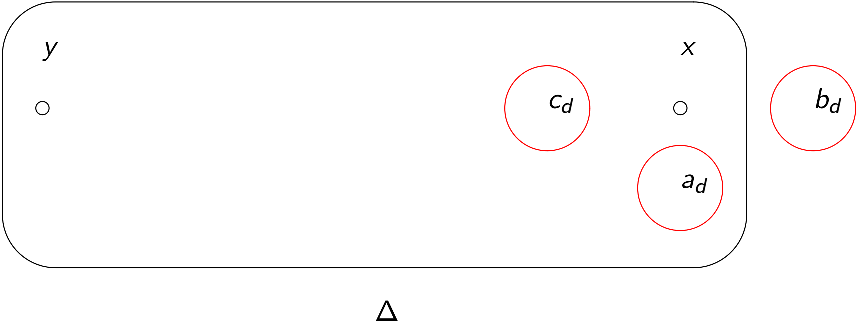


Definition. A subset $\Delta \subseteq X$ is weak-geodetically closed if for any weak-geodetic sequence x, z, y of Γ,

$$x,\ y \in \Delta \Longrightarrow z \in \Delta.$$

Weak-geodetically closed subgraphs are called strongly closed subgraphs in some literature. If a weak-geodetically closed subgraph Δ of diameter d is regular then it has valency $a_d + c_d = b_0 - b_d$, where a_d, c_d, b_0, b_d are intersection numbers of Γ. Furthermore Δ is distance-regular with intersection numbers $a_i(\Delta) = a_i(\Gamma)$ and $c_i(\Delta) = c_i(\Gamma)$ for $1 \le i \le d$.



Definition. Γ is said to be *i-bounded* whenever for all $x, y \in X$ with $\partial(x, y) \leq i$, there is a regular weak-geodetically closed subgraph of diameter $\partial(x, y)$ which contains x and y.

Note that a $(D-1)$-bounded distance-regular graph is clear to be D-bounded. The properties of D-bounded distance-regular graphs were studied in [—, D-bounded distance-regular graphs, European Journal of Combinatorics, 18(1997), 211-229], and these properties were used in the classification of classical distance-regular graphs of negative type [—, Classical distance-regular graphs of negative type, J. Combin. Theory Ser. B, 76(1999), 93-116].

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To state our main theorem we need more definitions.

A parallelogram of length i

A 4-tuple $xywz$ of vertices in X is a parallelogram of length i if $\partial(x, y) = \partial(w, z) = 1$, $\partial(x, w) = \partial(y, w) = \partial(w, z) = i - 1$ and $\partial(x, z) = i$.

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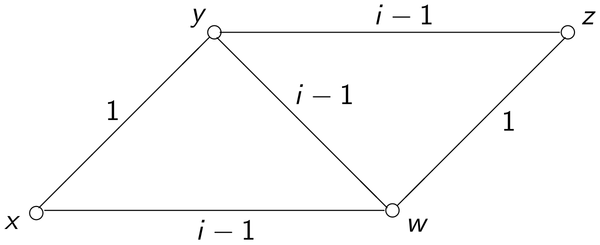

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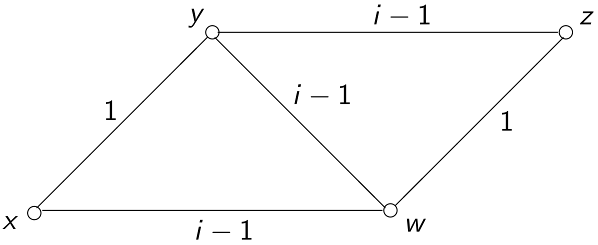

Note that if $c_i=1$ then there is no parallelogram of length i.

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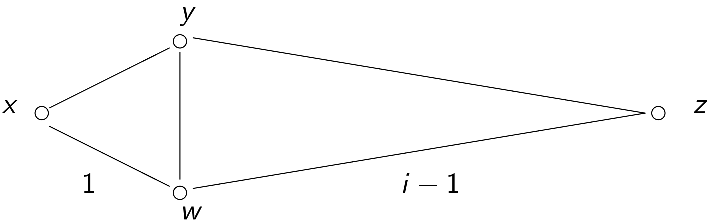

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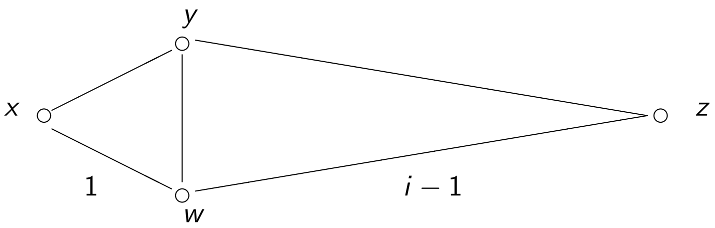

Note that if $c_i=1$ or $a_1=0$ then there is no kite of length i.

