Regular Graphs with Four Eigenvalues

Chih-wen Weng
Department of Applied Mathematics
National Chiao Tung University at Hsinchu
June 27, 2007

*A joint work with Tayuan Huang(NCTU), Yu-pei Huang(NCTU),
Shu-Chung Liu(NHCUE)

Abstract

Let $\Gamma = (X, R)$ denote a connected k-regular graph with v vertices and four distinct eigenvalues $k > \theta_1 > \theta_2 > \theta_3$. For a vertex $x \in X$, let $k_2(x)$ denote the number of vertices at distance 2 from x. We show there exists a rational function $f(v, k, \theta_1, \theta_2, \theta_3)$ in the variables v, k, θ_1, θ_2, θ_3 such that for any $x \in X$

$$k_2(x) \geq f(v, k, \theta_1, \theta_2, \theta_3).$$

Moreover the above equality holds for all $x \in X$ if and only if Γ is distance-regular with diameter 3.

Abstract

Let $\Gamma = (X, R)$ denote a connected k-regular graph with v vertices and four distinct eigenvalues $k > \theta_1 > \theta_2 > \theta_3$. For a vertex $x \in X$, let $k_2(x)$ denote the number of vertices at distance 2 from x. We show there exists a rational function $f(v, k, \theta_1, \theta_2, \theta_3)$ in the variables v, k, θ_1, θ_2, θ_3 such that for any $x \in X$

$$k_2(x) \geq f(v, k, \theta_1, \theta_2, \theta_3).$$

Moreover the above equality holds for all $x \in X$ if and only if Γ is distance-regular with diameter 3.

Abstract

Let $\Gamma = (X, R)$ denote a connected k-regular graph with v vertices and four distinct eigenvalues $k > \theta_1 > \theta_2 > \theta_3$. For a vertex $x \in X$, let $k_2(x)$ denote the number of vertices at distance 2 from x. We show there exists a rational function $f(v, k, \theta_1, \theta_2, \theta_3)$ in the variables v, k, θ_1, θ_2, θ_3 such that for any $x \in X$

$$k_2(x) \geq f(v, k, \theta_1, \theta_2, \theta_3).$$

Moreover the above equality holds for all $x \in X$ if and only if Γ is distance-regular with diameter 3.

Abstract

Let $\Gamma = (X, R)$ denote a connected k-regular graph with v vertices and four distinct eigenvalues $k > \theta_1 > \theta_2 > \theta_3$. For a vertex $x \in X$, let $k_2(x)$ denote the number of vertices at distance 2 from x. We show there exists a rational function $f(v, k, \theta_1, \theta_2, \theta_3)$ in the variables v, k, θ_1, θ_2, θ_3 such that for any $x \in X$

$$k_2(x) \geq f(v, k, \theta_1, \theta_2, \theta_3).$$

Moreover the above equality holds for all $x \in X$ if and only if Γ is distance-regular with diameter 3.

Outline

We show something more generally and treat the statement in the abstract as a special case.

t-Partially Distance-Regular Graphs

$\Gamma = (X, R)$ is t-PDRG whenever for each $i \leq t$, the i-th distance matrix A_i can be written as a polynomial of the adjacency matrix $A = A_1$ of degree i; that is $A_i = v_i(A)$ for some polynomial $v_i(x) \in \mathbb{R}[x]$ of degree i.

(Algebraic Definition)

Equivalent Conditions

$\Gamma = (X, R)$ is t-PDRG if and only if for $i \leq t$,

$$c_i := |\Gamma_1(x) \cap \Gamma_{i-1}(y)|,$$

$$a_{i-1} := |\Gamma_1(x) \cap \Gamma_{i-1}(z)|,$$

$$b_{i-1} := |\Gamma_1(x) \cap \Gamma_{i-1}(z)|$$

are constants subject to all vertices x, y with $\partial(x, y) = i$ and $\partial(x, z) = i - 1$.

(Combinatorial Definition)

c_i, a_{i-1}, b_{i-1} are called intersection numbers.

t-Partially Walk-Regular

Γ is t-PWR whenever for each integer $1 \leq i \leq t$, $(A^i)_{xx}$ is a constant, not depending on $x \in X$.

