More Examples of Pooling Spaces

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June 28, 2005

d-disjunct matrix

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Definition 0.1. An $n \times t$ matrix M over $\{0,1\}$ is d-disjunct if $d < t$ and for any one column j and any other d columns $j_1, j_2, \ldots, j_d$, there exists a row i such that $M_{ij} = 1$ and $M_{ij_s} = 0$ for $s = 1, 2, \ldots, d$.

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Example 0.2. A 2-disjunct matrix $M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Relation to Pooling Design

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A 4×6 1-disjunct matrix to detect the infected item $\mathbf{C}$ from $\{A, B, \mathbf{C}, D, E, F\}$:

$$\begin{pmatrix} \text{Tests/Items} & A & B & \mathbf{C} & D & E & F & & \text{Output} \\ \text{One} & 1 & 1 & 1 & 0 & 0 & 0 & \rightarrow & 1 \\ \text{Two} & 1 & 0 & 0 & 1 & 1 & 0 & \rightarrow & 0 \\ \text{Three} & 0 & 1 & 0 & 1 & 0 & 1 & \rightarrow & 0 \\ \text{Four} & 0 & 0 & 1 & 0 & 1 & 1 & \rightarrow & 1 \end{pmatrix}$$

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Reason 1. All the subsets of the set of items with size at most d have different outputs.

Reason 2. The tests with 0 outputs determine all the non-infected items.

Reason 3. The infected columns of are exactly those columns contained in the output vector (view vectors as subsets of $[n]$).

Construct d-disjunct matrices

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Theorem 0.3. *(Macula 1996) Let* $[m] := \{1, 2, \ldots, m\}$. *The incident matrix* W_{dk} *of* d*-subsets and* k*-subsets of* $[m]$ *is an* $\binom{m}{d} \times \binom{m}{k}$ d*-disjunct matrix.*

The subsets of $[m]$ when $m = 4$

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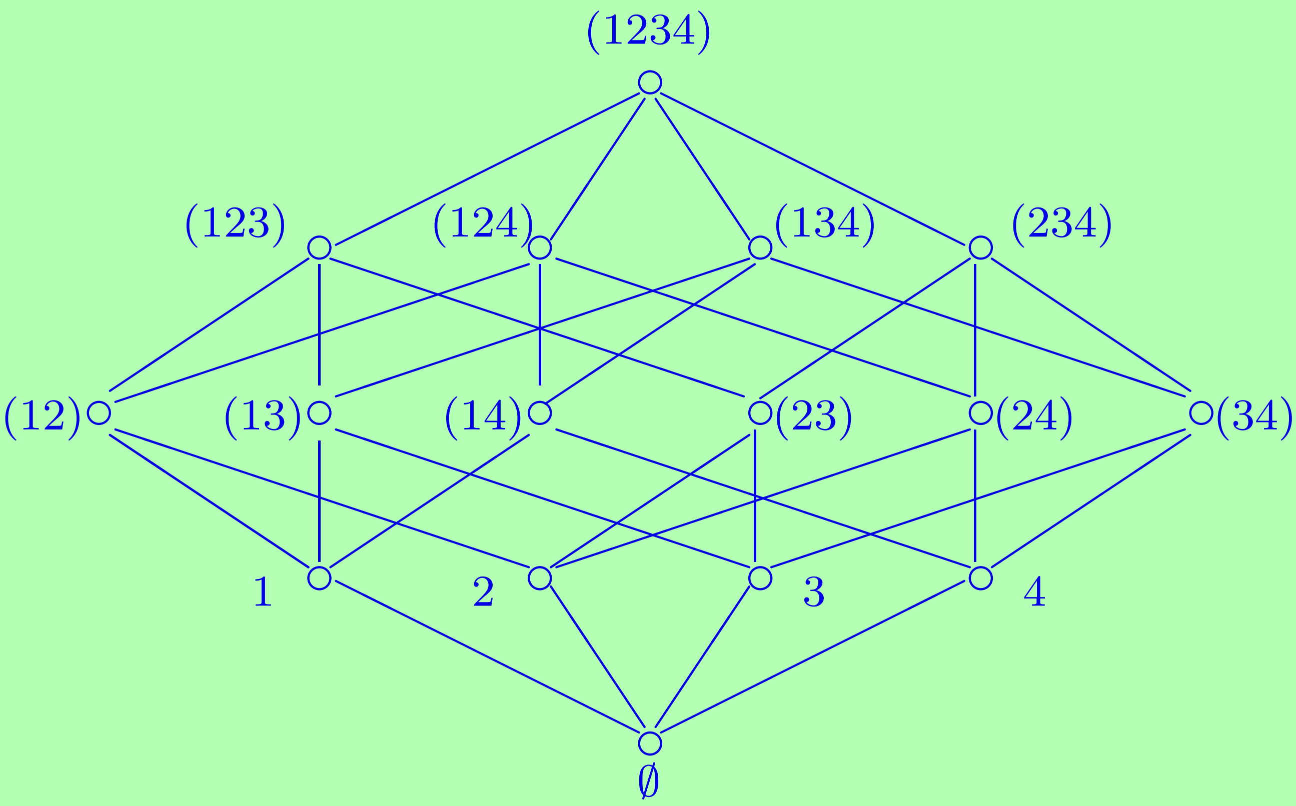