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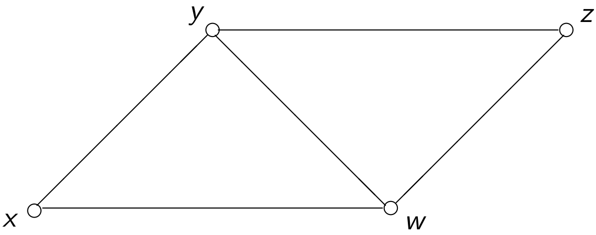

Main Theorem. Let Γ denote a distance-regular graph with diameter $D \geq 3$. Suppose the intersection number $a_2 \neq 0$. Fix an integer $2 \leq d \leq D-1$. Then the following two conditions (i), (ii) are equivalent:

(i) Γ is d-bounded.

(ii) Γ contains no parallelograms of any length up to $d+1$ and $b_1 > b_2$.

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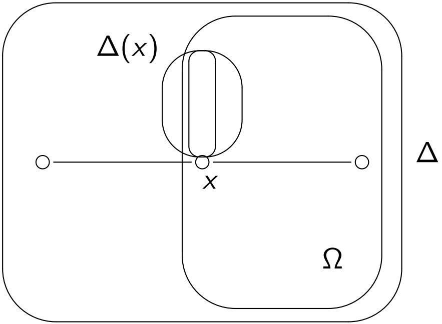
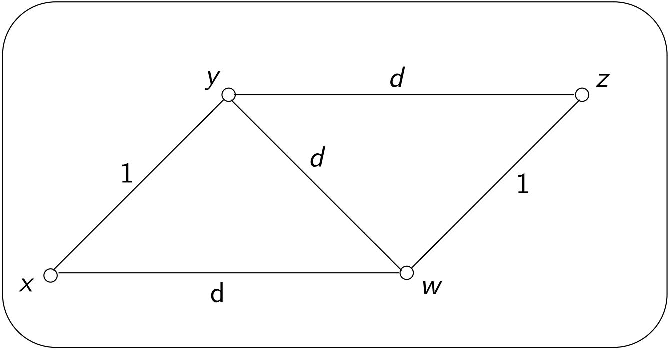


Use $\Omega(x) \subset \Delta(x)$ to obtain $b_0 - b_1 = |\Omega(x)| < |\Delta(x)| = b_0 - b_2$.

The Proof of d-bounded $\Rightarrow$ no parallelogram



$\Delta(x, w)$

If a parallelogram of length $d+1$ exists as shown above, then $x, y, z, w \in \Delta(x, w)$, but $\partial(x, z) = d + 1 > d = \text{diameter}(\Delta(x, w))$.

To prove the other direction "No parallelogram $\Rightarrow$ d-bounded," let's try first to find the nonexistence of many other configurations from the nonexistence of parallelogram.

Lemma

If Γ contains no parallelogram of any length up to $d+1$ then Γ contains no kite of any length up to $d+1$.

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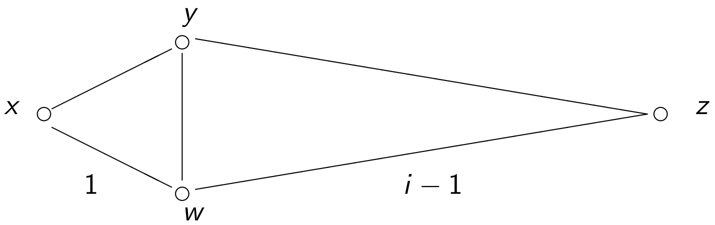

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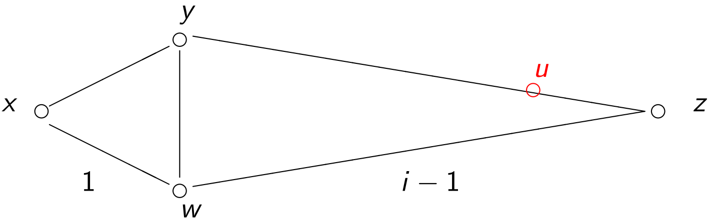

