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A Note on the Largest Real Eigenvalue of a Nonnegative Matrix

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Abstract

Let C be a nonnegative square matrix of order n and fix a 2-subset $\{a, b\}$ of $\{1, 2, \ldots, n\}$. Assume that the principal submatrix $C[a, b]$ of C restricted to $\{a, b\}$ is symmetric with constant diagonals. Let $C' = (c'_{ij})$ be a nonnegative square matrix of order n constructed from $C = (c_{ij})$ such that $C'[a, b] = C[a, b]$, $C'[[n] - \{a, b\}] = C[[n] - \{a, b\}]$, and for the remaining entries, $c'_{ib} = c_{ib} - t_i$, $c'_{ia} = c_{ia} + t_i$, $c'_{bj} = c_{bj} - s_j$ and $c'_{aj} = c_{aj} + s_j$ for some prescribed parameters t_i and s_j satisfying $\max(0, c_{ib} - c_{ia}) \leq t_i \leq c_{ib}$ and $\max(0, c_{bj} - c_{aj}) \leq s_j \leq c_{bj}$, where $i, j \in [n] - \{a, b\}$. We will show that the largest real eigenvalue of C is at most the largest real eigenvalue of C'.

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1 Introduction

The Kelmans transformation of an undirected graph was defined by A.K. Kelmans in 1981 [3]. We will generalize it to a matrix version.

Let $C = (c_{ij})$ be a nonnegative square matrix of order n and use the notation $[n] = \{1, 2, \ldots, n\}$. For $\alpha \subseteq [n]$, let $C[\alpha]$ denote the principal submatrix of C restricted to the entries in $\alpha \times \alpha$. Fix a 2-subset $\{a, b\}$ of $[n]$, and assume that C is *symmetric* on $\{a, b\}$, that is

$$C[a,b] = \begin{pmatrix} s & t \\ t & s \end{pmatrix} \tag{1}$$

for some scalars s, t. For every $i, j \in [n] - \{a, b\}$, choose t_i and s_j such that $\max(0, c_{ib} - c_{ia}) \leq t_i \leq c_{ib}$ and $\max(0, c_{bj} - c_{aj}) \leq s_j \leq c_{bj}$. We define a new matrix C' of order n from C by shifting the portion t_i of c_{ib} to c_{ia} and the portion s_j of c_{bj} to c_{aj} such that in the new matrix $C' = (c'_{ij})$ have $c'_{ia} \geq c'_{ib}$ and $c'_{aj} \geq c'_{bj}$, where $i, j \in [n] - \{a, b\}$. The following is an illustration of

$$C' = \begin{array}{c} \\ i \\ \\ a \\ b \end{array} \begin{array}{c} \begin{array}{ccc} j & a & b \end{array} \\ \left[\begin{array}{ccc} & & \\ * & c_{ia} + t_i & c_{ib} - t_i \\ & & \\ c_{aj} + s_j & s & t \\ c_{bj} - s_j & t & s \end{array}\right] \end{array} \qquad (i, j \in [n] - \{a, b\}).$$

Formally C' is defined from C by setting $C'[a, b] = C[a, b]$, $C'[[n] - \{a, b\}] = C[[n] - \{a, b\}]$, $c'_{ib} = c_{ib} - t_i$, $c'_{ia} = c_{ia} + t_i$, $c'_{bj} = c_{bj} - s_j$ and $c'_{aj} = c_{aj} + s_j$ for $i, j \in [n] - \{a, b\}$. The matrix C' is referred to as the *Kelmans transformation of C from b to a with respect to t_i and s_j.*

In the above setting, if $C = (c_{ij})$ is the adjacency matrix of an undirected graph G of order n (i.e., C is a symmetric binary matrix with zero diagonals), $t_i = \max(0, c_{ib} - c_{ia})$ and $s_j = \max(0, c_{bj} - c_{aj})$, then the Kelmans transformation C' of C from b to a with respect to t_i and s_j is essentially the Kelmans transformation of G defined by A.K. Kelmans.

It is well known that a nonnegative matrix has a real eigenvalue. P. Csikvári proved that the largest real eigenvalue will not be decreased after an Kelmans transformation of an undirected graph [2]. His method is by the Rayleigh quotient and can be directly extended to any symmetric matrices. Our main theorem is a generalization of this result to a nonnegative matrix which is not necessary to be symmetric.

