PhD Qualifying Exam in Numerical Analysis
Spring 2022

1. (15%) Given $n+1$ distinct points $x_0, \cdots, x_n$, we can define the characteristic polynomial $\ell_i(x)$ of degree n and the nodal polynomial $\omega_{n+1}(x)$ of degree $n+1$ as

$$\ell_i(x) = \prod_{j=0, j\neq i}^{n} \frac{x - x_j}{x_i - x_j}, \quad \omega_{n+1}(x) = \prod_{i=0}^{n} (x - x_i).$$

   (a) (5%) Prove that $w'_{n+1}(x_i) = \prod_{j=0, j\neq i}^{n} (x_i - x_j)$.

   (b) (5%) Prove that $\sum_{i=0}^{n} \ell_i(x) = 1$.

   (c) (5%) Prove that $\{\omega_{n+1}(x), \ell_i(x), i = 0, \cdots, n\}$ forms a basis for $\mathbb{P}_{n+1}$, polynomials of degree at most $n+1$.

2. (25%) Consider the iteration method

$$x^{(k+1)} = Bx^{(k)} + f,$$

   where $x^{(k)}$ and f are vectors in $\mathbb{R}^n$, $B \in \mathbb{R}^{n\times n}$ is a square matrix, and $x^{(0)}$ is a given initial guess. Assume that $\|B\| < 1$, where $\|B\|$ is a matrix norm induced by the vector norm $\|x\|$, show that:

   (a) (5%) The linear system $x = Bx + f$ has a unique solution.

   (b) (5%) The process is convergent to the solution of the linear system $x = Bx + f$.

   (c) (5%) The following inequality holds

$$\|x^{(k)} - x\| \leq \|(I - B)^{-1}\| \cdot \|x^{(k+1)} - x^{(k)}\|.$$

   (d) (5%) The following inequality holds

$$\|x^{(k)} - x\| \leq \|B\|^k \|x^{(0)}\| + \frac{\|B\|^k \cdot \|f\|}{1 - \|B\|}.$$

3. (15%) Consider the following quadrature formula

$$\int_0^\infty f(t)e^{-t}\, dt = af(1) + bf(2),$$

   where a and b are constants.

   (a) (7%) Determine the constants a and b such that the formula achieves the highest degree of exactness. You should also determine the degree of exactness of the formula.

   (b) (8%) Derive an appropriate error estimate for the quadrature formula.

4. (20%) Consider the Cauchy problem $\frac{dy}{dt} = f(t, y)$ on $[0, T]$ with $y(0) = Y_0$, where the function $f(t, y)$ is 2nd-order differentiable on t and y. Let u_n denote the solution obtained from some numerical method of the Cauchy problem and $u_0 = Y_0$.

   (a) (4%) What does it mean when we say the Cauchy problem is stable?

   (b) (4%) What does it mean when we say a numerical method for solving the Cauchy problem is convergent with order p?

   (c) (4%) Derive the two steps Adam-Bashforth method (AB2) in terms of u_n.

   (d) (8%) Prove that the AB2 method is convergent with order 2.

5. (20%) Consider the following 2nd-order wave equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0, \quad (x, t) \in \mathbb{R} \times \mathbb{R}_+,$$

   with initial condition $u(x, 0) = v(x)$ and $\frac{\partial u}{\partial t}(x, 0) = w(x)$.

   (a) (8%) By transforming the equation into a system of linear hyperbolic equations, prove the d'Alembert formula

$$u(x, t) = \frac{1}{2}\left(v(x + ct) + v(x - ct) + \frac{1}{c}\int_{x-ct}^{x+ct} w(\tau)d\tau \right).$$

   (b) (6%) Write down the Lax-Friedrichs (LF) scheme for the above equation. Describe the Courant–Friedrichs–Lewy (CFL) condition for the LF scheme.

   (c) (6%) Prove that the LF scheme is stable if the CFL condition holds.

6. (10%) Consider the following two points boundary value problem

$$-0.05\frac{d^2u}{dx^2} + \frac{du}{dx} = 0, \quad x \in (0, 1),$$

   with boundary values $u(0) = 0$ and $u(1) = 1$.

   (a) (6%) Write down the Galerkin finite element discretization for the equation using 5 linear elements.

   (b) (4%) How many elements are needed at least to prevent oscillatory solution? Explain your reason.