The Type of a Closed Walk

For a closed walk $x, x_1, x_2, \ldots, x_i, x$ of length $i + 1$ with base vertex x, we refer the type of the closed walk to be the sequence $\{\partial(x, x_1),\ \partial(x, x_2),\ \ldots,\ \partial(x, x_i)\}$.

Counting the Number of Clsoed Walks

The number of closed walks of type

$$\{1, 2, 3, 3, 2, 3, 2, 1\}$$

base on a fixed vertex is

$$b_0 \times b_1 \times b_2 \times a_3 \times c_3 \times b_2 \times c_3 \times c_2,$$

provided these intersection numbers are well-defined.

Proposition

Using the above counting we have

Proposition 0.1. *If Γ is t-Partially Distance-regular then Γ is $2t$-Partially Walk-Regular.*

The Gosset graph is a distance-regular graph on 56 of diameter 3. A cospetral mate of Gosset graph is obtained by doing "Godsil switching" on edges. This cospectral mate is walk-regular, but not distance-regular.

Question: Can you prove the above proposition algebraically.

Proposition

Using the above counting we have

Proposition 0.2. *If* Γ *is t-Partially Distance-regular then* Γ *is* $2t$*-Partially Walk-Regular.*

The Gosset graph is a distance-regular graph on 56 of diameter 3. A cospetral mate of Gosset graph is obtained by doing "Godsil switching" on edges. This cospectral mate is walk-regular, but not distance-regular.

Question: Can you prove the above proposition algebraically.

Proposition

Using the above counting we have

Proposition 0.3. *If* Γ *is t-Partially Distance-regular then* Γ *is* $2t$*-Partially Walk-Regular.*

The Gosset graph is a distance-regular graph on 56 of diameter 3. A cospetral mate of Gosset graph is obtained by doing "Godsil switching" on edges. This cospectral mate is walk-regular, but not distance-regular.

Question: Can you prove the above proposition algebraically.

Proposition

We also show

Proposition 0.4. *The Godsil switching of a walk-regular graph is still walk-regular.*

Counting the Closed Walks a little longer

$$
\begin{aligned}
& (A^{2t+2})_{xx} \\
= \; & \sum_{y \in X} ((A^{t+1})_{xy})^2 \\
= \; & \sum_{i=0}^{t-1} \sum_{y \in \Gamma_i(x)} ((A^{t+1})_{xy})^2 + \sum_{y \in \Gamma_t(x)} (A^{t+1})_{xy}^2 + \sum_{y \in \Gamma_{t+1}(x)} (A^{t+1})_{xy}^2.
\end{aligned}
$$

The first sum can be determined from intersection numbers if we assume t-PDRG.

Cauchy's Inequality

$$\sum_{y\in\Gamma_t(x)} (A^{t+1})^2_{xy} \quad \geq \quad \frac{1}{k_t(x)}\Big(\sum_{y\in\Gamma_t(x)} (A^{t+1})_{xy}\Big)^2,$$

$$\sum_{y\in\Gamma_{t+1}(x)} (A^{t+1})^2_{xy} \quad \geq \quad \frac{1}{k_{t+1}(x)}\Big(\sum_{y\in\Gamma_{t+1}(x)} (A^{t+1})_{xy}\Big)^2.$$

Equality holds iff the numbers $(A^{t+1})_{xy}$ is independent of y

Suppose Γ is t-PDRG

$$k_t := k_t(x), s_1 := \sum_{i=0}^{t-1} \sum_{y \in \Gamma_i(x)} ((A^{t+1})_{xy})^2$$

can be computed from the intersection numbers (independent of the choice of $x \in X$), and

$$s_2 := \sum_{y \in \Gamma_t(x)} (A^{t+1})_{xy}, s_3 := \sum_{y \in \Gamma_{t+1}(x)} (A^{t+1})_{xy}$$

can be computed from the intersection numbers and an additional constant $(A^{2t+1})_{xx}$.

Reformulate

Assume Γ is t-PDRG. Then

$$(A^{2t+2})_{xx} \geq s_1 + \frac{(s_2)^2}{k_t} + \frac{(s_3)^2}{k_{t+1}(x)}.$$

Theorem

Assume Γ is t-PDRG. Then

$$k_{t+1}(x) \geq \frac{(s_3)^2}{(A^{2t+2})_{xx} - s_1 - (s_2)^2/k_t}.$$

Furthermore Γ is $2(t+1)$-PWR and equality holds for each $x \in X$ if and only if Γ is $(t+1)$-PDRG.