Theorem 1.1. *Let $C = (c_{ij})$ denote a nonnegative square matrix of order n such that C is symmetric on $\{a, b\}$ for some $1 \leq a, b \leq n$. For every pair $i, j \in [n] - \{a, b\}$, choose t_i, s_j such that* $\max(0, c_{ib} - c_{ia}) \leq t_i \leq c_{ib}$ *and* $\max(0, c_{bj} - c_{aj}) \leq s_j \leq c_{bj}$. *Let C' be the Kelmans transformation from b to a with respect to t_i and s_j. Then the largest real eigenvalue of C is no larger than the largest real eigenvalue of C'.*

The symmetric condition for C on $\{a, b\}$ in Theorem 1.1 is necessary by the following counterexample. Consider the 4×4 matrices

$$C = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}, \quad C' = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

The matrix C' is obtained from C by applying Kelmans transformation from 4 to 3 with respect to $t_1 = 0$, $t_2 = 1$ and $s_1 = s_2 = 0$, while the matrix C is not symmetric on $\{3, 4\}$. By direct computing, the largest real eigenvalue of C is closed to 2.234 which is strictly greater than 2.148, the approximate of the largest real eigenvalue of C'.

It is well-known that the largest real eigenvalue of a nonnegative matrix is associated with a nonnegative eigenvector, called *Perron vector*. Moreover if the matrix is irreducible then its Perron vector is positive (e.g., [1, Theorem 2.2.1]). Our proof of Theorem 1.1, which will be in the section after the preliminaries, utilizes Perron vectors in a way inspired by [4].

2 Preliminaries

In the section, we shall introduce a few basis properties of Kelmans transformation of nonnegative matrices for later use. For a nonnegative square matrix M, let $\rho(M)$ denote the largest real eigenvalue of M. For two matrices M, N of the same size, we use the notation $M \leq N$ if $N - M$ is a nonnegative matrix. Let I_n denote the identity matrix of order n and E_{ij} denote the binary matrix of order n which has a unique 1 in the position ij. Note that $(I_n + E_{ij})^{-1} = I_n - E_{ij}$ and $((I_n + E_{ij})^{-1})^t = ((I_n + E_{ij})^t)^{-1}$.

Lemma 2.1. *Let $C = (c_{ij})$ denote a nonnegative square matrix of order n such that C is symmetric on $\{a, b\}$ for some $1 \leq a, b \leq n$. For every pair $i, j \in [n] - \{a, b\}$, choose t'_i, s'_j such that $\max(0, c_{ib} - c_{ia}) \leq t'_i \leq c_{ib}$ and $\max(0, c_{bj} - c_{aj}) \leq s'_j \leq c_{bj}$, and set $t''_i = c_{ia} - c_{ib} + t_i$ and $s''_j = c_{aj} - c_{bj} + s_j$. Then the following (i)-(iii) hold.*

- *(i)* $\max(0, c_{ia} - c_{ib}) \leq t''_i \leq c_{ia}$ *and* $\max(0, c_{aj} - c_{bj}) \leq s''_j \leq c_{aj}$*;*
- *(ii) If C' (resp. C'') is the Kelmans transformation from b to a (resp. from a to b) with respect to t'_i and s'_j (resp. with respect to t''_i and s''_j), then $\rho(C') = \rho(C'')$;*
- *(iii) As the notation C' in (ii), we have $(I_n + E_{ba})C(I_n + E_{ba})^t \leq (I_n + E_{ba})C'(I_n + E_{ba})^t$.*

Proof. (i) Since $t''_i = c_{ia} - c_{ib} + t'_i$ and

$$\max(0, c_{ia} - c_{ib}) = c_{ia} - c_{ib} + \max(0, c_{ib} - c_{ia}) \leq c_{ia} - c_{ib} + t'_i \leq c_{ia},$$

we have $\max(0, c_{ia} - c_{ib}) \leq t''_i \leq c_{ia}$. Similarly, we have $\max(0, c_{aj} - c_{bj}) \leq s''_j \leq c_{aj}$.

(ii) From the definition of $C' = (c'_{ij})$ and $C'' = (c''_{ij})$, we know $C''[a, b] = C'[a, b]$, $C''[[n] - \{a, b\}] = C'[[n] - \{a, b\}]$,

$$\begin{aligned} c''_{ib} =& c_{ib} + t''_i = c_{ib} + c_{ia} - c_{ib} + t_i = c'_{ia} \\ c''_{ia} =& c_{ia} - (c_{ia} - c_{ib} + t_i) = c'_{ib}, \end{aligned}$$

and similarly $c''_{bj} = c'_{aj}$, $c''_{aj} = c'_{bj}$ for $i, j \in [n] - \{a, b\}$. This shows that $C'' = P^{-1}C'P$, where $P = I - E_{aa} - E_{bb} + E_{ab} + E_{ba}$. Thus $\rho(C') = \rho(C'')$.

