交通大學應用數學系博士班資格考

實變分析試題（九十七年九月）

97.9.18

1. (15分) Let λ be the Lebesgue measure and let $\{E_k\}$ be a sequence of Lebesgue measurable subsets of $[0,1]$ satisfying
$$\lim_{k\to\infty} \lambda(E_k) = 1.$$
Show that, for any $0 < \varepsilon < 1$, there exists a subsequence $\{E_{k_n}\}$ such that
$$\lambda\left(\bigcap_{n=1}^{\infty} E_{k_n}\right) > \varepsilon.$$

2. (15分) Suppose that $f(x)$ is a Lebesgue integrable function on $[a,b]$. Prove that there exists a zero-measure subset $Z \subseteq [a,b]$ such that, for any real number α and $x \in [a,b] \setminus Z$, we have
$$\frac{d}{dx}\int_a^x |f(t)-\alpha|\,dt = |f(x)-\alpha|.$$

3. (15分) Let X be a measurable subset of $\mathbb{R}^n$ and f the Lebesgue integrable function on X. Show that
$$I := \left\{\int_E f(x)dx : \ E \text{ is a measurable subset of } X\right\}$$
is a closed interval. Point out the two end points of I.

4. (15分) Let $(X,\mathcal{B},\nu)$ and $(X,\mathcal{B},\mu)$ be σ-finite measure spaces, and let f be a $\mathcal{B}$-measurable and integrable over X with respect to ν. Let $\mathcal{B}_0$ be a σ-algebra satisfying $\mathcal{B}_0 \subseteq \mathcal{B}$ (f may not be $\mathcal{B}_0$-measurable). Show that there is a unique $\mathcal{B}_0$-measurable functions g, a unique nonnegative $\mathcal{B}_0$-measurable functions h, and a unique measure δ which is singular with respect to μ, such that
$$\int_X f\,d\nu = \int_X gh\,d\mu + \int_X g\,d\delta.$$

5. (16分) Let $1 \le p < \infty$ and f be a function such that f and its derivative f' are locally L^p-integrable on $\mathbb{R}$. Show that there is a constant C such that
$$|f(x)| \le C\left(\int_{|y-x|\le 1} \{|f'(y)| + |f(y)|\}^p dy\right)^{1/p} \qquad \text{for a.e. } x \in \mathbb{R}.$$

6. Given a measurable function f on $\mathbb{R}$, define a family of functions $\{B_\alpha(f)\}_{\alpha>0}$ by

$$B_\alpha(f)(x) = \int_{-\infty}^{x} e^{-\alpha(x-y)} f(y)dy, \qquad \alpha > 0,\ x \in \mathbb{R}.$$

Show the following properties of $B_\alpha(f)$:

(i) $\|B_\alpha(f)\|_{L^p} \le \alpha^{-1}\|f\|_{L^p}$ for $\alpha > 0$ and $1 \le p < \infty$; (8分)

(ii) $\lim_{\alpha\to\infty} \|\alpha B_\alpha(f) - f\|_{L^p} = 0$ for $\alpha > 0$ and $1 \le p < \infty$; (8分)

(iii) $\|\alpha B_\alpha(f)\|_{L^p}$ is a non-decreasing function of $\alpha \in (0,\infty)$. (8分)