Construct pooling designs from Hermitian forms graphs

Yu-Pei Huang, Chih-wen Weng(Speaker)
Department of Applied Mathematics
National Chiao Tung University
Hsinchu, Taiwan

June 7, 2007

b-disjunct matrix

Definition 0.1. An $n \times t$ matrix M over $\{0, 1\}$ is b-disjunct if $b < t$ and for any one column j and any other b columns $j_1, j_2, \ldots, j_d$, there exists a row i such that $M_{ij} = 1$ and $M_{ij_s} = 0$ for $s = 1, 2, \ldots, b$.

Example 0.2. A 2-disjunct matrix $M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Relation to Pooling Design

A 4×6 1-disjunct matrix to detect the infected item $\mathbf{C}$ from $\{A, B, \mathbf{C}, D, E, F\}$:

$$\begin{pmatrix} \text{Tests/Items} & A & B & \mathbf{C} & D & E & F & & \text{Output} \\ \text{One} & 1 & 1 & 1 & 0 & 0 & 0 & \rightarrow & 1 \\ \text{Two} & 1 & 0 & 0 & 1 & 1 & 0 & \rightarrow & 0 \\ \text{Three} & 0 & 1 & 0 & 1 & 0 & 1 & \rightarrow & 0 \\ \text{Four} & 0 & 0 & 1 & 0 & 1 & 1 & \rightarrow & 1 \end{pmatrix}$$

Relation to Pooling Design (conti.)

If the size of defected items at most b, then a b-disjunct matrix works for finding the defected items.

Why?

Reason 1. All the subsets of the set of items with size at most b have different outputs.

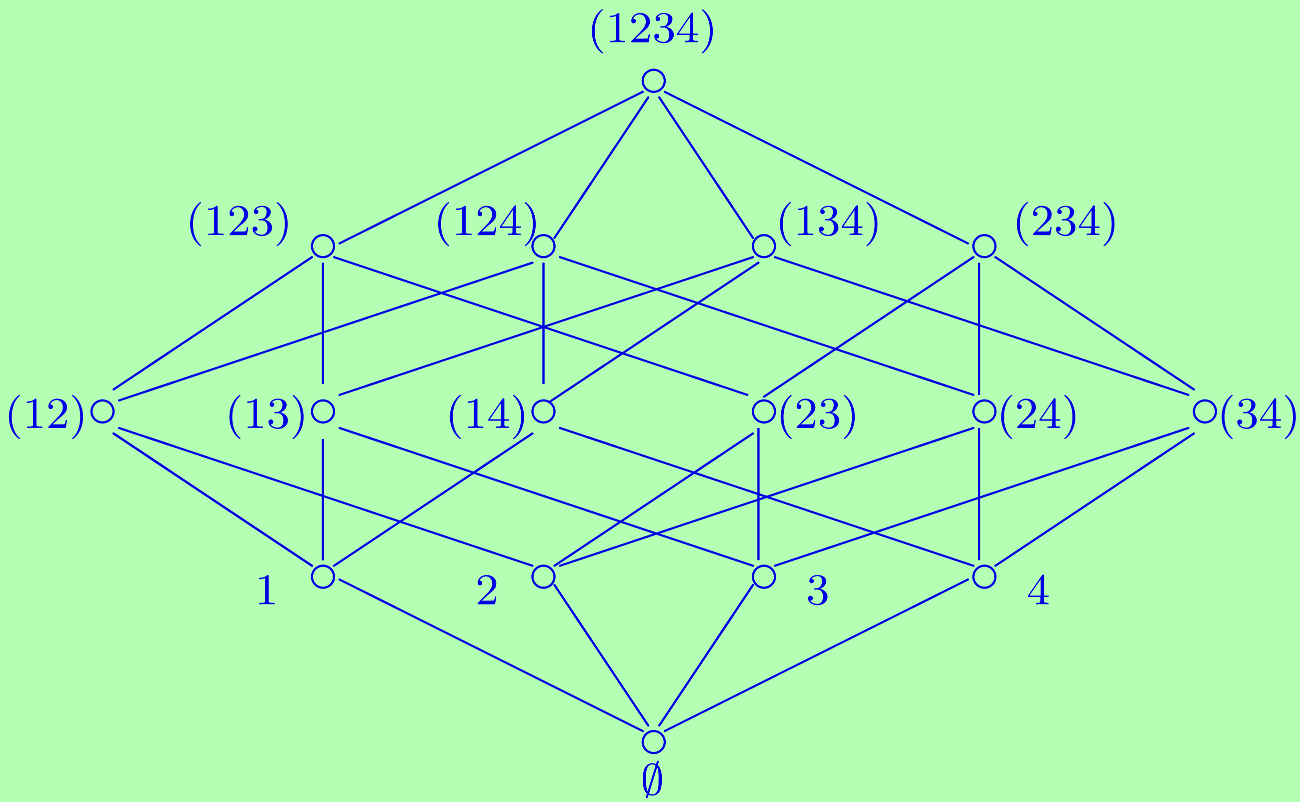
Reason 2. The tests with 0 outputs determine all the non-infected items.