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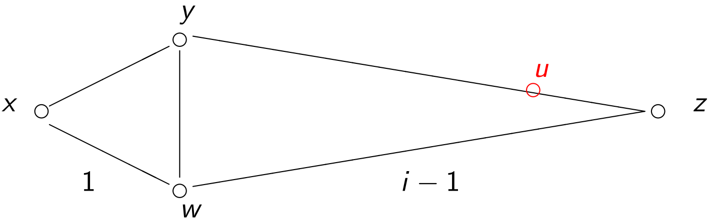
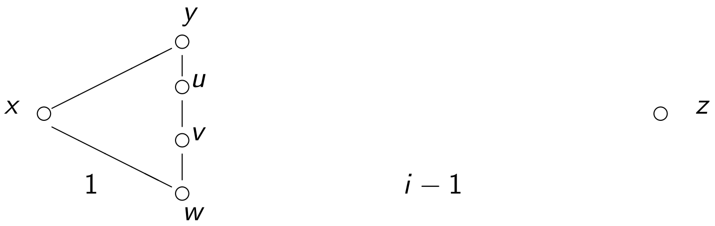


Pick u with $\partial(u, z) = 1$ and $\partial(y, u) = i - 2$. Then $xwuz$ is a parallelogram of length i, a contradiction.

Throughout the talk, we always assume that Γ does not contain parallelogram of any length up to $d+1$.

Non-existence configurations



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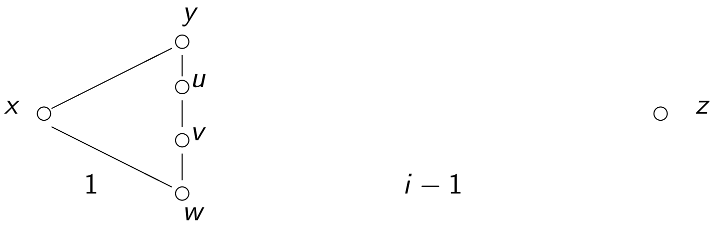

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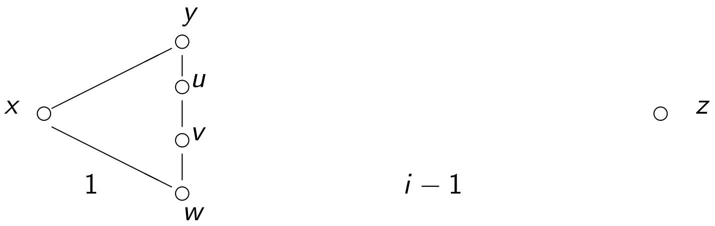

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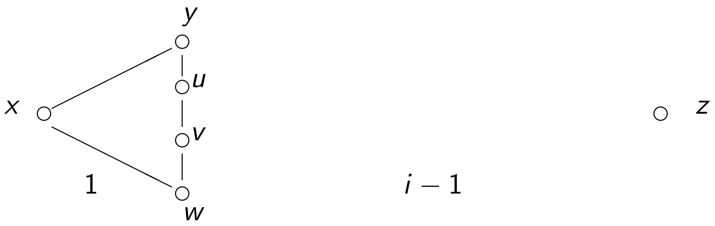

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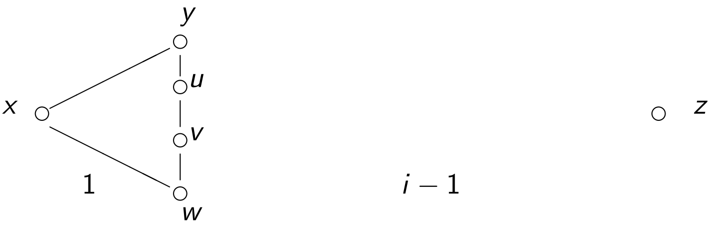

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3. Similarly, $\Gamma_1(z) \cap \Gamma_{i-2}(u) = \Gamma_1(z) \cap \Gamma_{i-2}(y)$.

