NCTU PhD Qualify Examination, Probability

September, 2008

97.9.18

20 points for each problem

1. Let X, Y be two independent r.v.'s. Calculate the convolution to prove that: (i) if both they are exponential with same parameter λ, then $X+Y$ is a Gamma distribution. (ii) if both they are Poisson, then $X+Y$ is also Poisson.

2. Let X be a symmetric r.v. such that, for $x > 1$, the tail distribution has the decay $P\{|X| > x\} \approx x^{-\alpha}$. (i) Discuss and prove the integrability of $|X|$, in cases $0 < \alpha \leq 1$ and $1 < \alpha < 2$. (ii) Discuss the possibility of $\alpha = 2$.

3. Let $(X_n, \mathcal{F}_n)$ be a martingale, and each X_n is in $L^2(dP)$. The difference is $\xi_{m,n} := X_n - X_m, m < n$. (i) Simplify $E[\xi_{m,n}^2|\mathcal{F}_m]$ to a manageable form. (ii) Prove that, if $\sum_n E\xi_{n-1,n}^2 < \infty$, then the martingale convergence holds both a.s. and in mean square.

4. Consider a simple branching model in which we start with one single particle, at time 0. After one unit time, it branches out a random number of sub-particles; each sub-particle then branches independently and with the same offspring distribution(branching mechanism) as the original one, $\{p_k, k \geq 0\}$. Let Z_n denote the population number of particles at time n. (i) Write Z_n in its recursive form. (ii) Prove that the sequence Z_n/μ^n form a martingale w.r.t. the natural filtration, where $\mu = \sum_k p_k$, the mean of the offspring distribution(assume the mean is finite and positive).

5. Let X_n (discrete time) be a finite-states irreducible aperiodic MC. (i) What means "irreducible aperiodic" ? (ii) Assume that X_n is also doubly stochastic, write and prove its unique stationary distribution. (iii) How about the (ii) if it is the infinite-states.