PhD Qualify Exam in Numerical Analysis
Spring 2015

1. 10% Find the constants A_0, A_1, A_2, x_0, x_1 and x_2 such that the Gaussian quadrature rule

$$\int_{-1}^{1} f(x)dx \approx A_0 f(x_0) + A_1 f(x_1) + A_2 f(x_2)$$

is exact for $f(x)$ in Π_5 which stands for the set of polynomials of degree less than or equal to 5.

2. 10% Let $g \in C^1[a,b]$ and $p_* \in (a,b)$ with $g(p_*) = p_*$ and $|g'(p_*)| > 1$. Show that the fixed point iteration given by $p_{k+1} = g(p_k)$, for $k = 0, 1, \ldots$, does not converge, no matter how close the initial approximation p_0 is to $p_* \neq p_0$.

3. 20% Let I be an interval of $\mathbb{R}$ containing t_0. Assume that f is uniformly Lipschitz continuous with respect to y. Consider the following initial value problem (IVP):

$$\begin{cases} y'(t) = f(t,y), & t \in I \\ y(t_0) = y_0. \end{cases}$$

Let u_j be the approximation at node t_j of the exact solution $y_j \equiv y(t_j)$ and let f_j denote the value $f(t_j, u_j)$ and set $u_0 = y_0$.

(a) Find the condition on the coefficients of the following 2-stage explicit Runge-Kutta method for the IVP such that the method is second-order accuracy:

$$u_{n+1} = u_n + hb_1 f_n + hb_2 f(t_n + hc_2, u_n + hc_2 f_n).$$

(b) From part (a), a second-order Runge-Kutta method can be obtain by taking $b_1 = b_2 = 1/2$ and $c_2 = 1$. Find the region of absolute stability of the method.

4. 15% Consider the square linear system $Ax = b$. Let D be the diagonal matrix consisting of the diagonal part of A. The parametric Jacobi method, called the relaxation of Jacobi iteration (JOR), is expressed by

$$x_{k+1} = x_k - \omega D^{-1}(Ax_k - b).$$

Show that if Jacobi iteration converges then JOR converges for $0 < \omega \leq 1$.

5. 20% Consider the following initial-boundary value problem of the 1-D heat equation:

$$\text{(IBVP)} \quad \begin{cases} u_t - u_{xx} = 0, & t > 0,\ 0 < x < 1, \\ u(x,0) = u_0(x), & 0 \le x \le 1, \\ u(0,t) = u(1,t) = 0, & t > 0. \end{cases}$$

(a) Find an explicit finite difference method to solve the IBVP and discuss the stability properties of the method.

(b) Construct an implicit finite difference method to improve the stability of the explicit method developed in part (a).

6. 10% Show that the vector x_* is a solution to the positive definite linear system $Ax = b$ if and only if x_* minimizes

$$\varphi(x) = x^T Ax - 2x^T b.$$

7. 15% Let f be defined on $[a, b]$, and let the nodes $a = x_0 < x_1 < x_2 = b$ be given. A quadratic spline interpolating function S consists of the quadratic polynomial

$$S_0(x) = a_0 + b_0(x - x_0) + c_0(x - x_0)^2 \quad \text{on } [x_0, x_1]$$

and the quadratic polynomial

$$S_1(x) = a_1 + b_1(x - x_1) + c_1(x - x_1)^2 \quad \text{on } [x_1, x_2],$$

such that

(i) $S(x_0) = f(x_0)$, $S(x_1) = f(x_1)$ and $S(x_2) = f(x_2)$,

(ii) $S \in C^1[x_0, x_2]$.

(a) Show that conditions (i) and (ii) lead to five equations in the six unknowns.

(b) Impose an additional condition so that the solution is unique.