Department of Applied Mathematics
National Chiao Tung University
Ph.D. Qualifying Examination
Discrete Mathematics
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Problem 1.(10%) A $(0,1)$-matrix is a matrix whose entries are all either 0 or 1. A principal submatrix of an $n \times n$ $A = (a_{ij})_{1 \leq i,j \leq n}$ is a submatrix B of the form $B = (a_{ij})_{i,j \in I}$ for some $I \subseteq \{1, 2, \ldots, n\}$. The size of B is the size of the set I. Let m be a given positive integer. Using Ramsey's Theorem to show that if n is large enough, every $n \times n$ $(0,1)$-matrix has a principal submatrix of size m, in which all the elements below the diagonal are the same, and all the elements above the diagonal are the same.

Problem 2.(10%) Let $A = (a_{ij})$ be an $n \times n$ matrix with nonnegative integers as entries, such that every row and column of A has sum ℓ. Prove that A is the sum of ℓ permutation matrices.

Problem 3. **(a)(5%)** Prove that a $k \times n$ Latin rectangle, $k < n$, can be extended to a $(k+1) \times n$ Latin rectangle.
(b)(10%) Let m and n be positive integers, $m < n$. Prove that $m \leq \frac{1}{2}n$ is a necessary and sufficient condition for the existence of a Latin square of order n containing a sub-Latin square of order m

Problem 4. A list assignment of a graph G is a function C that assigns a set $C(v)$ to each vertex v of G. A C-coloring of G is a function A defined on the vertex set of G such that $A(v) \in C(v)$ and such that $A(v)$ and $A(w)$ are different if v and w are adjacent in G. Let f be a function from the vertex set of G to the positive integers. We say that G is f-colorable if a C-coloring of G exists for every C with the property $|C(v)| \geq f(v)$ for every vertex v in G. If k is an integer, a graph G is called k-list-colorable if G is f-colorable for the constant function $f(v) = k$.
(a)(5%) Let D be a digraph for which every induced subgraph has a kernel. Prove that if $f(v) := 1 +$ outdegree(v) for every vertex v of D, then the underlying graph G of D is f-colorable.
(b)(10%) For a bipartite simple graph G, let $L(G)$ be the line graph of G and let $\chi(L)$ be the chromatic number of $L(G)$. Prove that $L(G)$ is $\chi(L)$-list-colorable.

Problem 5.(10%) Let $K_n - e$ denote the graph formed by taking complete graph K_n and removing one of its edges. Prove that $K_n - e$ has $(n-2)n^{n-3}$ different labeled spanning trees.

Problem 6. Let $c(n,k)$ denote the number of permutations $\pi \in S_n$ with exactly k cycles. This number is called a signless Stirling number of the first kind. For examples, $c(n,n) = 1$ and $c(n,1) = (n-1)!$.
(a)(10%) Prove that, for any positive integer n, we have

$$\sum_{k=1}^{n} c(n,k)x^k = x(x+1)\cdots(x+n-1).$$

(b)(10%) Use part (a) to prove that the total number of cycles in all permutations in S_n is equal to

$$n!\left(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}\right).$$

Problem 7.(10%) A simple graph F is said to be 2-connected when $|V(F)| \geq 3$ and the removal of any vertex from F does not result in a disconnected graph. Let G be a finite 2-connected simple graph with at least two edges and H a maximal proper 2-connected subgraph of G. Prove that G is the union of H and a path-graph P that joins distinct vertices of H and has none of its internal vertices in H.

Problem 8.(10%) Let $\chi_G(\lambda)$ be the number of proper colorings of a graph G with λ colors. For example, if T is a tree on n vertices, then $\chi_T(\lambda) = \lambda(\lambda-1)^{n-1}$. Let $\omega(G)$ be the number of acyclic orientations of a graph G. Prove that $\omega(G) = (-1)^{|V(G)|}\chi_G(-1)$.