交大應數系 97 學年度博士班入學考試離散數學題目

這份試卷有 6 大題, 共計 100 分. 請詳細標明題號並提供所有步驟,
證明題如須引用定理請詳述清楚.

1. [20%] Determine the number of integral solutions of the equation
$$x_1 + x_2 + x_3 + x_4 = 18$$
that satisfy
$$1 \le x_1 \le 5, \quad -2 \le x_2 \le 4, \quad 0 \le x_3 \le 5, \quad 3 \le x_4 \le 9.$$

2. [15%] Use the pigeonhole principle to prove that every sequence of 26 real numbers contains either an increasing subsequence of length 6 or a decreasing subsequence of length 6.

3. [15%] Determine the number h_n of ways to color the squares of a 1-by-n board with the colors red, white, and blue, where the number of red squares is even and there is at least one blue square.

4. [20%] Let G be a bipartite graph, let $\rho(G)$ denote the maximum number of edges in a matching in G and let $c(G)$ denote the minimum number of vertices in a cover of G. Show $\rho(G) = c(G)$. (Hint. A *matching* of G is a subset M of edges with the property that no two of the edges of M have a common vertex. A subset S of the vertex set of G is a *cover* if each edge of G has at least one of its two vertices in S.)

5. [10%] Let $F, G : \mathbb{N} \to \mathbb{Z}$ satisfy
$$G(n) = \sum_{k:k|n} F(k).$$
Show
$$F(n) = \sum_{k:k|n} \mu(n/k)G(k),$$
where
$$\mu(n) = \begin{cases} 1, & \text{if } n = 1; \\ (-1)^k, & \text{if } n \text{ is a product of } k \text{ distinct primes;} \\ 0, & \text{otherwise.} \end{cases}$$
and the symbol $k|n$ means that n is a multiple of k.

6. Let G be a finite group and let C be a set such that G acts on C, i.e. there exists a map $* : G \times C \longrightarrow C$ satisfying $g * c \in C$, $e * c = c$ and $g * (h * c) = (gh) * c$ for any $g, h \in G$ and $c \in C$, where e is the identity element of G. For $c \in C$, let $O_c := \{g * c \mid g \in G\}$ denote the *orbit* of c *under the action* of G and let $G_c := \{g \mid g * c = c\}$ denote the *stabilizer* (or *isotropy* subgroup) of c. For $g \in G$, let $C_g := \{c \in C \mid g * c = c\}$ denote the elements that are fixed by g.

   (a) [10%] Show that $|O_c| = \dfrac{|G|}{|G_c|}$ for $c \in C$.

   (b) [10%] Show that the number $N(G, C)$ of orbits in C is given by
$$N(G, C) = \frac{1}{|G|} \sum_{g \in G} |C_g|.$$