NCTU Department of Applied Mathematics
Qualifying Examination in Algebra
for the Ph. D. Program

Sep, 2020

Please justify all the steps in your arguments with clear and rigorous mathematical arguments. In the case when you need to use a specific mathematical theorem to justify your answer, please specify exactly which theorem or theory you are using.

1. Let G be a group. For each $x \in G$, define two mappings $\lambda_x, \rho_x : G \to G$ by $\lambda_x(g) = xg$ and $\rho_x(g) = gx$ for $g \in G$. Let $L = \{\lambda_x : x \in G\}$ and $R = \{\rho_x : x \in G\}$.
   (a) (5 %) Prove that L and R are groups under composition by the definition.
   (b) (5 %) Find an isomorphism $\phi : L \to R$ to prove that L is isomorphic to R.
2. Denote the cyclic group of order n by $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$.
   (a) (10 %) Prove that for $m, n \in \mathbb{N}$, $\mathbb{Z}_{mn} = \mathbb{Z}_m \times \mathbb{Z}_n$ if and only if $\gcd(m, n) = 1$.
   (b) (5 %) List all abelian groups of order 54 up to isomorphism.
   (c) (10 %) Express the following factor groups as the products of cyclic groups:
      (i) $\mathbb{Z}_6 \times \mathbb{Z}_9/\langle(0, 3)\rangle$,
      (ii) $\mathbb{Z}_6 \times \mathbb{Z}_9/\langle(1, 3)\rangle$,
      (iii) $\mathbb{Z}_6 \times \mathbb{Z}_9/\langle(2, 3)\rangle$,
      (iv) $\mathbb{Z}_6 \times \mathbb{Z}_9/\langle(3, 3)\rangle$,

      where $\langle(a, b)\rangle$ is the subgroup generated by (a, b), and verify your answer.
3. Let G be a group of order $399 = 3 \times 7 \times 19$.
   (a) (5 %) Find all possible the numbers of Sylow p-subgroups for each p.
   (b) (5 %) Show that G contains a cyclic subgroup of order $7 \times 19 = 133$.
   (c) (5 %) Find all possible group structures of G (in terms of generators and relations).

4. Let $R = \mathbb{Z}[\sqrt{-n}]$ where n is a positive integer.

(a) (5 %) Find a multiplicative norm on R.

(b) (5 %) Show that 2 is irreducible if $n \geq 3$.

(c) (5 %) Show that R is not a UFD if $n \geq 3$.

(d) (5 %) Show that if $n \leq 2$, R is an Euclidean domain.

5. Let $E = \mathbb{Q}(i, \sqrt{2}, \sqrt{3})$ be a finite extension over $\mathbb{Q}$, where $i^2 = -1$.

(a) (5 %) Show that $E/\mathbb{Q}$ is a Galois extension.

(b) (5 %) Show that $[E : \mathbb{Q}] = 8$.

(c) (5 %) Find a basis of E as a vector space over $\mathbb{Q}$.

(d) (5 %) Find the structure of the Galois group of $E/\mathbb{Q}$. (Show that the Galois group is isomorphic to some well-known group of order 8 like $\mathbb{Z}_8$, the dihedral group D_4, etc.)

(e) (5 %) Show that every subextension of $E/\mathbb{Q}$ is Galois.

(f) (5 %) Find all quadratic subextensions of $\mathbb{Q}$ in E.