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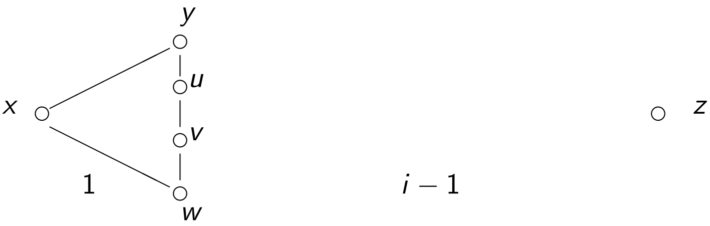

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4. Show $\Gamma_1(z) \cap \Gamma_{i-2}(v) = \Gamma_1(z) \cap \Gamma_{i-2}(y)$ to have a contradiction.

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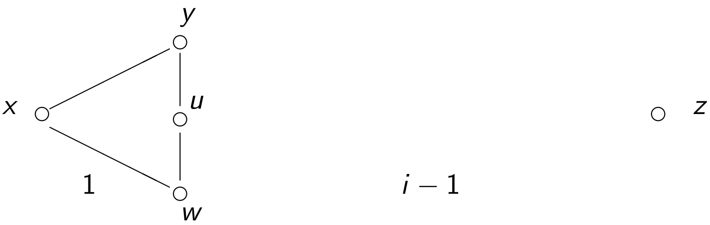

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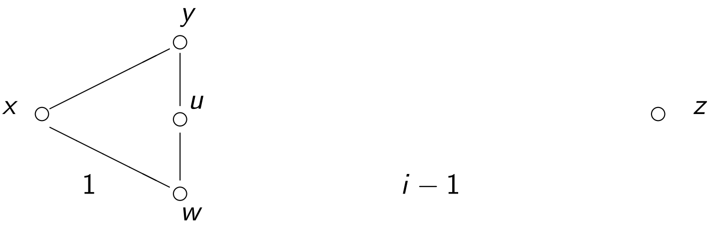
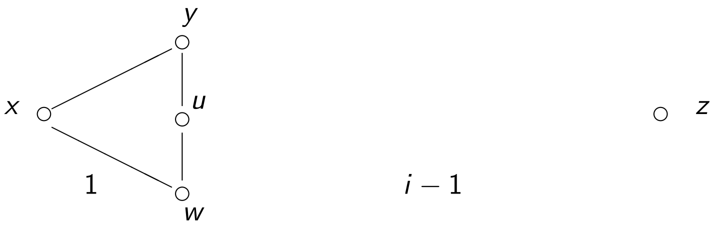


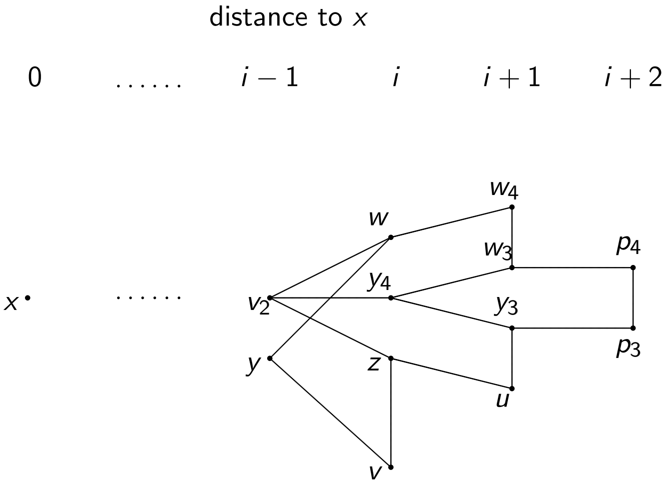
Proof.

