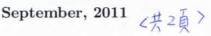
## Qualifying Exam.: Ordinary Differential Equations (常級)



Do five out of the following six problems. Each problem is 20 points. Show all your works to get full credits.

1. Consider the following predator-prey system

$$\begin{cases} x' = \gamma x \left( 1 - \frac{x}{K} \right) - \alpha xy \\ y' = y(\beta x - d) & \gamma, K, \alpha, \beta, d > 0 \\ x(0) > 0, & y(0) > 0. \end{cases}$$
 (1)

Show that the solutions x(t), y(t) of (1) are defined for all t>0 and the solutions are positive and bounded for all t > 0.

2. Consider the IVP

$$\begin{cases} x' = Ax = \begin{pmatrix} a & b \\ c & d \end{pmatrix} x, & x(t) \in \mathbb{R}^2, \\ x(0) = x_0 \neq 0. \end{cases}$$
 (2)

Assume that the real matrix A is nonsingular  $(ad - bc \neq 0)$ . Give necessary and sufficient condition(s) on matrix  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  such that the trajectory of IVP (2) is a circle on the phase plane. Give a mathematical proof.

3. (a) Solve  $x(t) = (x_1(t), x_2(t), x_3(t))^T \in \mathbb{R}^3$  of the linear system

$$x' = Ax = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} x. \tag{3}$$

where  $\lambda \in \mathbb{R}$ .

- (b) Analyze and draw the phase portrait of linear system (3) in the  $x_1x_2x_3$ -space. (Discuss the cases (i)  $\lambda > 0$ , (ii)  $\lambda = 0$ , (iii)  $\lambda < 0$ .)
- 4. Consider nonlinear undamped pendulum equation  $\theta'' + \sin \theta = 0$ . Let  $\omega = \theta'$ , we obtain the equivalent first order nonlinear system

$$\begin{cases} \theta' = \omega \\ \omega' = -\sin\theta. \end{cases} \tag{4}$$

**Problem.** Draw the global phase portrait of the system on the  $(\theta, \omega)$ -plane and verify your results. In particular,

- (a) Find and classify all equilibrium points (e.g. improper node, proper node, saddle point, spiral point, center,  $\cdots$ ) of nonlinear system (4). (Give proofs of the results.)
- (b) Find a Lyapunov function (energy function) for nonlinear system (4) and classify all the orbits by the Lyapunov function.
- (c) Show the existence of saddle connections (heteroclinic orbits) of nonlinear system (4).

5. Problem. Apply the following given Theorems 1 and 2 to prove Floquet's Theorem, parts (i) and (ii). You are not required to prove Theorems 1 and 2.

Consider the nonautonomous linear system

$$x' = A(t)x, (5)$$

where  $A(t) = (a_{ij}(t)) \in \mathbb{R}^{n \times n}$  is a continuous, T-periodic matrix (i.e., A(t) = A(t+T) for all t).

Theorem 1. Let  $\Phi(t)$  and  $\Psi(t)$  be two fundamental matrices of matrix equation

$$X' = A(t)X \tag{6}$$

where  $X = (x_{ij}(t)) \in \mathbb{R}^{n \times n}$  and  $X' = (x'_{ij}(t))$ . Then there exists a real, nonsingular matrix  $\tilde{P}$  such that  $\Psi(t) = \Phi(t)\tilde{P}$  for all t.

Theorem 2. Let  $B \in \mathbb{R}^{n \times n}$  be nonsingular. Then there exists  $A \in \mathbb{C}^{n \times n}$ , called logharithm of B, satisfying  $e^A = B$ .

(Floquet's Theorem) (i) If  $\Phi(t)$  is a fundamental matrix of (6), then so is  $\Phi(t+T)$ . Moreover there exists  $P(t) \in \mathbb{C}^{n \times n}$  which is nonsingular, differentiable and satisfies P(t) = P(t+T) for all t, and there exists  $R \in \mathbb{C}^{n \times n}$  such that  $\Phi(t) = P(t)e^{tR}$ .

(ii) Under the hypotheses of part (i), the nonautonomous linear system (5), under the linear change of coordinates  $y = P^{-1}(t)x$ , reduces to the autonomous linear system y' = Ry.

6. Consider the system

$$\begin{cases} x' = -2y + yz \\ y' = x - xz \\ z' = xy \end{cases}$$

**Problem.** (a) Investigate the stability, asymptotic stability, and instability of the origin. State the theorem used.

(b) Is the Hartman-Grobman theorem applicable to study the stability, asymptotic stability, and instability of the origin? (Why?)