Reason 3. The infected columns of are exactly those columns contained in the output vector (view vectors as subsets of $[n]$).

Construct b-disjunct matrices

Theorem 0.3. *(Macula 1996) Let* $[m] := \{1, 2, \ldots, m\}$. *The incident matrix* W_{bk} *of* b*-subsets and* k*-subsets of* $[m]$ *is an* $\binom{m}{b} \times \binom{m}{k}$ b*-disjunct matrix.*

The subsets of $[m]$ when $m = 4$



$W_{1,2}$ when $m = 4$

$$\begin{pmatrix} \frac{2\text{−subsets}}{1\text{−subsets}} & (12) & (13) & (14) & (23) & (24) & (34) \\ (1) & 1 & 1 & 1 & 0 & 0 & 0 \\ (2) & 1 & 0 & 0 & 1 & 1 & 0 \\ (3) & 0 & 1 & 0 & 1 & 0 & 1 \\ (4) & 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

$(b;d)$-disjunct matrix

Definition 0.4. An $n \times t$ matrix M over $\{0,1\}$ is $(b;d)$-disjunct if $b < t$ and for any one column j and any other b columns $j_1, j_2, \ldots, j_b$, there exist d rows $i_1, i_2, \ldots, i_d$ such that $M_{i_u j} = 1$ and $M_{i_u j_v} = 0$ for $u = 1, 2, \ldots, d$ and $v = 1, 2, \ldots, b$.

A $(b;d)$-disjunct matrix can be used to construct a pooling design that can find the set of defected item of size at most b with $\lfloor \frac{d-1}{2} \rfloor$ errors allowed in the output.

Example of $(b; d)$-disjunct matrix

Theorem 0.5. *(Huang and Weng 2004) Macula's b-disjunct matrix W_{bk} is $(b-1; k-b+1)$-disjunct.*

Posets

Definition 0.6. A poset P is ranked if there exists a function $\text{rank} : P \to \mathbb{N} \cup \{0\}$ such that for all elements $x, y \in P$,

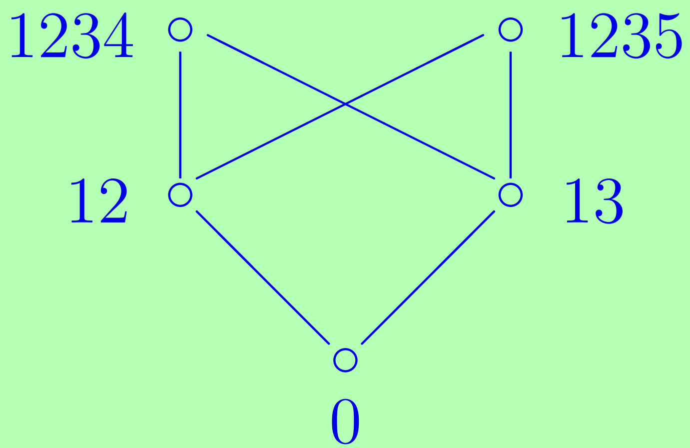
$$y \text{ covers } x \Rightarrow \text{rank}(x) - \text{rank}(y) = 1.$$

Let P_i denote the elements of rank i in P. P is atomic if each elements w is the least upper bound of the set $P_1 \cap \{y \leq w | y \in P\}$.

Pooling Spaces

Definition 0.7. (Huang and Weng 2004) A pooling space is a ranked poset P that the for each element $w \in P$ the subposet induced on $w^+ := \{y \geq w | y \in P\}$ is atomic.

A Nonexample of Pooling Spaces



Every interval in P is atomic, but P is not a pooling space.

b-Disjunct Matrices in Pooling Spaces

Theorem 0.8. *(Huang and Weng 2004) Let P be a pooling space. Then the incident matrix M_{bk} of rank b elements P_b and rank k elements P_k is a b-disjunct matrix. In fact, M_{bk} is $(b'; d_{b'})$-disjunct matrix for some large integer $d_{b'}$ depending on $b' \leq b$ and P. (We can reduce the disjunct value b to increase the error-correctable value d)*

Examples of Pooling Spaces

Hamming matroids, the attenuated spaces, quadratic polar spaces, alternating polar spaces, quadratic polar spaces (two types), Hermitian polar spaces (two types). These are called quantum matroids. More generally, projective and affine geometries, contraction lattices of a graph are also pooling spaces. All these are called geometric lattices.