Non-existence configurations

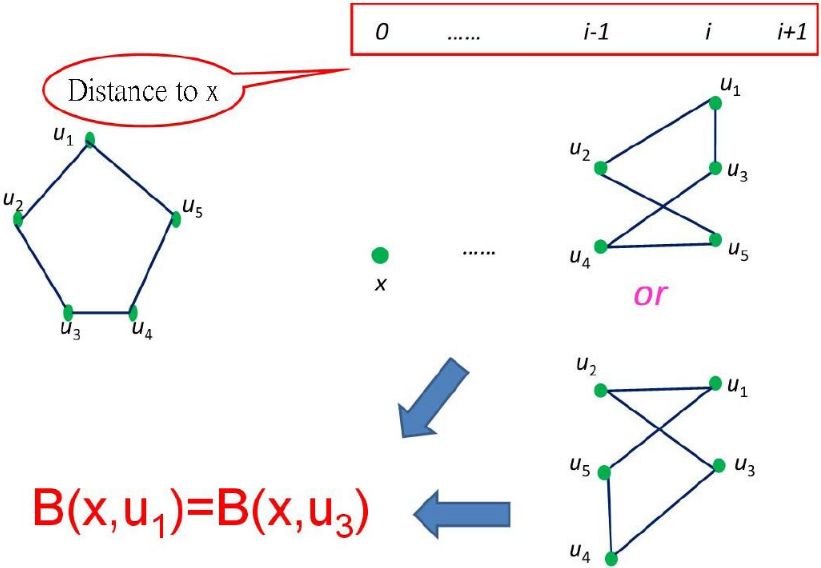


Proof. Skip

Non-existence configurations



From the nonexistence of many configurations, we can show



$B(x,u_1)=B(x,u_3)$

We need a theory to reduce the load of the proof.

Definition. Assume $x \in \Delta \subseteq X$.. The subset Δ is weak-geodetically closed with respect to x if for any weak-geodetic sequence x, z, y of Γ,

$$x,\ y \in \Delta \Longrightarrow z \in \Delta.$$

Theorem

Let Γ be a distance-regular graph with diameter $D \geq 3$. Let Ω be a regular subgraph of Γ with valency γ and set $d := \min\{i \mid \gamma \leq c_i + a_i\}$. Then the following (i),(ii) are equivalent.

(i) *Ω is weak-geodetically closed with respect to at least one vertex $x \in \Omega$.*

(ii) *Ω is weak-geodetically closed with diameter d.*

In this case $\gamma = c_d + a_d$. □

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([Theorem 4.6 in —, Weak-geodetically closed subgraphs in distance-regular graphs, Graphs and Combinatorics, 14(1998), 275-304])

The construction

Definition

For any vertex $x \in X$ and any subset $\Pi \subseteq X$, define $[x, \Pi]$ to be the set

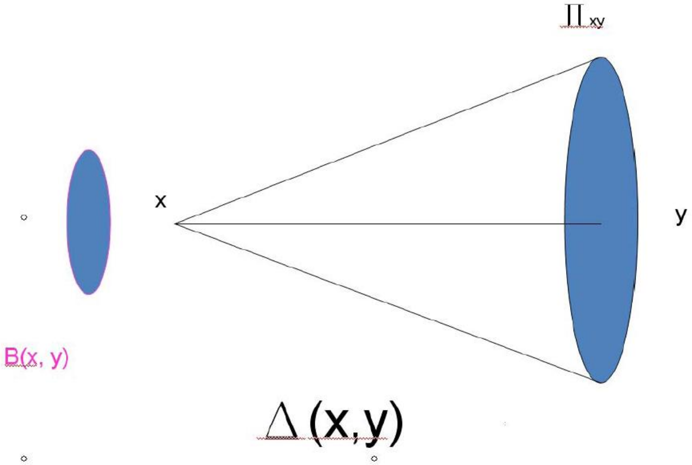
$\{v \in X \mid \text{there exists } y' \in \Pi, \text{ such that the sequence } x, v, y' \text{ is geodetic }\}$.

For any $x, y \in X$ with $\partial(x, y) = d$, set

$$\Pi_{xy} := \{y' \in \Gamma_d(x) \mid B(x, y) = B(x, y')\}$$

and

$$\Delta(x, y) = [x, \Pi_{xy}].$$


∏xy
x
y
B(x, y)
Δ(x,y)

We shall prove that for any vertices $x, y \in X$ with $\partial(x, y) = d$ the following statements W_d, R_d hold.

(W_d) $\Delta(x, y)$ is weak-geodetically closed with respect to x, and

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To prove W_d in the case $c_2 > 1$, we use induction on d and induction on $d - \partial(x, z)$ to show $v \in \Delta(x, y)$ for any $z \in \Delta(x, y)$ and $v \in A(z, x)$ in the following picture.

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The idea is to show $B(x, s) = B(x, u)$ and use this to show $s \in \Delta(x, y)$. Then $v \in \Delta(x, y)$ by the construction.

Precisely, we need to show the following.

Proposition

For any vertices $x, y \in X$ *with* $\partial(x, y) = d$ *and for any vertex* $z \in \Delta(x, y) \cap \Gamma_i(x)$, *where* $1 \leq i \leq d$, *we have the following (i), (ii).*

(i) $A(z, x) \subseteq \Delta(x, y)$.