(iii) The Kelmans transformation from b to a moves a nonnegative portion of row b of C to row a, but the multiplication of $(I_n + E_{ba})$ from the left will add the whole column a into the column b. Similarly for the column part. Hence (iii) follows. □

3 The proof of Theorem 1.1

Let C and C' be as described in the assumption of Theorem 1.1. To prove $\rho(C) \leq \rho(C')$ we might assume $s \geq t$ in (1), since if we know this and $s < t$ then applying the matrix $C + (t - s)I_n$ to as the role of C and considering the corresponding Kelmans transformation $C' + (t - s)I_n$, we still have

$$\rho(C) = \rho(C + (t - s)I_n) - (t - s) \leq \rho(C' + (t - s)I_n) - (t - s) = \rho(C').$$

Let row vector $w^t = (w_i)$ denote the left Perron vector for $\rho(C)$ of C.

We first assume $w_a \geq w_b$. Set $v^t = w^t Q^{-1}$, where $Q = I_n + E_{ba}$. Thus

$$v^t QC = w^t C = \rho(C)w^t = \rho(C)v^t Q. \tag{2}$$

Note that v is nonnegative since $v_i = w_i \geq 0$ for $i \neq a$ and $v_a = w_a - w_b \geq 0$.

For $\epsilon > 0$, let $C'^{\epsilon} = C' + \epsilon J_n$, where J_n is the matrix of order n with entries all 1's. By Lemma 2.1(iii) and using $Q^{-1}C'Q^t \leq Q^{-1}C'^{\epsilon}Q^t$, we have

$$QCQ^t \leq QC'^{\epsilon}Q^t. \tag{3}$$

From the constriction of C'^{ϵ} and the assumption $s \geq t$ in the beginning, the matrix $(Q^t)^{-1}C'^{\epsilon}Q^t$ is nonnegative. Let u^{ϵ} denote a right Perron vector for $(Q^t)^{-1}C'^{\epsilon}Q^t$. Since C'^{ϵ} and $(Q^t)^{-1}C'^{\epsilon}Q^t$ are similar, we have $\rho(C'^{\epsilon}) = \rho((Q^t)^{-1}C'^{\epsilon}Q^t)$ and $(Q^t)^{-1}C'^{\epsilon}Q^t u^{\epsilon} = \rho(C'^{\epsilon})u^{\epsilon}$, which implies

$$C'^{\epsilon}Q^t u^{\epsilon} = \rho(C'^{\epsilon})Q^t u^{\epsilon}. \tag{4}$$

Because of the irreducibility of C'^{ϵ}, $Q^t u^{\epsilon} > 0$.

Multiplying the nonnegative vector u^{ϵ} from the right to both terms of (3) and applying (4),

$$QCQ^t u^{\epsilon} \leq QC'^{\epsilon} Q^t u^{\epsilon} = \rho(C'^{\epsilon}) QQ^t u^{\epsilon}. \tag{5}$$

Multiplying the nonnegative row vector v^t from the left to the first and last terms in (5) and using (2), we have

$$\rho(C) v^t QQ^t u^{\epsilon} = v^t QCQ^t u^{\epsilon} \leq \rho(C'^{\epsilon}) v^t QQ^t u^{\epsilon}. \tag{6}$$

As $w^t = v^t Q$ nonnegative and $Q^t u^{\epsilon}$ positive, $v^t QQ^t u^{\epsilon}$ is positive. Deleting the positive term $v^t QQ^t u^{\epsilon}$ from both sides of (6), we have $\rho(C) \leq \rho(C'^{\epsilon})$ for any $\epsilon > 0$ and by continuity

$$\rho(C) \leq \lim_{\epsilon \to 0^+} \rho(C'^{\epsilon}) = \rho(C').$$

Next assume $w_b \geq w_a$. Let C'' denote the Kelmans transformation of C from a to b with respect to $c_{ia} - c_{ib} + t_i$ and $c_{aj} - c_{bj} + s_j$. By the previous case, we have $\rho(C) \leq \rho(C'')$, and by Lemma 2.1(ii), $\rho(C'') = \rho(C')$. Hence $\rho(C) \leq \rho(C')$. □

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