$W_{d,k}$ when $m = 4$

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$$\begin{pmatrix} \frac{2\text{−subsets}}{1\text{−subsets}} & (12) & (13) & (14) & (23) & (24) & (34) \\ (1) & 1 & 1 & 1 & 0 & 0 & 0 \\ (2) & 1 & 0 & 0 & 1 & 1 & 0 \\ (3) & 0 & 1 & 0 & 1 & 0 & 1 \\ (4) & 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

(d, s)-disjunct matrix

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Definition 0.4. An $n \times t$ matrix M over $\{0, 1\}$ is (d, s)-disjunct if $d < t$ and for any one column j and any other d columns $j_1, j_2, \ldots, j_d$, there exist s rows $i_1, i_2, \ldots, i_s$ such that $M_{i_u j} = 1$ and $M_{i_u j_v} = 0$ for $u = 1, 2, \ldots, s$ and $v = 1, 2, \ldots, d$.

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A (d, s)-disjunct matrix can be used to construct a pooling design that can find the set of defected item of size at most d with $\lfloor \frac{s-1}{2} \rfloor$ errors allowed in the output.

As an error-correcting code

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Remark 0.5. *Let M be an $n \times t$ (d, s)-disjunct matrix over $\{0, 1\}$. Let C denote the set consisting of all the boolean sum of at most d columns of M. Then $C \subseteq F_2^n$ has cardinality* $\binom{t}{0} + \binom{t}{1} + \cdots + \binom{t}{d}$ *and minimum distance s.*

Decoding algorithm

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Theorem 0.6. *(Huang and Weng 2003) Let* M *be an* $n \times t$ (d, s)*-disjunct matrix over* $\{0, 1\}$*. Suppose the output vector* O *has at most* $\lfloor \frac{s-1}{2} \rfloor$ *errors. Then a column of* M *with at most* $\lfloor \frac{s-1}{2} \rfloor$ *elements not in* O *is an infected column.*

Example of (d, s)-disjunct matrix

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Theorem 0.7. *(Huang and Weng 2004) Macula's d-disjunct matrix W_{dk} is $(d-1, k-d+1)$-disjunct.*

Posets

Posets

Definition 0.8. A poset P is ranked if there exists a function $\text{rank} : P \to \mathbb{N} \cup \{0\}$ such that for all elements $x, y \in P$,

$$y \text{ covers } x \Rightarrow \text{rank}(x) - \text{rank}(y) = 1.$$

Let P_i denote the elements of rank i in P. P is atomic if each elements w is the least upper bound of the set $P_1 \cap \{y \leq w | y \in P\}$.

Pooling Spaces

Pooling Spaces

Definition 0.9. A pooling space is a ranked poset P that the for each element $w \in P$ the subposet induced on $w^+ := \{y \geq w | y \in P\}$ is atomic.

A Nonexample of Pooling Spaces

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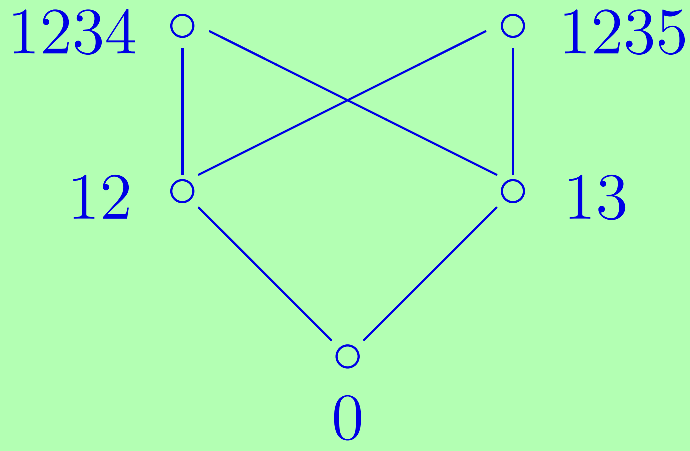

Every interval in P is atomic, but P is not a pooling space.

d-disjunct matrices in Pooling Spaces

d-disjunct matrices in Pooling Spaces

Theorem 0.10. *(Huang and Weng 2004) Let P be a pooling space. Then the incident matrix P_{dk} of rank d elements P_d and rank k elements P_k is a d-disjunct matrix. In fact, P_{dk} is $(d', s_{d'})$-disjunct matrix for some large integer $s_{d'}$ depending on $d' \leq d$ and P.*

Examples of Pooling Spaces

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Hamming matroids, the attenuated spaces, quadratic polar spaces, alternating polar spaces, quadratic polar spaces (two types), Hermitian polar spaces (two types). These are called quantum matroids.

