PhD Qualifying Exam in Numerical Analysis

Fall 2013

1. (20%) Let $\boldsymbol{A} = (a_{ij}) \in \mathbb{R}^{n\times n}$, $a_{ii} \neq 0$ for $i = 1, \cdots, n$, be a given nonsingular matrix and $\boldsymbol{b} = (b_1, \cdots, b_n)^\top \in \mathbb{R}^n$ a given vector. The basic concept of iterative methods for solving the system of linear equations $\boldsymbol{Ax} = \boldsymbol{b}$ is to convert the system into an equivalent system of the form $\boldsymbol{x} = \boldsymbol{Bx} + \boldsymbol{f}$ and lead to the iterations

$$\boldsymbol{x}^{(k+1)} = \boldsymbol{Bx}^{(k)} + \boldsymbol{f} \quad \text{for } k \geq 0.$$

   (a) Prove *in detail* that the sequence of vectors $\{\boldsymbol{x}^{(k)}\}$ converges to the solution of $\boldsymbol{Ax} = \boldsymbol{b}$ for any choice of $\boldsymbol{x}^{(0)} \in \mathbb{R}^n$ if and only if $\rho(\boldsymbol{B}) < 1$, where $\rho(\boldsymbol{B})$ is the spectral radius of $\boldsymbol{B}$.

   (b) Find $\boldsymbol{B}$ and $\boldsymbol{f}$ for the Gauss-Seidel method and show that if $\boldsymbol{A}$ is diagonally dominant, then the Gauss-Seidel method converges for any initial guess.

2. (20%) Assume that $f \in C^4([a,b])$.

   (a) Derive the following Simpson formula by using the Lagrange interpolation of degree 2:

$$\int_a^b f(x)dx \approx \frac{b-a}{6}\Big(f(a) + 4f(\frac{a+b}{2}) + f(b)\Big),$$

   where the quadrature error is $-\dfrac{1}{90}\Big(\dfrac{b-a}{2}\Big)^5 f^{(4)}(\xi)$ for some $\xi \in (a,b)$.

   (b) Let $a = x_0 < x_1 < \cdots < x_n = b$ be a uniform partition of $[a,b]$ with mesh size $h = (b-a)/n$ and n be even. Derive the composite Simpson formula for approximating $\int_a^b f(x)dx$ with error term $\dfrac{-1}{180}(b-a)h^4 f^{(4)}(\xi)$ for some $\xi \in (a,b)$.

3. (20%) Let I be an interval of $\mathbb{R}$ containing t_0. Assume that f is uniformly Lipschitz continuous with respect to y. Consider the following initial value problem (IVP):

$$\begin{cases} y'(t) &= f(t, y(t)), \quad t \in I, \\ y(t_0) &= y_0. \end{cases}$$

   Let u_j be the approximation at node t_j of the exact solution $y_j := y(t_j)$ and let f_j denote the value $f(t_j, u_j)$ and set $u_0 = y_0$.

   (a) Find the conditions on the coefficients of the following general 2-stage explicit Runge-Kutta method for the IVP such that the method is of second-order accuracy:

$$u_{n+1} = u_n + hb_1 f_n + hb_2 f(t_n + hc_2, u_n + hc_2 f_n).$$

   (b) From part (a), a second-order Runge-Kutta method can be obtained by taking $b_1 = b_2 = 1/2$ and $c_2 = 1$. Find the region of absolute stability of the method.

4. (20%) Let us consider the following two-point boundary value problem (BVP):

$$\begin{cases} -\alpha u''(x) + \beta u'(x) + u(x) = f(x), & 0 < x < 1, \\ u(0) = u(1) = 0, \end{cases}$$

where α and β are two positive constants and f is a given smooth function.

(a) Derive the weak formulation of the BVP on the Hilbert space $H_0^1(0,1)$ and prove the weak problem has a unique solution $u \in H_0^1(0,1)$.

(b) Prove that the piecewise linear finite element solution u_h of the BVP satisfies the following error estimate provided $u \in H_0^1(0,1) \cap H^2(0,1)$:

$$\|u - u_h\|_{H_0^1(0,1)} \leq Ch^1 \|u\|_{H^2(0,1)}.$$

5. (20%) Let us consider the following scalar hyperbolic problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} + a\dfrac{\partial u}{\partial x} = 0, & x \in \mathbb{R},\ t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}, \end{cases}$$

where a is a positive constant. The following Lax-Wendroff method is a second-order accurate scheme for the above problem:

$$u_j^{n+1} = u_j^n - \frac{\lambda}{2}a(u_{j+1}^n - u_{j-1}^n) + \frac{\lambda^2}{2}a^2(u_{j+1}^n - 2u_j^n + u_{j-1}^n),$$

where $\lambda := \Delta t/\Delta x$, Δt is the time step and Δx is the spatial grid size.

(a) Use the Taylor series expansion on $u(x, t + \Delta t)$ and the standard centered approximations to $u_x(x,t)$ and $u_{xx}(x,t)$ to derive the Lax-Wendroff method.

(b) Prove by the *Von Neumann stability analysis* that the Lax-Wendroff method is stable provided that $\left|\dfrac{a\Delta t}{\Delta x}\right| \leq 1$.