Hermitian Forms Graphs

Let q denote a prime power, and let H denote the set of $D \times D$ Hermitian matrices over the field $GF(q^2)$. The Hermitian forms graph $Her_q(D) = (X, R)$ is the graph with vertex set $X = H$ and vertices $x, y \in R$ iff $\mathrm{rk}(x - y) = 1$ for $x, y \in X$.

Properties

It is well known that $Her_q(D)$ is distance-regular with diameter D and intersection numbers

$$c_i = \frac{q^{i-1}(q^i - (-1)^i)}{q+1},$$
$$b_i = \frac{q^{2D} - q^{2i}}{q+1}$$

for $0 \leq i \leq D$. Note that

$$|X| = |H| = q^{D^2}.$$

Many Hermitian Forms Graphs

Let $\Gamma = (X, R)$ be the Hermitian forms graph $Her_q(D)$. Then for two vertices $x, y \in X$ with distance t, there exists a subgraph $\Delta(x, y)$ such that $\Delta(x, y)$ is isomorphic to the Hermitian forms graph $Her_q(t)$.

The Poset P

Fix a Hermitian forms graph $\Gamma = Her_q(D)$, and let $P = P(Her_q(D))$ denote the poset consisting of $\Delta(x, y)$ for any $x, y \in X$, and $\Delta \leq \Delta'$ in P iff $\Delta \supseteq \Delta'$ for $\Delta, \Delta' \in P$. Note that Δ is isomorphic to $Her_q(t)$ iff Δ had rank $D - t$ for $\Delta \in P$.

The Binary Matrix $M_q(D, k, r)$

Let P_r and P_k denote the rank r elements and rank k elements of $P = P(Her_q(D))$.

Let $M = M_q(D, k, r)$ denote the incidence matrix of P_r and P_k, i.e. M is a binary matrix with rows and columns indexed by P_r and P_k respectively such that

$$M_{\Omega\Delta} = \begin{cases} 0, & \text{if } \Omega \not\leq \Delta, \text{ (i.e. } \Delta \not\subseteq \Omega); \\ 1, & \text{if } \Omega \leq \Delta, \text{ (i.e. } \Delta \subseteq \Omega). \end{cases}$$

Main Result

Suppose $k - r \geq 2$ and set $p := \dfrac{q^2(q^{2k-2} - 1)}{q^{2k-2r} - 1} + 1$. Then $M_q(D, k, r)$ is $(b; d)$-disjunct of size

$$\begin{bmatrix} D \\ r \end{bmatrix}_{q^2} q^{r(2D-r)} \times \begin{bmatrix} D \\ k \end{bmatrix}_{q^2} q^{k(2D-k)},$$

for any $1 \leq b < p$ and

$$d = q^{2k-2r} \begin{bmatrix} k-1 \\ r-1 \end{bmatrix}_{q^2} - (b-1) q^{2k-2r-2} \begin{bmatrix} k-2 \\ r-1 \end{bmatrix}_{q^2}.$$

A Special Case

Suppose $D \geq k \geq 2$. Then $M_q(D, k, 1)$ is $(q^2;\ q^{2k-4})$-disjunct matrix of size

$$\begin{bmatrix} D \\ 1 \end{bmatrix}_{q^2} q^{(2D-1)} \times \begin{bmatrix} D \\ k \end{bmatrix}_{q^2} q^{k(2D-k)}.$$

The Transpose of $M_q(D, k, r)$

Suppose $k - r \geq 2$. Then the transpose of $M_q(D, k, r)$ is $(b; d)$-disjunct of size

$$\begin{bmatrix} D \\ k \end{bmatrix}_{q^2} q^{k(2D-k)} \times \begin{bmatrix} D \\ r \end{bmatrix}_{q^2} q^{r(2D-r)},$$

where b, d are defined in the next page.

The Transpose of $M_q(D, k, r)$ (conti.)

b is any positive integer such that

$$d = q^{(k-r)(2D-k-r)} \begin{bmatrix} D-r \\ k-r \end{bmatrix}_{q^2} - bq^{(k-r-1)(2D-k-r-1)} \begin{bmatrix} D-r-1 \\ k-r-1 \end{bmatrix}_{q^2}$$

is positive.

Another Special Case

The transpose of $M_q(D, D, D-1)$ is $(q-s; s)$-disjunct of size

$$q^{D^2} \times \begin{bmatrix} D \\ 1 \end{bmatrix}_{q^2} q^{D^2-1},$$

where $1 \leq s \leq q-1$.

Thank You!