National Chiao Tung University
Departmet of Applied Mathematics
Discrete Mathematics Qualifying Examination
September 2013

Problem 1. (10pts)
Explain the following terminologies/theorems *and give an example* for each. (You do not need to prove them. Write down clearly your notation, definitions and statements.)

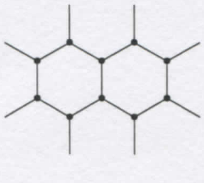
(1) Stirling numbers of the second kind.
(2) Fischer's inequality (of BIBD).
(3) Dilworth's theorem (on a poset).
(4) Euler's pentagon identity (about $\prod_{k=1}^{\infty}(1-x^k)$).
(5) König's theorem (on a bipartite graph)

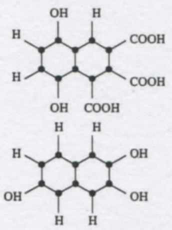
Problem 2. (10pts)
Solve the following recurrence relation: $a_1 = 8$,

$$a_n = 3a_{n-1} - 4n + 3 \times 2^n.$$

Problem 3. (10pts) Prove Dirac's theorem: If a simple graph G on $n > 2$ vertices has all vertices of degree at least $\frac{n}{2}$, then it contains a Hamiltonian circuit.

Problem 4. (10pts)
Theoretically how many different compounds (化合物) are there if one may attach H, OH, or $COOH$ to each of the eight bonds (鍵) of an Naphthalene (雙苯環)? (The Naphthalene and its eight bounds are illustrated in the figure and two compounds are also given. Two compounds are regarded as the same if one is congruent with the other by a rotation and/or a reflection.)





Problem 5. (15pts)
Given $n \geq m$, find the close form of

$$\text{(i)} \sum_{k=0}^{m} \binom{n}{k}\binom{n-k}{m-k} \qquad \text{(ii)} \sum_{k=0}^{m} (-1)^k \binom{n}{k}\binom{n-k}{m-k}.$$

Problem 6. (15pts)
Given $k < n$. Prove that a $k \times n$ Latin rectangle can be completed to an $n \times n$ Latin square.

Problem 7. (15pts)
Let n be a positive integer. How many n-element multisets on $\mathbb{Z}/(n+1)\mathbb{Z}$ whose elements sum to 0? Justify your answer.

Problem 8. (15pts)
Let k, t be relatively prime integers greater than 1. Starting from the permutation $(1, 2, \ldots, n)$ of the numbers $1, 2, \ldots, n$, we may exchange two numbers if their difference is either k or t. Prove that we can obtain any permutation of $1, 2, \ldots, n$ with such steps *if and only if* $n \geq k + t - 1$.