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2. Fix a graph G. The partitions of the vertices of G with connected blocks, ordered by refinement.

Note. 1 is the special case of 2 with G the complete graph.

Connected partitions of the 4-cycle

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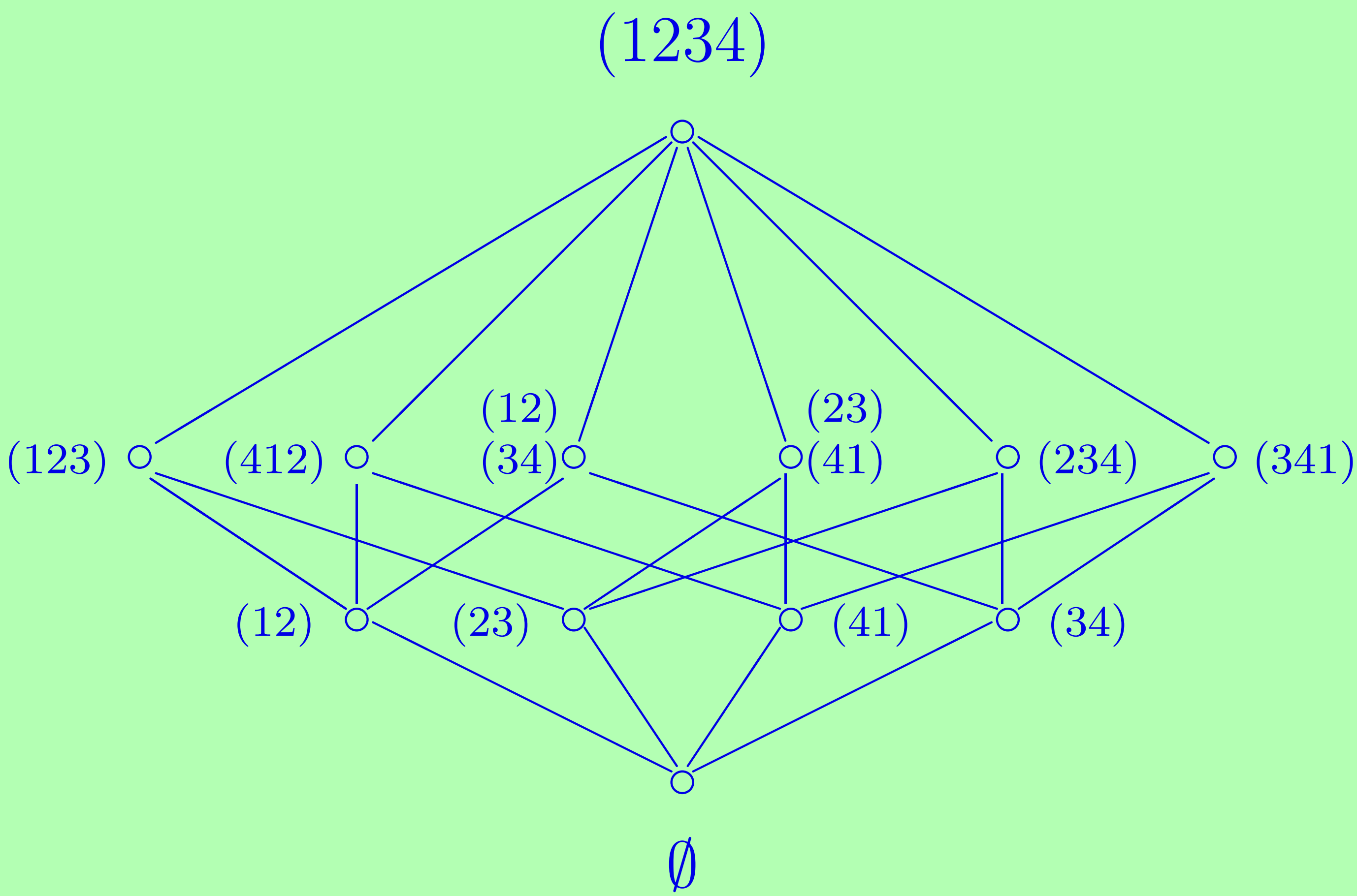

Combinatorial Geometry

Definition 0.11. A combinatorial geometry is a pair $(X, \mathcal{F})$ where X is a set of points and where $\mathcal{F}$ is a family of subsets of X called flats such that

(1) $\mathcal{F}$ is closed under intersection;

(2) $\emptyset$, X, $\{x\} \in \mathcal{F}$ for all $x \in X$;

(3) For $E \in \mathcal{F}$, $E \neq X$, the flats that cover E in $\mathcal{F}$ partition the remaining points.

Combinatorial Geometry is a Pooling Space

Theorem 0.12. *Let P be a combinatorial geometry. Then $(P, \subseteq)$ is a pooling space.*

The end

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Thank You!