PhD Qualify Exam in Numerical Analysis

Fall 2017

(1) Consider following linear system

$$Ax \equiv \begin{bmatrix} 2 & \alpha & -1 \\ \alpha & 2 & 1 \\ -1 & 1 & 4 \end{bmatrix} x = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \equiv b.$$

(a) (10%) Show that A is positive definite if $-2 < \alpha < \frac{3}{2}$.

(b) (10%) Compute the spectrum and the spectral radius of the matrix A with $\alpha = -1$.

(c) (10%) Compute the condition number of A in part (b) with $\|\cdot\|_\infty$.

(d) (10%) Please write down the algorithms of the Jacobi and Gauss-Seidel methods for solving the linear system with $\alpha = -1$.

(e) (10%) Compare the convergence of the Jacobi and Gauss-Seidel methods in (d).

(f) (10%) Let $g(x) = x^\top Ax - 2x^\top b$ with $\alpha = -1$. Compute the steepest descent direction v_0 of $g(x)$ at $x_0 = [1, 1, 1]$.

(2) (10%) When we apply Newton's method with any initial vector x_0 to solve a linear system $Ax = b$ with nonsingular A, please prove that it only needs one iteration to get the solution of the linear system.

(3) (10%) Determine constants a, b, c, and d that will produce a quadrature formula

$$\int_{-1}^{1} f(x)dx = af(-1) + bf(1) + cf'(-1) + df'(1)$$

that has degree of precision 3.

(4) (10%) A natural cubic spline S is defined by

$$S(x) = \begin{cases} S_0(x) = 1 + B(x-1) - D(x-1)^3, & \text{if } 1 \leq x, 2, \\ S_1(x) = 1 + b(x-2) - \frac{3}{4}(x-2)^2 + d(x-2)^3, & \text{if } 2 \leq x \leq 3. \end{cases}$$

If S interpolates the data $(1, 1)$, $(2, 1)$, and $(3, 0)$, find B, D, b, and d.

(5) (10%) Discretize the equation

$$\begin{cases} -u''(x) + 2u(x) = f(x), \text{ for } 0 < x < 1, \\ u(0) = u(1) = 0, \end{cases} \tag{1}$$

with a given function f by using the centered difference method with uniform mesh. Please explicitly write down the coefficient matrix A of the resulting linear system $Au = b$ and prove that A is symmetric positive definite.