Question: Can you prove this theorem algebraically.

Theorem

Assume Γ is t-PDRG. Then

$$k_{t+1}(x) \geq \frac{(s_3)^2}{(A^{2t+2})_{xx} - s_1 - (s_2)^2/k_t}.$$

Furthermore Γ is $2(t+1)$-PWR and equality holds for each $x \in X$ if and only if Γ is $(t+1)$-PDRG.

Question: Can you prove this theorem algebraically.

Corollary

Assume Γ is 1-PDRG (i.e. k-regular). Then

$$k_2(x) \geq \frac{(k^2 - k - (A^3)_{xx})^2}{(A^4)_{xx} - k^2 - ((A^3)_{xx})^2/k}.$$

Furthermore Γ is 4-PWR and equality holds for each $x \in X$ if and only if Γ is 2-PDRG.

k-Regular Graphs with 4 Eigenvalues

$$A^3 - (\theta_1 + \theta_2 + \theta_3)A^2 + (\theta_2\theta_3 + \theta_3\theta_1 + \theta_1\theta_2)A - \theta_1\theta_2\theta_3$$

$$= \frac{(k-\theta_1)(k-\theta_2)(k-\theta_3)}{|X|}J. \qquad \text{(Hoffman polynomial)}$$

A^3_{xx} is determined from eigenvalues and $|X|$.

$$AJ = kJ.$$

A^4_{xx} is determined from eigenvalues and $|X|$.

k-Regular Graphs with 4 Eigenvalues

$$A^3 - (\theta_1 + \theta_2 + \theta_3)A^2 + (\theta_2\theta_3 + \theta_3\theta_1 + \theta_1\theta_2)A - \theta_1\theta_2\theta_3$$

$$= \frac{(k-\theta_1)(k-\theta_2)(k-\theta_3)}{|X|}J. \qquad \text{(Hoffman polynomial)}$$

A^3_{xx} is determined from eigenvalues and $|X|$.

$$AJ = kJ.$$

A^4_{xx} is determined from eigenvalues and $|X|$.

k-Regular Graphs with 4 Eigenvalues

$$A^3 - (\theta_1 + \theta_2 + \theta_3)A^2 + (\theta_2\theta_3 + \theta_3\theta_1 + \theta_1\theta_2)A - \theta_1\theta_2\theta_3$$

$$= \frac{(k-\theta_1)(k-\theta_2)(k-\theta_3)}{|X|} J. \qquad \text{(Hoffman polynomial)}$$

A^3_{xx} is determined from eigenvalues and $|X|$.

$$AJ = kJ.$$

A^4_{xx} is determined from eigenvalues and $|X|$.

k-Regular Graphs with 4 Eigenvalues

$$A^3 - (\theta_1 + \theta_2 + \theta_3)A^2 + (\theta_2\theta_3 + \theta_3\theta_1 + \theta_1\theta_2)A - \theta_1\theta_2\theta_3$$

$$= \frac{(k-\theta_1)(k-\theta_2)(k-\theta_3)}{|X|}J. \qquad \text{(Hoffman polynomial)}$$

A^3_{xx} is determined from eigenvalues and $|X|$.

$$AJ = kJ.$$

A^4_{xx} is determined from eigenvalues and $|X|$.

Abstract

Let $\Gamma = (X, R)$ denote a connected k-regular graph with v vertices and four distinct eigenvalues $k > \theta_1 > \theta_2 > \theta_3$. For a vertex $x \in X$, let $k_2(x)$ denote the number of vertices at distance 2 from x. We show there exists a rational function $f(v, k, \theta_1, \theta_2, \theta_3)$ in the variables v, k, θ_1, θ_2, θ_3 such that for any $x \in X$

$$k_2(x) \geq f(v, k, \theta_1, \theta_2, \theta_3).$$

Moreover the above equality holds for all $x \in X$ if and only if Γ is distance-regular with diameter 3.

Conjecture

Assume $\Gamma = (X, R)$ is $2t$-partially walk-regular, where t is strictly less than the diameter of Γ. Then there exist a function f of spectrums such that

$$k_2(x) + \cdots + k_t(x) \geq f$$

and equality holds for each $x \in X$ iff Γ is t-PDRG.

Thank You