NYCU Department of Applied Mathematics
Qualifying Examination in Discrete Mathematics
For the Ph.D. Program February 2023

<u>Note</u>: The proofs and statements must be detailed. When you quote some theorems, please prove them.

1. (a) Give the definition of an *Eulerian circuit* and the definition of an *Eulerian graph.* (3%)
   (b) Prove that a finite graph G with no isolated vertices (but possibly with multiple edges) is Eulerian if and only if it is connected and every vertex has even degree. (12%)

2. (a) Give the definition of a *Hamiltonian circuit.* (3%)
   (a) Prove that if a simple graph G on n vertices has all vertices of degree at least $n/2$, then it has a Hamiltonian circuit. (12%)

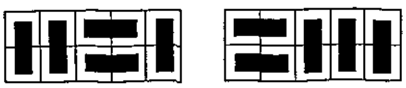
3. Let K_n denote the complete graph of order n.
   A red (respectively, blue) K_3 is a K_3 whose edges are all colored red (respectively, blue).
   (a) Prove that if the edges of K_6 are colored red or blue, then, no matter how edges are colored, there is a red K_3 or there is a blue K_3. (12%)
   (b) Prove that the outcome stated in (a) may not occur if we replace K_6 with K_5. (3%)

4. Let S be a subset of vertices of a graph, let $N(v)$ be the set of all vertices adjacent to a given vertex v, and let $N(S) = \cup_{v\in S} N(v)$. Let G be a bipartite graph with bipartition X, Y.
   (a) Give the definition of a complete matching from X to Y. (3%)
   (b) Prove that G has a complete matching from X to Y if and only if $|N(S)| \geq |S|$ for every $S \subseteq X$. (12%)

5. (a) Prove the Euler's formula, i.e., for a drawing of a connected planar graph G, $n - e + f = 2$, where n, e, and f are, respectively, the number of vertices, edges, and faces. (8%)
   (b) Prove that every simple planar graph of order $n \geq 3$ has at most $3n - 6$ edges. (4%)
   (c) Prove that K_5 is not planar. (2%)

6. A derangement of $1, 2, \ldots, n$ is a permutation $i_1 i_2 \ldots i_n$ of $1, 2, \ldots, n$ such that $i_1 \neq 1, i_2 \neq 2, \ldots, i_n \neq n$. Denote by D_n the number of derangements of $1, 2, \ldots, n$.
   List all derangements of $1, 2, 3$ and prove that for $n \geq 1$,
   $$D_n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right). \quad (13\%)$$

7. Let g_n be the number of different ways to put n blocks, each of size 1×2, over a $2 \times n$ array, such that each block is put either vertically or horizontally, and, no two blocks overlap. For example, the followings are two ways to put blocks in a 2×5 array.



   (a) Find a recurrence relation that g_n satisfies and verify your answer. (6%)
   (b) Solve your recurrence relation and give an explicit formula for g_n. (7%)