94學年度博士班資格考 實變

QUALIFYING EXAMINATION ON REAL ANALYSIS

1. Let λ denote the Lebesgue measure on $\mathbb{R}$ and $f:\mathbb{R}\to\mathbb{R}$ be Lebesgue integrable.

   (a) Suppose that $\int_{[a,b]} f\,d\lambda = 0$ for all $[a,b]\subseteq\mathbb{R}$. Show that $f=0$ λ-a.e. (10pts)

   (b) Suppose $f(x)>0$, $\forall x\in[c,d]$ for some $[c,d]\subseteq\mathbb{R}$. Show that $\int_{[c,d]} f\,d\lambda > 0$. (Hint: For each $n\in\mathbb{N}$, let H_n be the closure of the set of points $x\in[c,d]$ such that $f(x)>\frac{1}{n}$ and apply the Baire Category Theorem.) (10pts)

2. Let p and q be conjugate indices and let $f\in L^p(\mathbb{R})$, $g\in L^q(\mathbb{R})$. Show that the function $F(t)=\int_{\mathbb{R}} f(x+t)g(x)\,d\lambda(x)$, $t\in\mathbb{R}$, is a continuous function of t. (10pts)

3. Let $L^1[0,1]=\{f:[0,1]\to\mathbb{R} : f \text{ is Lebesgue integrable}\}$, $C[0,1]=\{f:[0,1]\to\mathbb{R} : f \text{ is continuous}\}$ and $P[0,1]=\{f:[0,1]\to\mathbb{R} : f \text{ is a polynomial}\}$.

   (a) Show that $C[0,1]$ is dense in $L^1[0,1]$. (10pts)

   (b) Show that $P[0,1]$ is dense in $L^1[0,1]$. (10pts)

4. In the Hilbert space $L^2[0,1]$ define the operator $T=\frac{d}{dx}$ on the subspace AC of absolutely continuous functions.

   (a) Show that T is not bounded. (7pts)

   (b) Show that T is closed, that is, if $f_n\in AC$, $\|f_n-f\|_2\to 0$ and $\|Tf_n-g\|_2\to 0$ as $n\to+\infty$, then $f\in AC$ and $Tf=g$. (8pts)

5. Let X denote a countable infinite set with the discrete topology, and let μ denote a measure on the power set 2^X with the following two properties:

   (i) $\mu(\{x\})>0$ for every $x\in X$; (ii) $\mu(X)=1$.

   (a) Show that for any $\epsilon>0$ there exists a nonempty subset E of X such that $\mu(E)<\epsilon$. (5pts)

   (b) Suppose that $f:X\to\mathbb{R}$ belongs to $L^1(\mu)$. Show that for each $\epsilon>0$ there exists a $\delta>0$ such that $\int_E |f|\,d\mu<\epsilon$ for all $E\subseteq X$ with $\mu(E)<\delta$. (5pts)

   (c) Show that there exists $f\in L^1(\mu)$ such that f is unbounded. (5pts)

6. Let λ denote the Lebesgue measure on $\mathbb{R}$ and μ be a finite Borel measure on $(0,+\infty)$ such that

   (i) $\mu\ll\lambda$, and

   (ii) $\mu(aB)=\mu(B)$ for each $a>0$ and each Borel subset B of $(0,+\infty)$.

   If the Radon-Nikodym derivative $\frac{d\mu}{d\lambda}$ is continuous, then show that there exists a constant $c\geq 0$ such that $\frac{d\mu}{d\lambda}(x)=\frac{c}{x}$ for each $x>0$. (10pts)

7. Let $f(x,y)=\begin{cases} \left(x-\frac{1}{2}\right)^{-3}, & \text{if } 0<y<\left|x-\frac{1}{2}\right|; \\ 0, & \text{otherwise.}\end{cases}$

   Are the iterated integrals $\int_0^1\int_0^1 f(x,y)\,dx\,dy$ and $\int_0^1\int_0^1 f(x,y)\,dy\,dx$ equal? Comment on the relation between this and the Fubini's Theorem. (10pts)