(ii) *For any vertex* $w \in \Gamma_i(x) \cap \Gamma_2(z)$ *with* $B(x, w) = B(x, z)$, *we have* $w \in \Delta(x, y)$.

In particular (W_d) *holds.*

Precisely, we need to show the following.

Proposition

For any vertices $x, y \in X$ with $\partial(x, y) = d$ and for any vertex $z \in \Delta(x, y) \cap \Gamma_i(x)$, where $1 \leq i \leq d$, we have the following (i), (ii).

(i) $A(z, x) \subseteq \Delta(x, y)$.

(ii) *For any vertex $w \in \Gamma_i(x) \cap \Gamma_2(z)$ with $B(x, w) = B(x, z)$, we have $w \in \Delta(x, y)$.*

In particular (W_d) holds.

The proof is quite technical. To prove (i) we need (ii) to help. The nonexistence of many configurations listed before is used in the proof of (ii).

The following proves R_d.

Proposition

For any vertices $x, y \in X$ *with* $\partial(x, y) = d$, $\Delta(x, y)$ *is regular with valency* $a_d + c_d$.

The following proves R_d.

Proposition

For any vertices $x, y \in X$ *with* $\partial(x, y) = d$, $\Delta(x, y)$ *is regular with valency* $a_d + c_d$.

Proof.

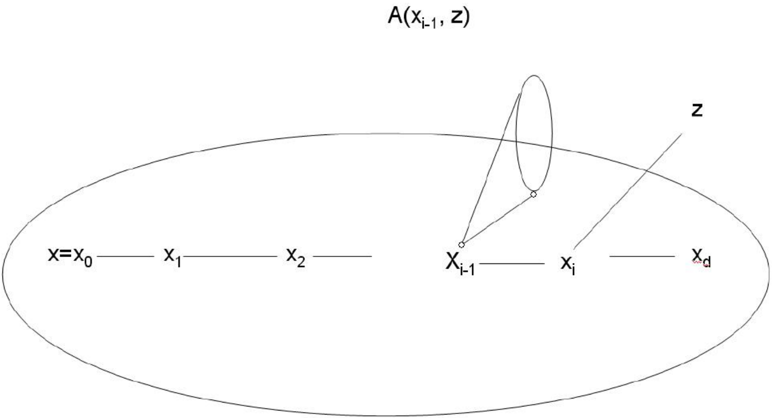
(Sketch) Since each vertex in $\Delta = \Delta(x, y)$ appears in a sequence of vertices $x = x_0, x_1, \ldots, x_d$ in Δ, where $\partial(x, x_j) = j$, $\partial(x_{j-1}, x_j) = 1$ for $1 \leq j \leq d$, and $x_d \in \Pi_{xy}$, it suffices to show

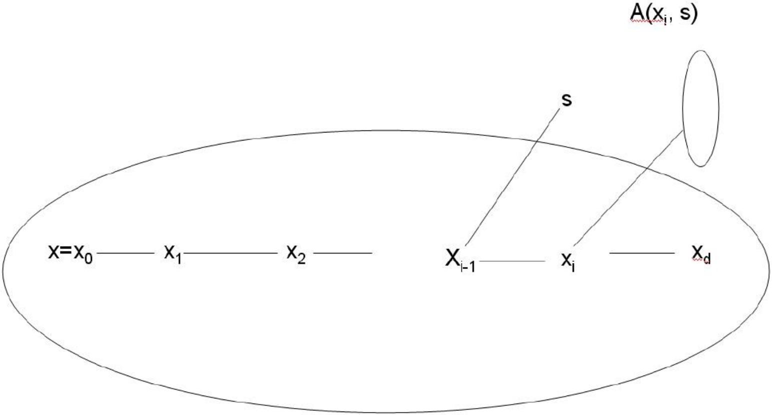
$$|\Gamma_1(x_i) \cap \Delta| = a_d + c_d \tag{1}$$

for $1 \leq i \leq d-1$. We show (1) holds for $i = 0, d$, and for each integer $1 \leq i \leq d$, we use W_d to show

$$|\Gamma_1(x_{i-1}) \setminus \Delta| \leq |\Gamma_1(x_i) \setminus \Delta|.$$

□


A(x_{i-1}, z)
z
x=x_0
x_1
x_2
X_{i-1}
x_i
x_d


A(x_i, s)
s
x=x_0
x_1
x_2
X_{i-1}
x_i
X_d

Thank you for your attention.

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