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Comment

Corrigendum of a Theorem on New Upper Bounds for the α-Indices. Comment on Lenes et al. New Bounds for the α-Indices of Graphs. *Mathematics* 2020, *8*, 1668

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Abstract: We give a family of counterexamples of a theorem on a new upper bound for the α-indices of graphs in the paper that is mentioned in the title. We also provide a new upper bound for corrigendum.

Keywords: nonnegative matrix; graph; index; α-index

MSC: 05C50; 15A42

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The Statement, Counterexamples, and Corrigendum

Let C be an $n \times n$ real symmetric matrix. The *index* of C, denoted by $\rho(C)$, is the largest eigenvalue of C. Let $G = (V, E)$ be a connected graph of order $n = |V|$ and size $m = |E|$ with adjacency matrix $A(G)$ and diagonal matrix $D(G)$ of degree sequence. Nikiforov [1] proposed the following matrix:

$$A_\alpha(G) = \alpha D(G) + (1-\alpha)A(G),$$

where $0 \le \alpha \le 1$. The α-*index* of G, denoted by $\rho_\alpha(G)$, is the index of $A_\alpha(G)$. E. Lenes, E. Mallea-Zepeda, and J. Rodríguez [2] (Theorem 4) gave the following upper bound for $\rho_\alpha(G)$.

$$\rho_\alpha(G) \le \frac{\delta - 1 + \alpha + \sqrt{(\delta+1-\alpha)^2 + 4(2m - n\delta)(1-\alpha)}}{2}, \tag{1}$$

where δ is the minimum degree of G.

The upper bound of $\rho_\alpha(G)$ in (1) is not true by the following family of counterexamples.

Example 1. *It was shown in [1] that the α-index of the star graph $K_{1,n-1}$ is*

$$\rho_\alpha(K_{1,n-1}) = \frac{1}{2}\left(\alpha n + \sqrt{\alpha^2 n^2 + 4(n-1)(1-2\alpha)}\right).$$

Suppose $\alpha \neq 0$. Then

$$\lim_{n\to\infty} \frac{\rho_\alpha(K_{1,n-1})}{\alpha n} = 1. \tag{2}$$

Applying $\delta = 1$ for $K_{1,n-1}$ with $n \ge 2$ in (1), we find

$$\rho_\alpha(K_{1,n-1}) \le \frac{1}{2}\left(\alpha + \sqrt{(2-\alpha)^2 + 4(n-2)(1-\alpha)}\right) \sim n^{\frac{1}{2}}.$$

Hence

$$\lim_{n\to\infty} \frac{\rho_\alpha(K_{1,n-1})}{\alpha n} = 0,$$

a contradiction to (2).

We follow the idea of the proof of the inequality (1) in [2] and give the following corrected version.

Theorem 1. *Let G be a connected graph of order n and size m with maximum degree Δ and minimum degree δ. Then*

$$\rho_\alpha(G) \le \frac{\alpha\Delta + (1-\alpha)(\delta-1) + \sqrt{(\alpha\Delta + (1-\alpha)(\delta-1))^2 + 4(1-\alpha)(2m-(n-1)\delta)}}{2}$$

for $0 \le \alpha < 1$. Equality holds if and only if G is regular, or $\alpha = 0$ and every vertex in G has degree $n-1$ or δ.

Proof. Let G have the degree sequence $\Delta = d_1 \ge d_2 \ge \cdots \ge d_n = \delta$, and $r_i(C)$ denote the i-th row sum of an $n \times n$ matrix C. Note that $r_i(A_\alpha(G)) = \alpha d_i + (1-\alpha)d_i = d_i$, and

$$\begin{aligned} r_i(A_\alpha(G)^2) =& \alpha d_i^2 + (1-\alpha)\sum_{ij\in E} d_j = \alpha d_i^2 + (1-\alpha)(2m - d_i - \sum_{j\ne i, ij\notin E} d_j) \\ \le& \alpha\Delta d_i + (1-\alpha)(2m - d_i - (n-d_i-1)\delta) \\ =& (\alpha\Delta + (1-\alpha)(\delta-1))d_i + (1-\alpha)(2m-(n-1)\delta). \end{aligned} \tag{3}$$

Therefore, for $1 \le i \le n$,

$$r_i(A_\alpha(G)^2 - (\alpha\Delta + (1-\alpha)(\delta-1))A_\alpha(G)) \le (1-\alpha)(2m-(n-1)\delta). \tag{4}$$

Note that $A_\alpha^2(G) - (\alpha\Delta + (1-\alpha)(\delta-1))A_\alpha(G)$ has eigenvalue $\rho_\alpha^2(G) - (\alpha\Delta + (1-\alpha)(\delta-1))\rho_\alpha(G)$ associated with a nonnegative eigenvector which is also a $\rho_\alpha(G)$ eigenvector of $A_\alpha(G)$. By [3],

$$\rho_\alpha^2(G) - (\alpha\Delta + (1-\alpha)(\delta-1))\rho_\alpha(G) \le (1-\alpha)(2m-(n-1)\delta),$$

with equality if, and only if, the equality in (4) (or equivalently in (3)) holds for every $1 \le i \le n$. Solving the above quadratic inequality of $\rho_\alpha(G)$ and studying the equality, the theorem follows. □

Theorem 1 is a generalization of a result of Hong, Shu, and Fang [4]. It is worth mentioning that many different upper bounds of $\rho_\alpha(G)$ are already given in [5–7].

Remark 1. *If we give an additional assumption in Theorem 1*

$$t := \min_{i\in V} \sum_{j\ne i, ij\notin E} (d_j - \delta),$$

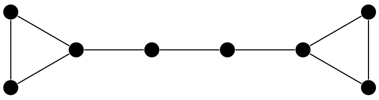
then with little modification of the proof in line (3), we have

$$\rho_\alpha(G) \le \frac{\alpha\Delta + (1-\alpha)(\delta-1) + \sqrt{(\alpha\Delta + (1-\alpha)(\delta-1))^2 + 4(1-\alpha)(2m-(n-1)\delta - t)}}{2}.$$

The above equality holds if, and only if, (i) G is regular, or (ii) $\alpha = 0$ and

$$t = \sum_{j\ne i, ij\notin E} (d_j - \delta) \qquad \text{for } i \in V. \tag{5}$$

Theorem 1 is a special case of Remark 1 with $t = 0$. The following is a graph that satifies (5) with $\delta = 2$ and $t = 1$. It is of independent interest to find all graphs that satisfy (5).



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