PhD Qualifying Exam in Numerical Analysis

Spring 2014

1. (20%) Consider the following linear Cauchy problem:

$$\begin{cases} y'(t) &= \lambda y(t), \quad t > 0, \\ y(0) &= 1, \end{cases}$$

with $\lambda \in \mathbb{C}$.

(a) Give the definitions of the "absolute stability" and the "region of absolute stability" of a numerical method for approximating the linear Cauchy problem.

(b) Find the regions of absolute stability of the forward Euler method and the backward Euler method.

2. (20%) Consider the following boundary value problem:

$$\begin{cases} -(\alpha u')'(x) + (\beta u')(x) + (\gamma u)(x) = f(x), \quad 0 < x < 1, \\ u(0) = u(1) = 0, \end{cases}$$

where α, β and γ are smooth functions on $[0,1]$ with $\alpha(x) \geq \alpha_0 > 0$ for any $x \in [0,1]$ and $f \in L^2(0,1)$.

(a) Derive the weak formulation of the boundary value problem on the Hilbert space $H_0^1(0,1) := \{v \in L^2(0,1) : \ v' \in L^2(0,1), v(0) = v(1) = 0\}$.

(b) Show that the weak formulation has a unique solution u in $H_0^1(0,1)$ if

$$-\frac{1}{2}\beta'(x) + \gamma(x) \geq 0 \quad \forall\, x \in [0,1].$$

(c) Let V_h be a finite-dimensional vector subspace of $H_0^1(0,1)$ and let $\{\varphi_1, \cdot \quad, \varphi_N\}$ be a basis of V_h. Please describe the Galerkin method for finding the numerical solution $u_h \in V_h$ of the boundary value problem and find the associated stiffness matrix.

(d) Prove the following result: there exists a constant $C > 0$ such that

$$|u - u_h|_{H^1(0,1)} \leq C \min_{v_h \in V_h} |u - v_h|_{H^1(0,1)},$$

where $| \quad |_{H^1(0,1)}$ is the norm of space $H_0^1(0,1)$,

$$|v|_{H^1(0,1)} := \left\{ \int_0^1 |v'(x)|^2 dx \right\}^{1/2}$$

3. (20%) Consider the following 1-D heat problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu \dfrac{\partial^2 u}{\partial x^2} = 0, \quad 0 < x < 1, \ t > 0, \\ u(0,t) = u(1,t) = 0, \quad t > 0, \\ u(x,0) = u_0(x), \quad 0 \leq x \leq 1, \end{cases}$$

where constant $\nu > 0$ is the thermal conductivity.

(a) Derive an implicit finite difference method for the 1-D heat problem by using the central difference formula for the spatial variable and the backward difference formula for the temporal variable.

(b) Show that the implicit finite difference method derived in part (a) is unconditionally stable.

Hint: The eigenvalues λ_i of the following $n \times n$ tridiagonal matrix

$$A = \begin{bmatrix} 1+2s & -s & & & & \\ -s & 1+2s & -s & & & \\ & -s & 1+2s & -s & & \\ & & \ddots & \ddots & \ddots & \\ & & & -s & 1+2s & -s \\ & & & & -s & 1+2s \end{bmatrix}$$

are given by $\lambda_i = 1 + 2s(1 - \cos\theta_i)$, $\theta_i = \dfrac{i\pi}{n+1}$, $1 \le i \le n$, $s = \dfrac{\nu \Delta t}{h^2}$, Δt is the time step and h is the spatial grid size.

4. (20%) Consider the linear system $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A} = (a_{ij}) \in \mathbb{R}^{n\times n}$, $a_{ii} \neq 0 \ \forall\, i$, is a given nonsingular matrix and $\mathbf{b} = (b_1, \quad , b_n)^\top \in \mathbb{R}^n$ a given vector.

   (a) Describe the basic concept of linear iterative methods for solving the linear system and find the so-called iterative matrix associated with the Jacobi method.

   (b) Prove that if $\|\mathbf{I} - \mathbf{D}^{-1}\mathbf{A}\| < 1$ for any matrix norm then the sequence $\{\mathbf{x}^{(k)}\}$ generated by the Jacobi method converges to the solution of $\mathbf{Ax} = \mathbf{b}$ for any choice of $\mathbf{x}^{(0)} \in \mathbb{R}^n$, where matrix $\mathbf{D}$ is the diagonal part of $\mathbf{A}$.

5. (20%) Assume that $f \in C^2([a,b])$.

   (a) Derive the following Trapezoidal formula by using Lagrange interpolating polynomial of degree 1 of f:

$$\int_a^b f(x)dx \approx \frac{b-a}{2}\Big(f(a) + f(b)\Big).$$

   (b) Show that the quadrature error is given by

$$E(f) = -\frac{(b-a)^3}{12} f''(\xi), \quad \text{for some } \xi \in (a,b).$$