博士班資格考 - 實變分析

97年2月

\# 1. (20 pt) Let $f : \mathbb{R} \to \mathbb{R}$ be Riemann integrable over $\mathbb{R}$. Let $g : \mathbb{R} \to \mathbb{R}$ be Lebesgue integrable over $\mathbb{R}$.

(1) Must f be Lebesgue integrable over $\mathbb{R}$? (10 pt)

(2) Must g be Riemann integrable over $\mathbb{R}$? (10 pt)

Prove or disprove your answers.

\# 2. (20 pt) Let $u \in L^6(\Omega)$ and $v \in L^4(\mathbb{R}) \cap L^6(\mathbb{R})$, where Ω is a bounded domain in $\mathbb{R}^n, n \geq 2$.

(1) Can there exist C_1 a positive constant independent of u such that

$$\left(\int_\Omega u^6\right)^2 \leq C_1 \left(\int_\Omega u^4\right)^3 ?$$

(10 pt)

(2) Can there exist C_2 a positive constant independent of v such that

$$\left(\int_\mathbb{R} v^4\right)^3 \leq C_2 \left(\int_\mathbb{R} v^6\right)^2 ?$$

(10 pt)

Prove or disprove your answers.

\# 3. (20 pt) Let $f \in L^1(\mathbb{R})$. For $\xi \in \mathbb{R}$, let $\hat{f}(\xi) = \displaystyle\int_0^\infty e^{-\xi^2 x} f(x)\, dx$. Answer the following questions:

(1) Can $\hat{f} \in L^1(\mathbb{R})$? (10 pt)

(2) Can $\hat{f}$ be differentiable? (10 pt)

Prove or disprove all your answers.

\# 4. (20 pt) Let $u : B_1 \to \mathbb{R}$ be a smooth function satisfying $u(x) = 0$ for $|x| = 1$, where B_1 is the unit ball in $\mathbb{R}^2$ with center at origin. Can there exist a positive constant C independent of u such that

$$\int_{B_1} u^2\, dx \leq C \int_{B_1} u_r^2\, dx$$

hold? Here (r, θ) is the polar coordinate and u_r is the associated partial derivative. Prove or disprove your answer.

5. (20 pt) Let $\{\nu_j\}_{j=1}^{\infty}$ be a sequence of Radon measures satisfying

$$\|\nu_j\|_\infty \leq M\,, \quad j = 1, 2, 3, \cdots\,,$$

where M is a positive constant independent of j, and the norm $\|\cdot\|_\infty$ is defined by

$$\|\nu\|_\infty = \sup_{f \in C_0^\infty(\mathbb{R}^n)} \frac{\int_{\mathbb{R}^n} f\, d\nu_j}{\int_{\mathbb{R}^n} f\, d\mu}\,.$$

Here μ is the standard Lebesgue measure, and $C_0^\infty(\mathbb{R}^n)$ is the collection of smooth functions with compact support. Can there exist ν_* a Radon measure such that $\nu_j \rightharpoonup \nu_*$ i.e.

$$\int_{\mathbb{R}^n} f\, d\nu_j \to \int_{\mathbb{R}^n} f\, d\nu_*\,, \quad \forall f \in C_0^\infty(\mathbb{R}^n)$$

(up to a subsequence) as $j \to \infty$? Prove or disprove your answer.